Problem Set #4

CO 639: Quantum Error Correction
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Due Tues., Mar. 9

Problem 1. Phase Error Correction

a) Consider the 3-qubit phase error-correcting code with stabilizer generated by $X \otimes X \otimes I$ and $I \otimes X \otimes X$. Write down a fault-tolerant syndrome measurement circuit for this code, including any necessary ancilla verification.

b) Which of the following gates, when performed transversally on the code, give a valid encoded operation: CNOT, Hadamard, Phase P? (That is, do $\mathrm{CNOT}^{\otimes 3}$, $H^{\otimes 3}$, and $P^{\otimes 3}$ preserve the coding space?) What logical gates are performed by those valid transversal operations? Can you find any other transversal operations?

c) Find a 3-qubit phase error-correcting code with a different set of transversal operations.

d) Suppose you have a phase error-correcting code for which transversal Hadamard returns you to the code space. Is the transversal Hadamard fault-tolerant?

Problem 2. Transversal Operations For Any Stabilizer Code

a) Given an $n \times n$ matrix A_{ij} of 0s and 1s, define a transformation A on the n-qubit Pauli group as follows. Let X_i and Z_i be X and Z acting on the ith qubit. Then

$$A(X_i) = \prod_{j=1}^{n-1} X_j^{A_{ij}} \qquad (1)$$

$$A(Z_i) = \prod_{j=1}^{n-1} Z_j^{A_{ij}}. \qquad (2)$$

When does this transformation correspond to conjugation by some unitary operation U? Of those As which correspond to unitary U, which are in the Clifford group?

b) Show that whenever A corresponds to a Clifford group operation, it defines a gate that acts transversally on an arbitrary stabilizer code.

c) Show that any gate which acts transversally on an arbitrary stabilizer code corresponds to a transformation A as given in part a.

d) Find a nontrivial gate that acts transversally on any stabilizer code.

Problem 3. Repeating Syndrome Measurement

a) Suppose we use Shor error correction for the 7-qubit code, including ancilla verification, but without repeating the syndrome measurement. Give an example of a place where a single error somewhere in the circuit can cause two errors in the final encoded state.

b) Now suppose we use Steane error correction, including the ancilla verification, but do not repeat the syndrome measurement. Can you find a place where a single error somewhere in the circuit can cause two errors in the final encoded state?

c) For the five-qubit code, draw the circuit to measure the logical $\overline{X}$ operator using the Shor cat state method. (You may omit the verification of the cat state.) Give an example of a place where a single error in the circuit can cause us to have the wrong measurement outcome.

d) Suppose we repeat the measurement by performing this circuit twice and get the same result both times. Show that we could still have the wrong outcome, even if only a single error occurred in the circuit and/or initial state. Can this be remedied? That is, can you find a way of measuring the encoded $\overline{X}$ operator for the five-qubit code that is robust against single errors? (Of course, two errors can cause the code to fail anyway.)

Problem 4. Measurements and Stabilizers

Suppose we start a system with the state $|\psi\rangle \otimes |0\rangle$, measure $Y \otimes X$, and then measure $I \otimes Y$.

a) What Pauli operations do we need to perform following each of the measurements to steer the state into the +1-eigenstate of each measured operator?

b) Compute the action of the above series of operations (with Pauli corrections) on the $\overline{X}$ and $\overline{Z}$ operators. Describe the overall action in terms of standard gates.

c) Suppose we had started with the input state $|\psi\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2}$ and then performed the same two measurements. What would have happened then?

Problem 5. Compressed Teleportation Constructions

a) Find a two-qubit circuit that allows Alice to transmit one qubit to Bob using 1 CNOT gate from Alice to Bob (Alice has the control qubit and Bob has the target qubit) and 1 bit of classical communication (plus as many single-qubit Clifford group gates as you like).

b) Find another circuit for the same task, but this time with a CNOT gate from Bob to Alice.

c) Let the $\pi/8$ rotation gate be the diagonal matrix $\mathrm{diag}(1, e^{i\pi/4})$ (so the $\pi/8$ gate is a square root of P). Note that $\pi/8 \otimes I$ commutes with CNOT (although $I \otimes \pi/8$ does not). Use this fact and one of the compressed teleportation circuits from parts a and b to find a single-qubit ancilla and corresponding fault-tolerant circuit that allows us to perform a $\pi/8$ gate on an encoded state of the 7-qubit code (or any other code which allows all transversal Clifford group operations).