Searching an ordered list with a quantum computer

Andrew Childs
Waterloo

based on joint work with Andrew Landahl and Pablo Parrilo
(quant-ph/0608161, PRA 07) and with Troy Lee (arXiv:0708.3396)

Query complexity

Problem: Compute a function $f : S \to \Sigma$

Input set: $S \subseteq \{0,1\}^n$ Output set: Σ

Fix some (unknown) input $x \in S$. Given a black box for the bits of x, how many queries are required to compute $f(x)$?

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Example: Unstructured search (aka OR)

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Deterministic classical query complexity: n
Randomized classical query complexity: $\Theta(n)$
Quantum query complexity: $\Theta(\sqrt{n})$

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In the above example, we have $x =$

0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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Theorem [C., Lee 07] This is asymptotically optimal among all adversary lower bounds.

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1. Upper bounds by semidefinite programming

Symmetry

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Definition: An *automorphism* of $f : S \to \Sigma$ is a permutation $\pi \in S_n$ satisfying

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The automorphisms of f form a group, $\mathrm{Aut}(f)$. This group structure can be exploited both when designing algorithms for computing f and when proving lower bounds showing that f is hard to compute.

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Recall ordered search function: e.g., for $n = 4$, the inputs are

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$$\begin{aligned} S' = \{&11110000, 01111000, 00111100, 00011110, \\ &00001111, 10000111, 11000011, 11100001\} \end{aligned}$$

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Note that an algorithm for this problem gives an algorithm for the original ordered search problem.

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FGGS polynomials

Consider exact algorithms for ordered search that are translation-invariant (no loss of generality), with no workspace and with no "null query" (possible loss of generality).

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A k-query algorithm corresponds to a solution of the following:

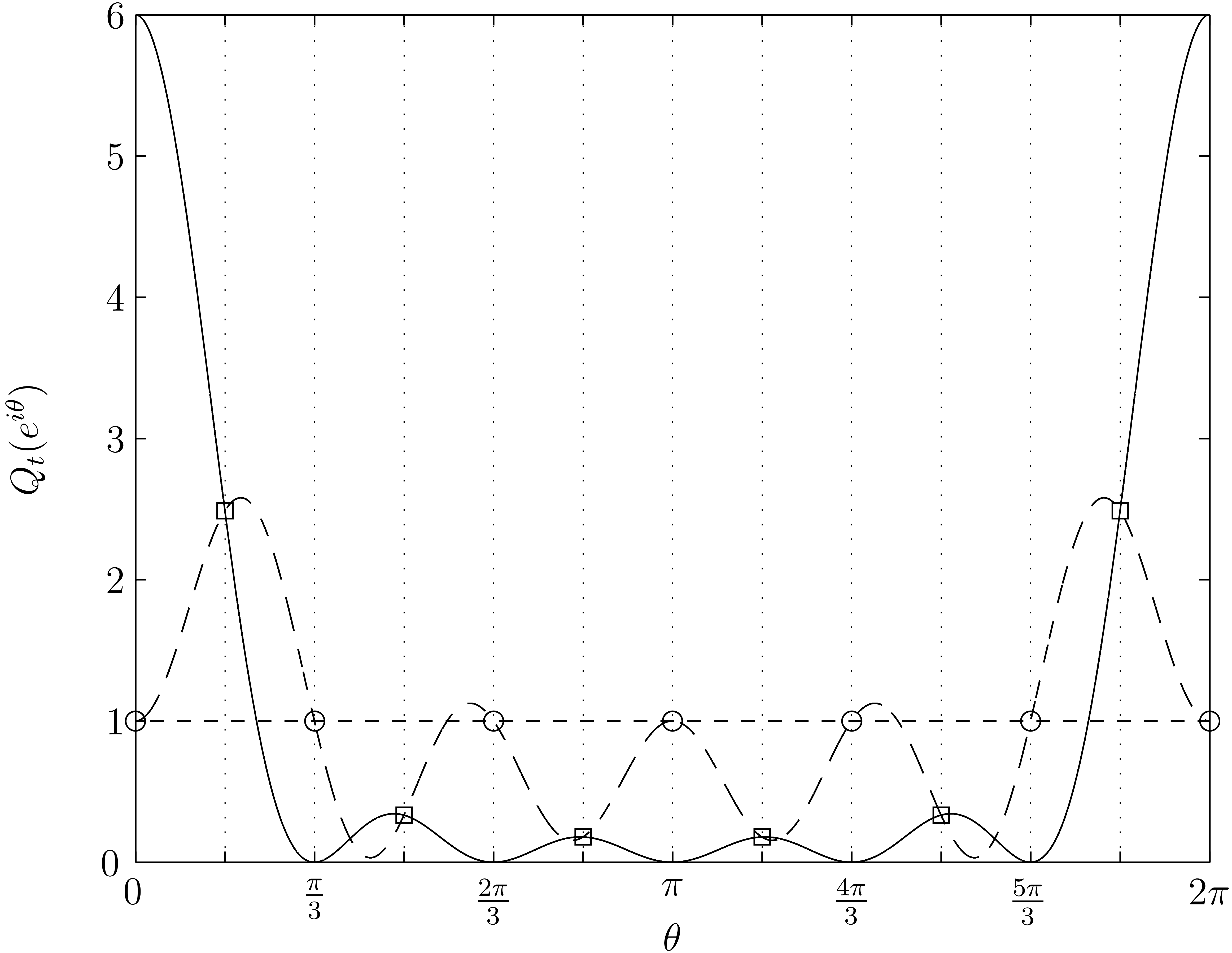
Find Laurent polynomials of degree $n-1$, $Q(z) = \sum_{i=-n-1}^{n-1} q_i \, z^i$, that are *symmetric* ($q_i = q_{-i}$) and *non-negative* ($Q_t(e^{i\theta}) \geq 0$), satisfying

$$Q_0(z) = \sum_{i=-(n-1)}^{n-1} (1 - \tfrac{|i|}{n}) z^i$$

$$Q_t(z) = Q_{t-1}(z) \qquad \text{at } z^n = (-1)^t, \; t = 1, 2, \ldots, k$$

$$Q_k(z) = 1$$

$$\frac{1}{2\pi} \int_0^{2\pi} Q_t(e^{i\theta}) \, \mathrm{d}\theta = 1 \qquad t = 0, 1, \ldots, k$$



$Q_t(e^{i\theta})$
6
5
4
3
2
1
0
0
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
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2π
θ

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- There are fast (classical) algorithms to solve semidefinite programs numerically (using so-called *interior point methods*).
- From a primal SDP (say, a minimization problem), we can construct a dual SDP (a maximization problem).
  The minimum value of the primal SDP equals the mazimum value of the dual SDP.
  A particular solution of the primal gives an upper bound; a particular solution of the dual gives a lower bound.

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Find $n \times n$ symmetric positive semidefinite matrices $Q_1, \ldots, Q_{k-1}$ satisfying

$$\begin{aligned} Q_0 &= E/n \\ \mathcal{T}_t\, Q_t &= \mathcal{T}_t\, Q_{t-1} && t = 1, 2, \ldots, k \\ Q_k &= I/n \\ \operatorname{tr}\, Q_t &= 1 && t = 0, 1, \ldots, k \end{aligned}$$

where E is the matrix with every entry equal to 1, and

$$\mathcal{T}_t\, Q := \sum_{i=1}^{n-t} Q_{i,i+t} + (-1)^t \sum_{i=1}^{t} Q_{i,i+n-t}\,.$$

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Proof: Uses Fejér-Riesz theorem to relate non-negative polynomials to positive semidefinite matrices.

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However, for $k=2,3$ we know these are best possible (by solving a different SDP that characterizes *general* quantum query algorithms [Barnum, Saks, Szegedy 03]).

II. Optimality of adversary lower bounds

The quantum adversary method

$$\mathrm{ADV}(f) := \max_{\substack{\Gamma \geq 0 \\ \Gamma \neq 0}} \frac{\|\Gamma\|}{\max_i \|\Gamma_i\|}$$

where Γ is an $|S| \times |S|$ matrix

entries $\Gamma[x,y]$ correspond to pairs of inputs $x, y \in S$

$\Gamma[x,y] = 0$ if $f(x) = f(y)$

$$\Gamma_i[x,y] := \begin{cases} 0 & x_i = y_i \\ \Gamma[x,y] & x_i \neq y_i \end{cases}$$

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Theorem [Ambainis 00]: (Q. query complexity of f) $\geq \frac{1}{2}\mathrm{ADV}(f)$.

Proof idea: Define a progress measure for algorithms. It starts at 0 and must reach $\|\Gamma\|$ for the algorithm to succeed; the maximum change per query is $2 \max_i \|\Gamma_i\|$.

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Solving this SDP can be simplified using symmetry.

Automorphism principle [Høyer, Lee, Špalek 07]: If π is an automorphism of f, then we can choose an optimal adversary matrix Γ satisfying $\Gamma[x,y] = \Gamma[\pi(x), \pi(y)]$ for all pairs of inputs x, y.

Furthermore, if the automorphism group is transitive, then the uniform vector is a principal eigenvector of Γ, and all $\|\Gamma_i\|$ are equal.

Symmetric ordered search ≈ Ordered search

Now consider the problem of identifying the input in the symmetrized ordered search problem on $2n$ bits (not just mod n). Then the automorphism group is simply cyclic with $2n$ elements. We call this problem OSP_n.

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Symmetric ordered search ≈ Ordered search

Now consider the problem of identifying the input in the symmetrized ordered search problem on $2n$ bits (not just mod n). Then the automorphism group is simply cyclic with $2n$ elements. We call this problem OSP_n.

This problem looks similar to the original ordered search problem, but maybe its query complexity is dramatically different!

In fact, the query complexity differs by at most 1.

Reduction, original → symmetric: $x' = x_1 x_2 \ldots x_n \bar{x}_1 \bar{x}_2 \ldots \bar{x}_n$

Reduction, symmetric → original: $x' = \begin{cases} x_1 x_2 \ldots x_n & x_n = 1 \\ x_{n+1} x_{n+2} \ldots x_{2n} & x_n = 0 \end{cases}$

one extra query

Asymptotically, this is negligible.

Adversary SDP for ordered search

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} \rotatebox{90}{11110000} & \rotatebox{90}{01111000} & \rotatebox{90}{00111100} & \rotatebox{90}{00011110} & \rotatebox{90}{00001111} & \rotatebox{90}{10000111} & \rotatebox{90}{11000011} & \rotatebox{90}{11100001} \end{array} & \\
\Gamma = & \begin{bmatrix}
0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\
\gamma_1 & 0 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_2 & \gamma_8 & 0 & \gamma_{14} & \gamma_{15} & \gamma_{16} & \gamma_{17} & \gamma_{18} \\
\gamma_3 & \gamma_9 & \gamma_{14} & 0 & \gamma_{19} & \gamma_{20} & \gamma_{21} & \gamma_{22} \\
\gamma_4 & \gamma_{10} & \gamma_{15} & \gamma_{19} & 0 & \gamma_{23} & \gamma_{24} & \gamma_{25} \\
\gamma_5 & \gamma_{11} & \gamma_{16} & \gamma_{20} & \gamma_{23} & 0 & \gamma_{26} & \gamma_{27} \\
\gamma_6 & \gamma_{12} & \gamma_{17} & \gamma_{21} & \gamma_{24} & \gamma_{26} & 0 & \gamma_{28} \\
\gamma_7 & \gamma_{13} & \gamma_{18} & \gamma_{22} & \gamma_{25} & \gamma_{27} & \gamma_{28} & 0
\end{bmatrix} & \begin{array}{l} 11110000 \\ 01111000 \\ 00111100 \\ 00011110 \\ 00001111 \\ 10000111 \\ 11000011 \\ 11100001 \end{array}
\end{array}
$$

Adversary SDP for ordered search

By the automorphism principle, we can assume

$$\Gamma = \begin{array}{cc} \begin{array}{cccccccc} \rotatebox{90}{11110000} & \rotatebox{90}{01111000} & \rotatebox{90}{00111100} & \rotatebox{90}{00011110} & \rotatebox{90}{00001111} & \rotatebox{90}{10000111} & \rotatebox{90}{11000011} & \rotatebox{90}{11100001} \end{array} & \\ \begin{bmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 \\ \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & \gamma_1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 \end{bmatrix} & \begin{array}{l} 11110000 \\ 01111000 \\ 00111100 \\ 00011110 \\ 00001111 \\ 10000111 \\ 11000011 \\ 11100001 \end{array} \end{array}$$

Spectral norm achieved by uniform eigenvector: $\gamma_n + 2\sum_{i=1}^{n-1} \gamma_i$

Adversary SDP for ordered search

Also by the automorphism principle, it suffices to consider

$$\Gamma_8 = \begin{array}{c} \begin{array}{cccccccc} \rotatebox{90}{11110000} & \rotatebox{90}{01111000} & \rotatebox{90}{00111100} & \rotatebox{90}{00011110} & \rotatebox{90}{00001111} & \rotatebox{90}{10000111} & \rotatebox{90}{11000011} & \rotatebox{90}{11100001} \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ 0 & 0 & 0 & 0 & \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 \\ 0 & 0 & 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 \\ 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \\ \gamma_3 & \gamma_4 & \gamma_3 & \gamma_2 & 0 & 0 & 0 & 0 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_3 & 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 11110000 \\ 01111000 \\ 00111100 \\ 00011110 \\ 00001111 \\ 10000111 \\ 11000011 \\ 11100001 \end{array} \end{array}$$

In general, $\|\Gamma_{2n}\| = \|\mathrm{Toeplitz}(\gamma_n, \gamma_{n-1}, \ldots, \gamma_1)\|$.

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$$\max \gamma_n + 2\sum_{i=1}^{n-1} \gamma_i \quad \text{subject to} \quad \|\mathrm{Toeplitz}(\gamma_n, \ldots, \gamma_1)\| \leq 1,\ \gamma_i \geq 0$$

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Høyer, Neerbek, Shi: Let $\gamma_i = \begin{cases} \frac{1}{\pi i} & i = 1, 2, \ldots, \lfloor n/2 \rfloor \\ 0 & \text{otherwise} \end{cases}$

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Hilbert matrix: $\left\| \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & & \\ \frac{1}{3} & & & \\ \vdots & & & \end{bmatrix} \right\| = \pi$

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Theorem.

$$\mathrm{ADV}(\mathrm{OSP}_{2m}) = 2\sum_{i=0}^{m-1}\left(\frac{\binom{2i}{i}}{4^i}\right)^2 = \frac{2}{\pi}(\ln 16m + \gamma) + O(\tfrac{1}{m})$$

$$\mathrm{ADV}(\mathrm{OSP}_{2m+1}) = 2\sum_{i=0}^{m-1}\left(\frac{\binom{2i}{i}}{4^i}\right)^2 + \left(\frac{\binom{2m}{m}}{4^m}\right)^2$$

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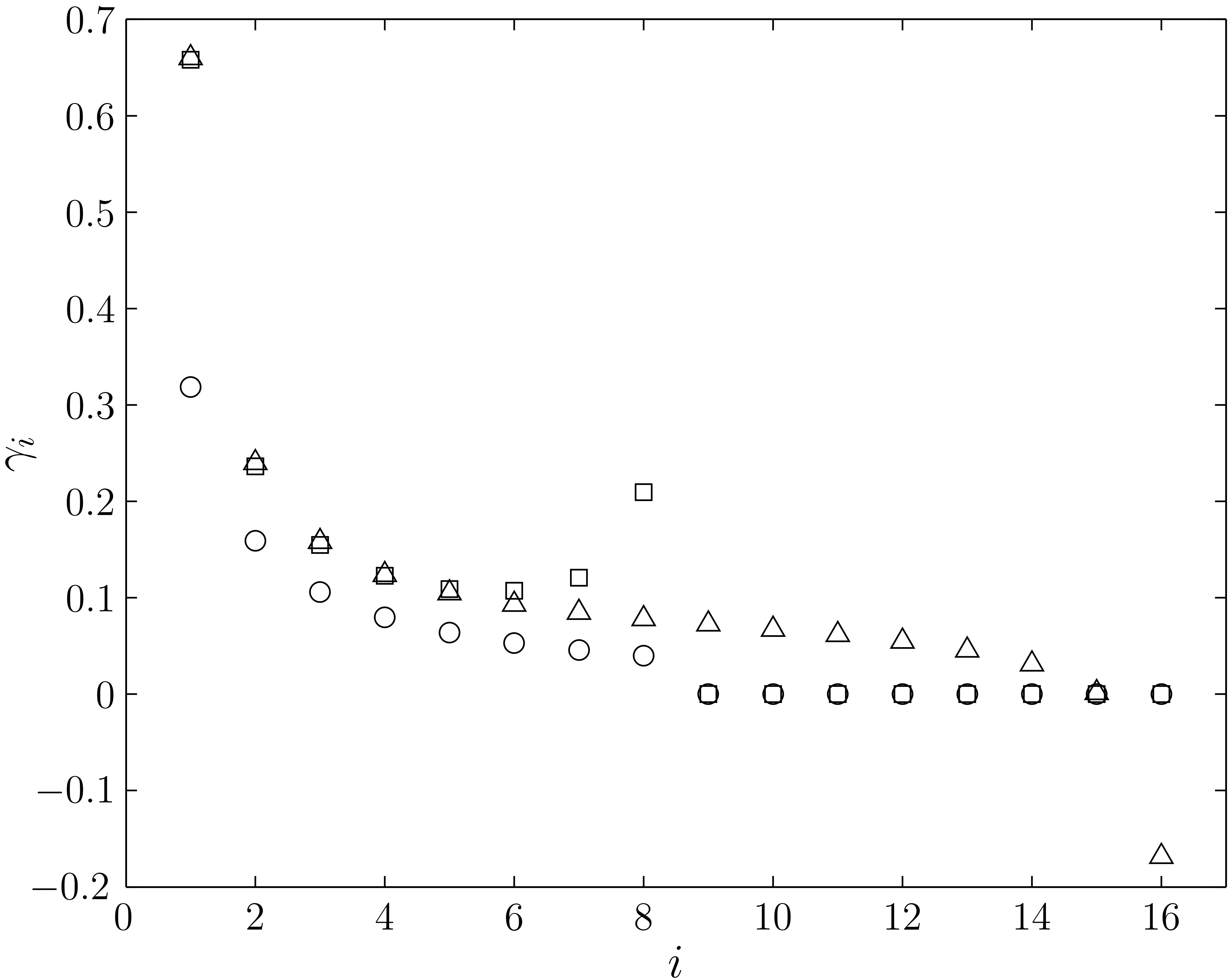
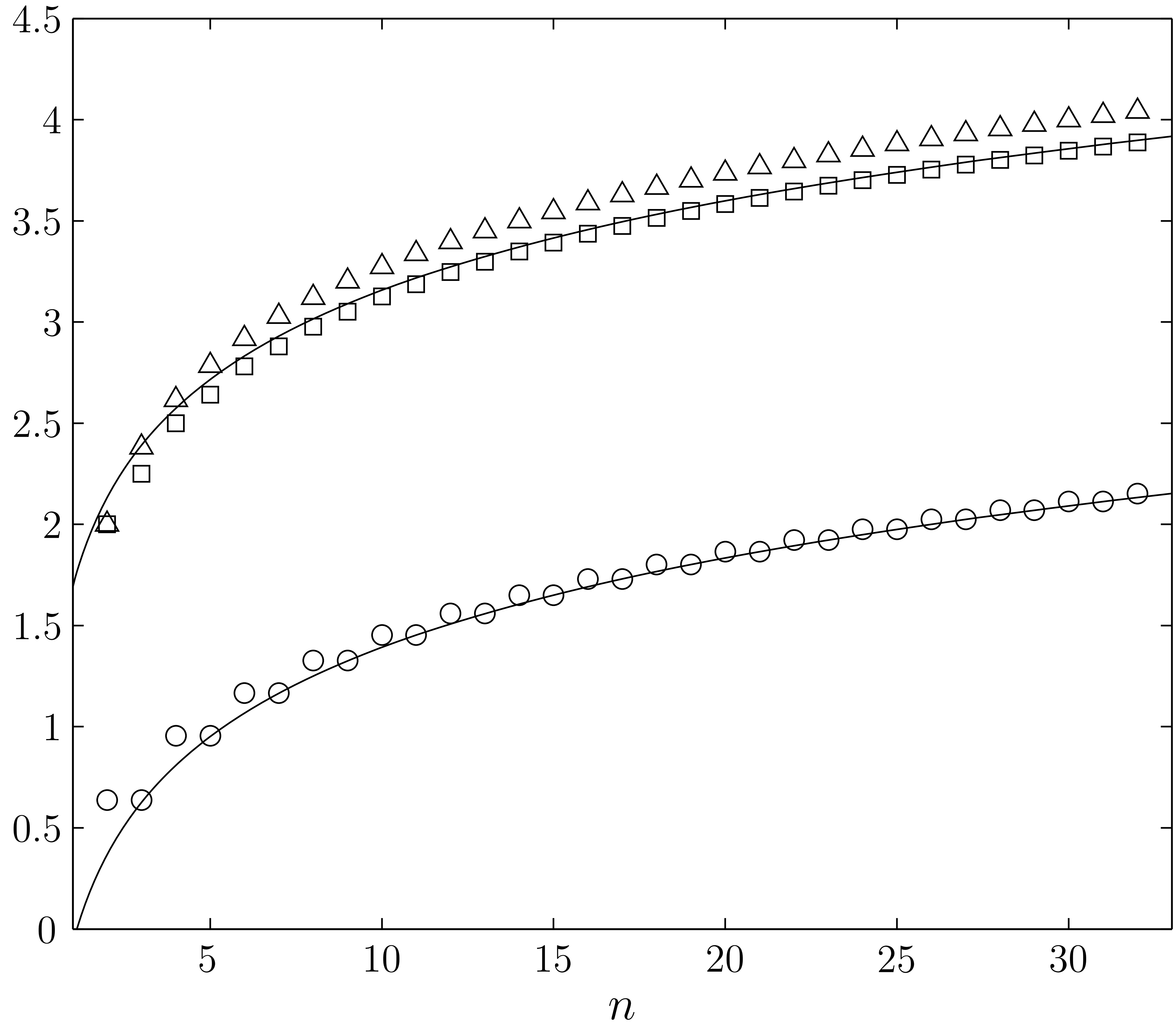
Primal is more technical but uses similar ideas.



0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
γ_i
0
2
4
6
8
10
12
14
16
i


4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
5
10
15
20
25
30
n

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(Quantum query complexity of f) $\geq \frac{1}{2}\mathrm{ADV}^{\pm}(f) \geq \frac{1}{2}\mathrm{ADV}(f)$.

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Furthermore, there are functions for which the negative adversary gives a significantly better lower bound.

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We give a general analysis of the spectra of such matrices.

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- Can we find optimal adversary lower bounds for other problems? (Element distinctness?)

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Recall $\xi_i := \dfrac{\binom{2i}{i}}{4^i}$

Proposition. For any j, $\displaystyle\sum_{i=0}^{j} \xi_i \, \xi_{j-i} = 1.$

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Asymptotic analysis

$$\mathrm{ADV}(\mathrm{OSP}_n) = 2 \sum_{i=0}^{\frac{n}{2}-1} \left(\frac{\binom{2i}{i}}{4^i} \right)^2$$

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For comparison, the HNS bound says

$$\mathrm{ADV}(\mathrm{OSP}_n) \geq \frac{2}{\pi}(\ln n + \gamma - \ln 2) + O(1/n)$$