Computergrafik

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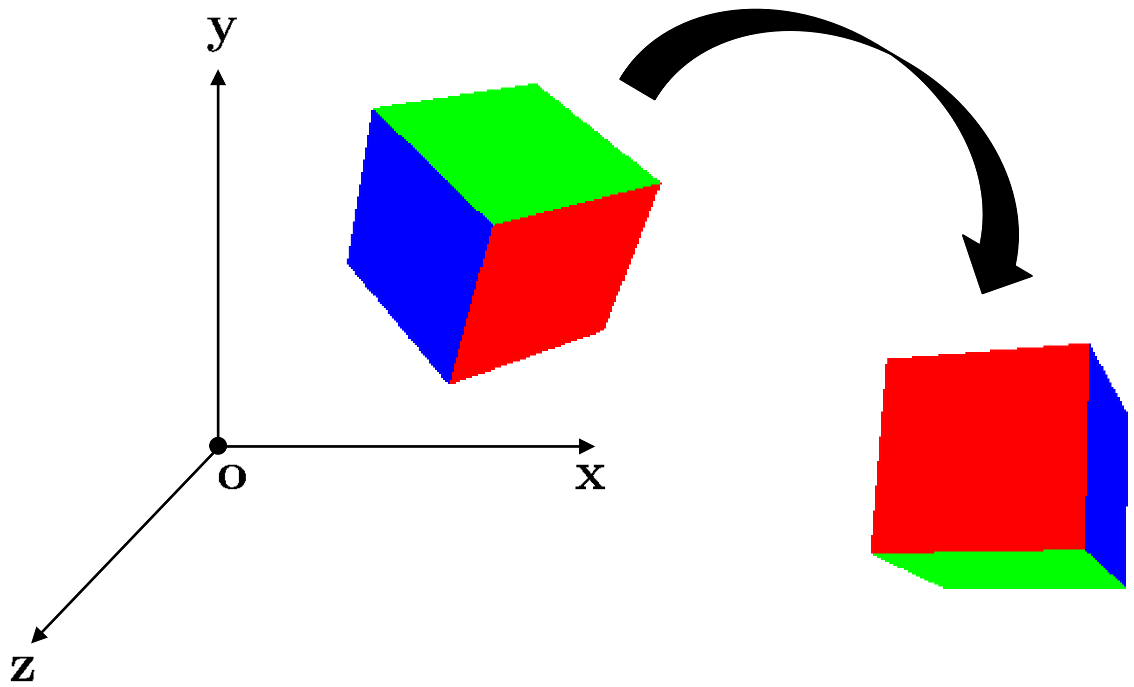
Today

Transformations & matrices

- Introduction
- Matrices
- Homogeneous coordinates
- Affine transformations
- Concatenating transformations
- Change of coordinates
- Common coordinate systems

Introduction

- Goal: Freely position rigid objects in 3D space



- What are the degrees of freedom to position a rigid object?
- How to express mathematically?

Today

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Matrices

- Matrix algebra to implement geometric transformations
  - Position 3D objects in space
  - Specify location, orientation

Matrices

Abstract point of view

- Mathematical objects with set of operations
  - Addition, subtraction, multiplication, multiplicative inverse, etc.
- Similar to integers, real numbers, etc.

But

- Properties of operations are different
  - E.g., multiplication is not commutative
- Represent different intuitive concepts
  - Scalar numbers represent distances
  - Matrices can represent coordinate systems, rigid motions, in 3D and higher dimensions, etc.

Matrices

Practical point of view

- Rectangular array of numbers

$$\mathbf{M} = \begin{bmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m,1} & m_{2,2} & \dots & m_{m,n} \end{bmatrix} \in \mathbf{R}^{m \times n}$$

- Square matrix if $\mathbf{m} = \mathbf{n}$
- In graphics often $\mathbf{m} = \mathbf{n} = 3, \mathbf{m} = \mathbf{n} = 4$

Matrix addition

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \dots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \dots & a_{2,n} + b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & a_{2,2} + b_{2,2} & \dots & a_{m,n} + b_{m,n} \end{bmatrix}$$

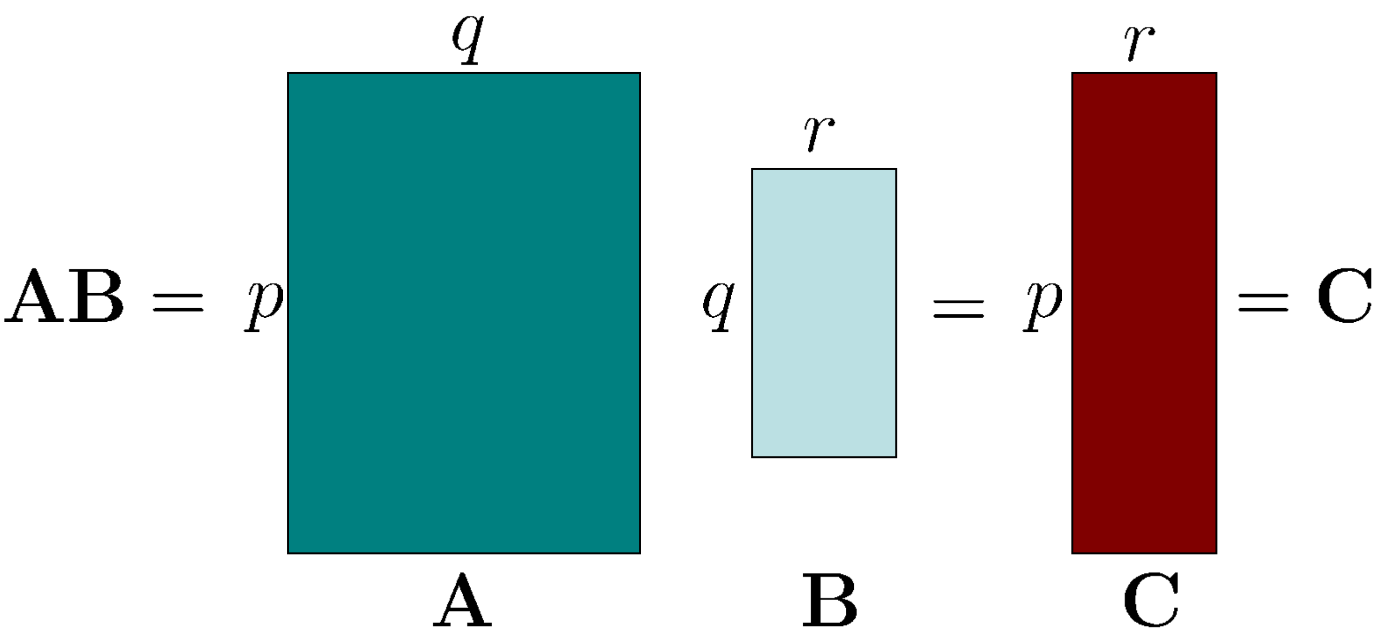
$$\mathbf{A}, \mathbf{B} \in \mathbf{R}^{m \times n}$$

Multiplication with scalar

$$s\mathbf{M} = \mathbf{M}s = \begin{bmatrix} sm_{1,1} & sm_{1,2} & \dots & sm_{1,n} \\ sm_{2,1} & sm_{2,2} & \dots & sm_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ sm_{m,1} & sm_{2,2} & \dots & sm_{m,n} \end{bmatrix}$$

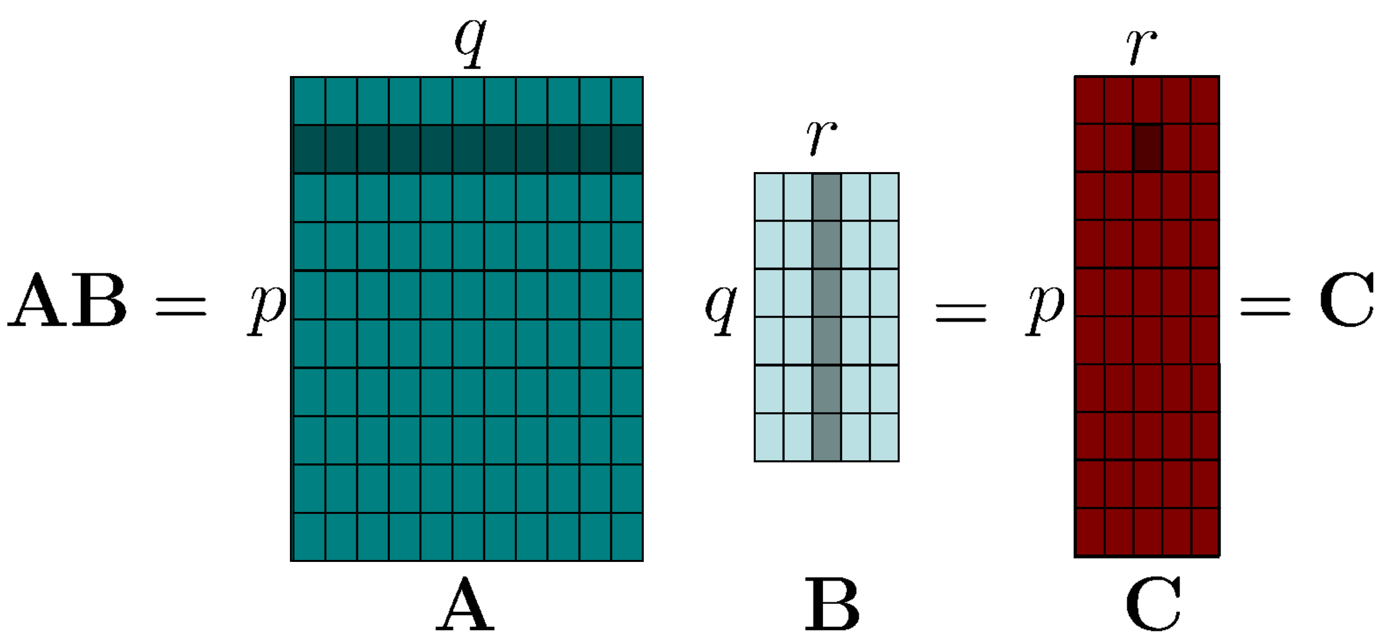
Matrix multiplication

$$\mathbf{A}\mathbf{B} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$



Matrix multiplication

$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$



Matrix multiplication

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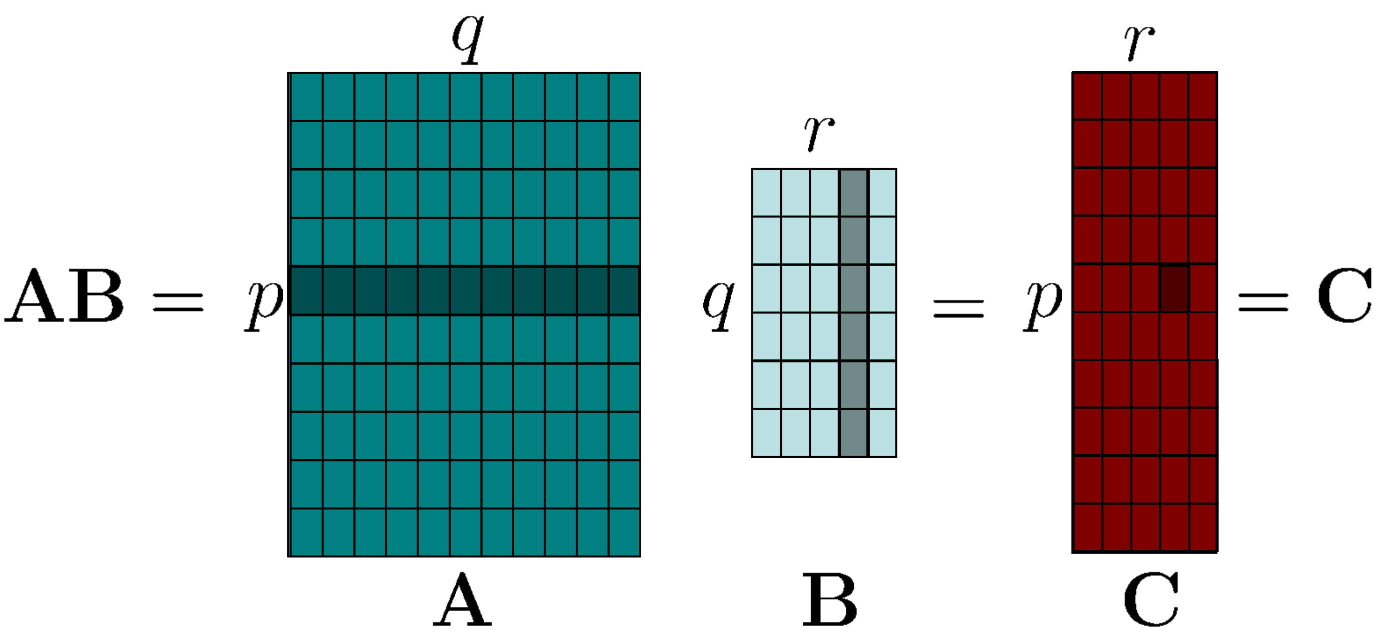
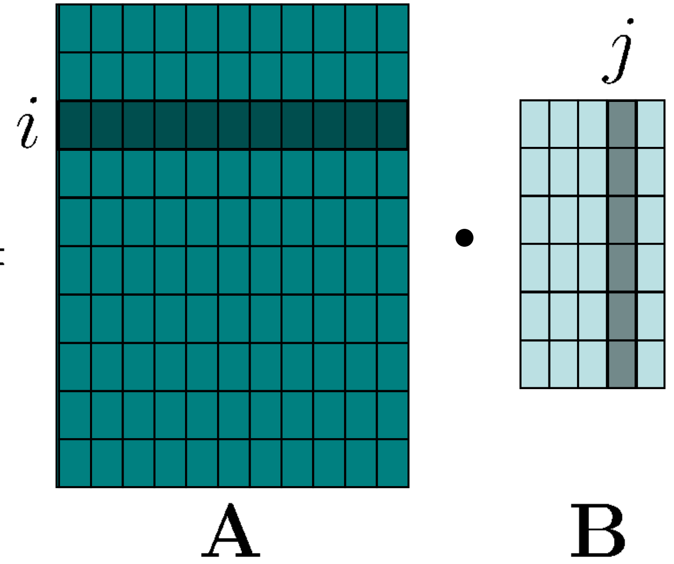


Matrix multiplication

$$\mathbf{AB} = \mathbf{C}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{B} \in \mathbf{R}^{q,r}, \mathbf{C} \in \mathbf{R}^{p,r}$$

$$\boxed{(\mathbf{AB})_{i,j} = \mathbf{C}_{i,j} = \sum_{k=1}^{q} a_{i,k} b_{k,j}, \quad i \in 1..p, j \in 1..r}$$

$$(\mathbf{AB})_{i,j} = \mathbf{C}_{i,j} =$$



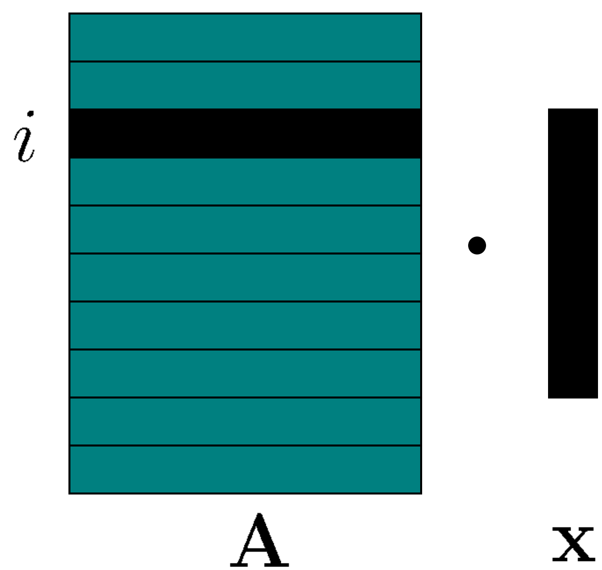
Matrix multiplication

Special case: matrix-vector multiplication

$$\mathbf{A}\mathbf{x} = \mathbf{y}, \quad \mathbf{A} \in \mathbf{R}^{p,q}, \mathbf{x} \in \mathbf{R}^{q}, \mathbf{y} \in \mathbf{R}^{p}$$

$$(\mathbf{A}\mathbf{x})_i = \mathbf{y}_i = \sum_{k=1}^{q} a_{i,k} x_k$$

$$(\mathbf{A}\mathbf{x})_i = \mathbf{y}_i =$$



Linearity

- Distributive law holds

$$\mathbf{A}(s\mathbf{B} + t\mathbf{C}) = s\mathbf{A}\mathbf{B} + t\mathbf{A}\mathbf{C}$$

  i.e., matrix multiplication is linear
  http://en.wikipedia.org/wiki/Linear_map

- But multiplication is not commutative,

$$\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$$

  in general

Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbf{R}^{n \times n}$$

$$\mathbf{MI} = \mathbf{IM} = \mathbf{M}, \quad \text{for any } \mathbf{M} \in \mathbf{R}^{n \times n}$$

Matrix inverse

Definition

If a square matrix M is non-singular, there exists a unique inverse M^{-1} such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

- Note

$$(\mathbf{MPQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\mathbf{M}^{-1}$$

- Computation
  - Gaussian elimination, Cramer’s rule
  - Review in your linear algebra book, or quick summary http://www.maths.surrey.ac.uk/explore/emmaspages/option1.html

Java vs. OpenGL matrices

- OpenGL (underlying 3D graphics API used in the Java code, more later)
  http://en.wikipedia.org/wiki/OpenGL
  - Matrix elements stored in array of floats float M[16];
  - "Column major" ordering

$$\begin{bmatrix} m[0] & m[4] & m[8] & m[12] \\ m[1] & m[5] & m[9] & m[13] \\ m[2] & m[6] & m[10] & m[14] \\ m[3] & m[7] & m[11] & m[15] \end{bmatrix}$$

- Java base code
  - "Row major" indexing
  - Conversion from Java to OpenGL convention hidden somewhere in basecode!

$$\begin{bmatrix} m(0,0) & m(0,1) & m(0,2) & m(0,3) \\ m(1,0) & m(1,1) & m(1,2) & m(1,3) \\ m(2,0) & m(2,1) & m(2,2) & m(2,3) \\ m(3,0) & m(3,1) & m(3,2) & m(3,3) \end{bmatrix}$$

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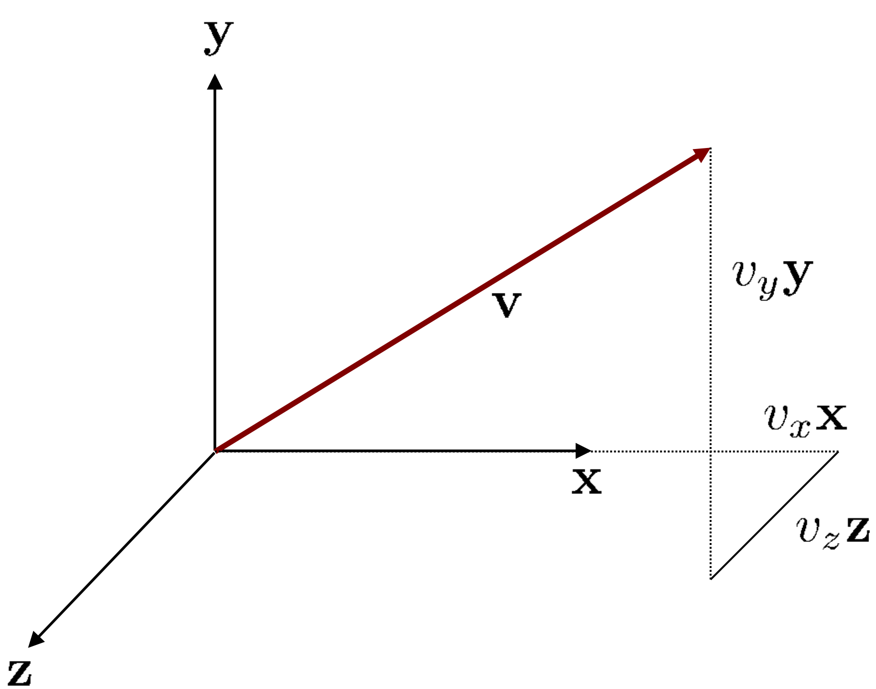
Vectors & coordinate systems

- Vectors defined by orientation, length
- Describe using three basis vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$



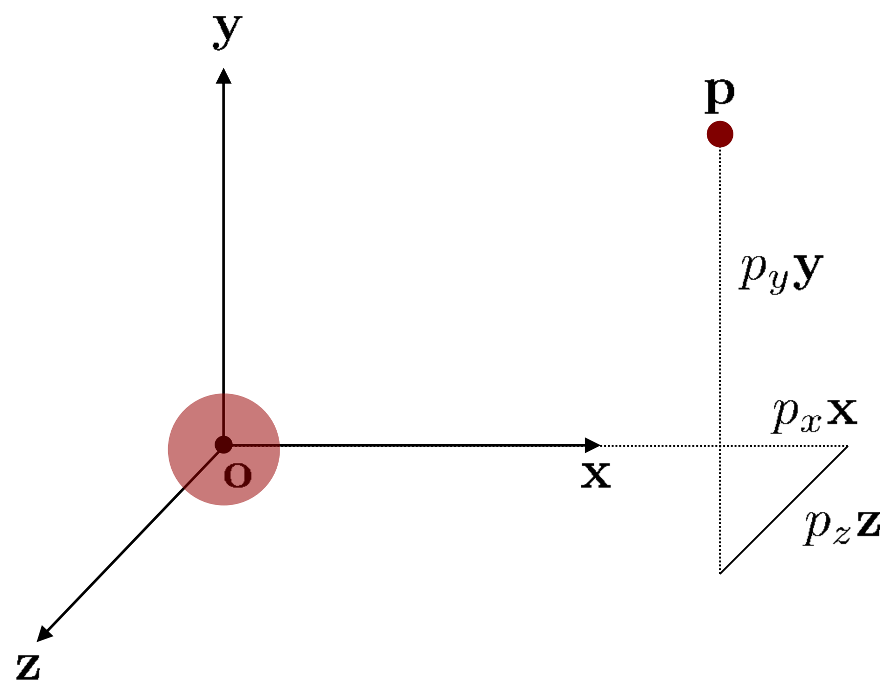
$$\mathbf{v} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z}$$

Points in 3D

- How do we represent 3D points?
- Are three basis vectors enough to define the location of a point?

Points in 3D

- Describe using three basis vectors and reference point, origin



$$\mathbf{p} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z} + \mathbf{o}$$

Vectors vs. points

- Vectors

$$\mathbf{v} = v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z} + \boxed{0 \cdot \mathbf{o}} \qquad \begin{bmatrix} v_x \\ v_y \\ v_z \\ \boxed{0} \end{bmatrix}$$

- Points

$$\mathbf{p} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z} + \boxed{1 \cdot \mathbf{o}} \qquad \begin{bmatrix} p_x \\ p_y \\ p_z \\ \boxed{1} \end{bmatrix}$$

- Representation of vectors and points using 4th coordinate is called homogeneous coordinates

Homogeneous coordinates

- Represent an affine space
  http://en.wikipedia.org/wiki/Affine_space
- Intuitive definition
  - Affine spaces consist of a vector space and a set of points
  - There is a subtraction operation that takes two points and returns a vector
  - Axiom I: for any point $\mathbf{a}$ and vector $\mathbf{v}$, there exists point $\mathbf{b}$, such that $(\mathbf{b}-\mathbf{a}) = \mathbf{v}$
  - Axiom II: for any points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we have $(\mathbf{b}-\mathbf{a})+(\mathbf{c}-\mathbf{b}) = \mathbf{c}-\mathbf{a}$

Affine space

Vector space,

http://en.wikipedia.org/wiki/Vector_space

- [xyz] coordinates
- represents vectors

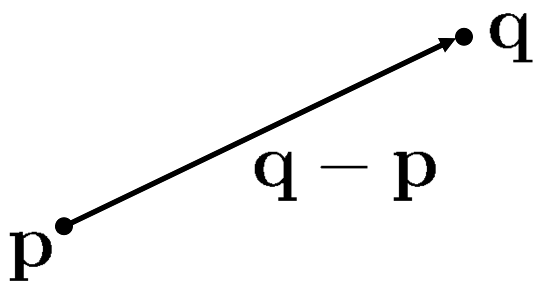
Affine space

http://en.wikipedia.org/wiki/Affine_space

- [xyz1], [xyz0] homogeneous coordinates
- distinguishes points and vectors

Homogeneous coordinates

- Subtraction of two points yields a vector



- Using homogeneous coordinates

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} \qquad \mathbf{q} - \mathbf{p} = \begin{bmatrix} q_x - p_x \\ q_y - p_y \\ q_z - p_z \\ 0 \end{bmatrix}$$

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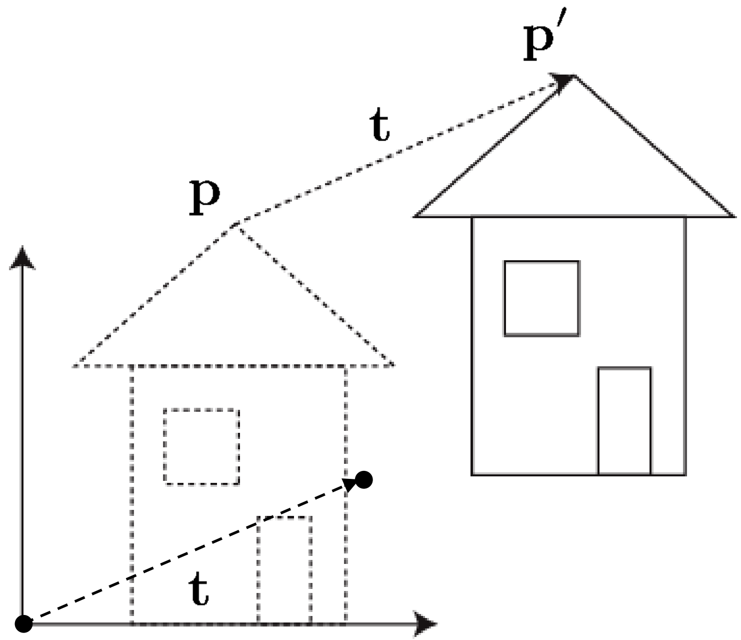
Affine transformations

- Transformation, or mapping: function that maps each 3D point to a new 3D point „f: $\mathbf{R}^3$ -> $\mathbf{R}^3$“
- Affine transformations: class of transformations to position 3D objects in space
- Affine transformations include
  - Rigid transformations
    - Rotation
    - Translation
  - Non-rigid transformations
    - Scaling
    - Shearing

Affine transformations

- Definition: mappings that preserve colinearity and ratios of distances
  http://en.wikipedia.org/wiki/Affine_transformation
  - Straight lines are preserved
  - Parallel lines are preseverd
- Linear transformations + translation
- Nice: All desired transformations (translation, rotation) implemented using homogeneous coordinates and matrix-vector multiplication

Translation



Point

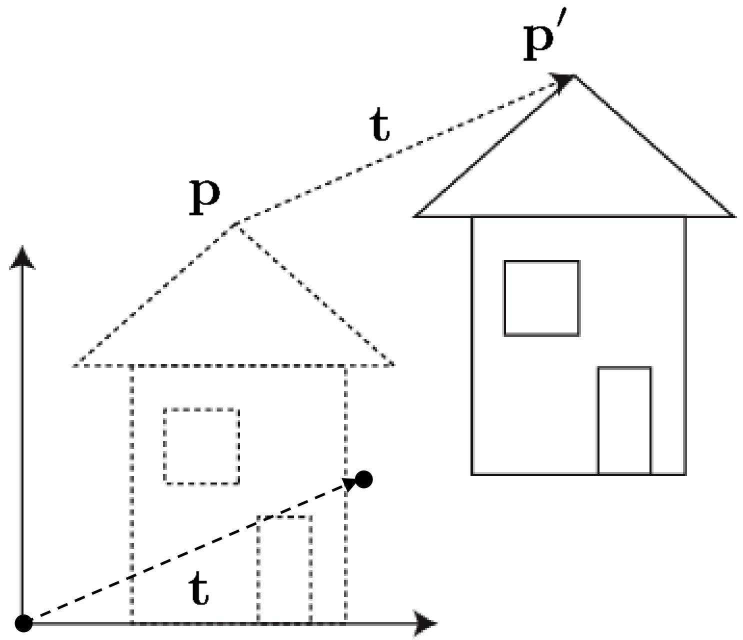
$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Vector

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \\ 0 \end{bmatrix}$$

$$\mathbf{p}' = \mathbf{p} + \mathbf{t} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{bmatrix}$$

Matrix formulation



Point

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Vector

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$$\mathbf{p}' = \mathbf{p} + \mathbf{t} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix}}_{\mathbf{p}'} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}(\mathbf{t})} \underbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}}_{\mathbf{p}}$$

$$\mathbf{p}' = \mathbf{T}(\mathbf{t})\mathbf{p}$$

Matrix formulation

- Inverse translation

$$\mathbf{T}(\mathbf{t})^{-1} = \mathbf{T}(-\mathbf{t})$$

$$\mathbf{T}(\mathbf{t}) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}(-\mathbf{t}) = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Verify that

$$\mathbf{T}(-\mathbf{t})\mathbf{T}(\mathbf{t}) = \mathbf{T}(\mathbf{t})\mathbf{T}(-\mathbf{t}) = \mathbf{I}$$

Note

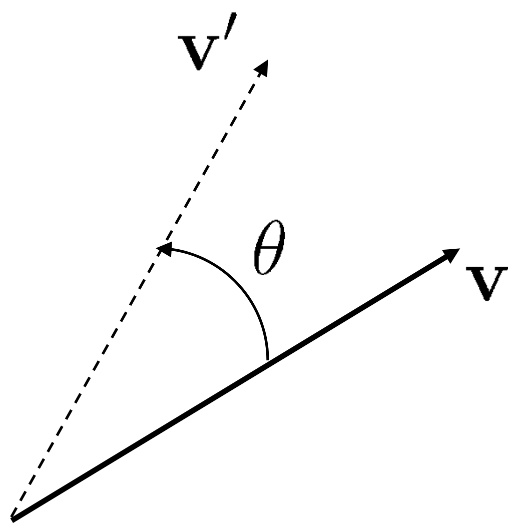
- What happens when you translate a vector?

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = ?$$

Rotation

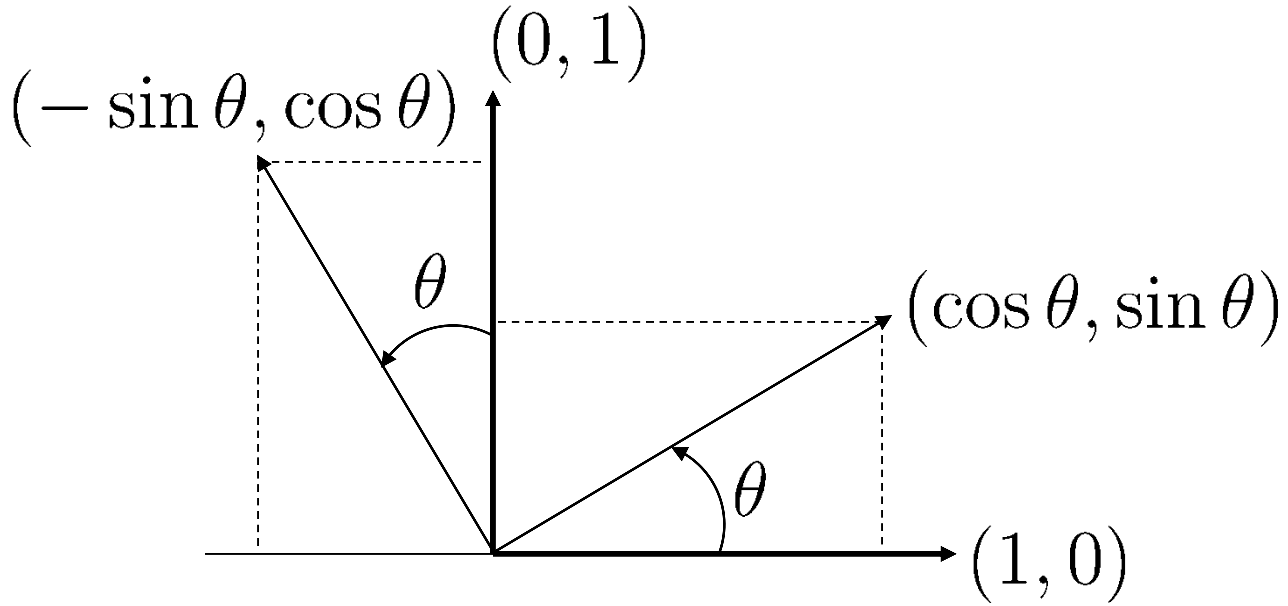
First: rotating a vector in 2D

- Convention: positive angle rotates counterclockwise
- Express using rotation matrix $\mathbf{R}(\theta)$

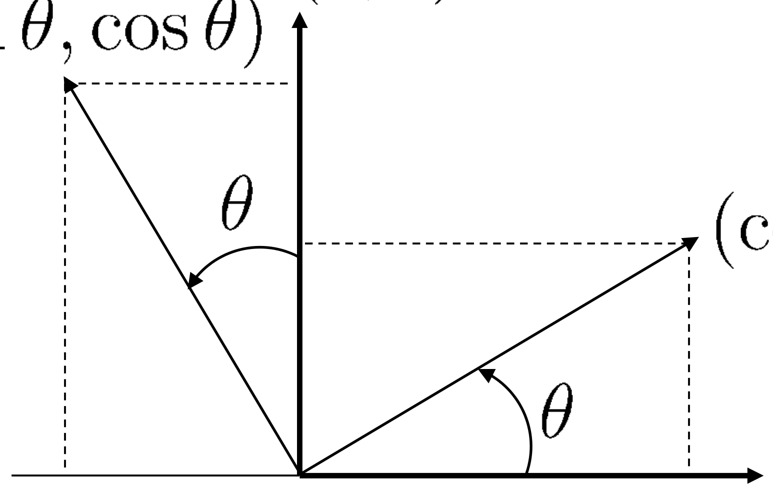


$$\mathbf{v}' = \mathbf{R}(\theta)\mathbf{v}$$

Rotating a vector in 2D



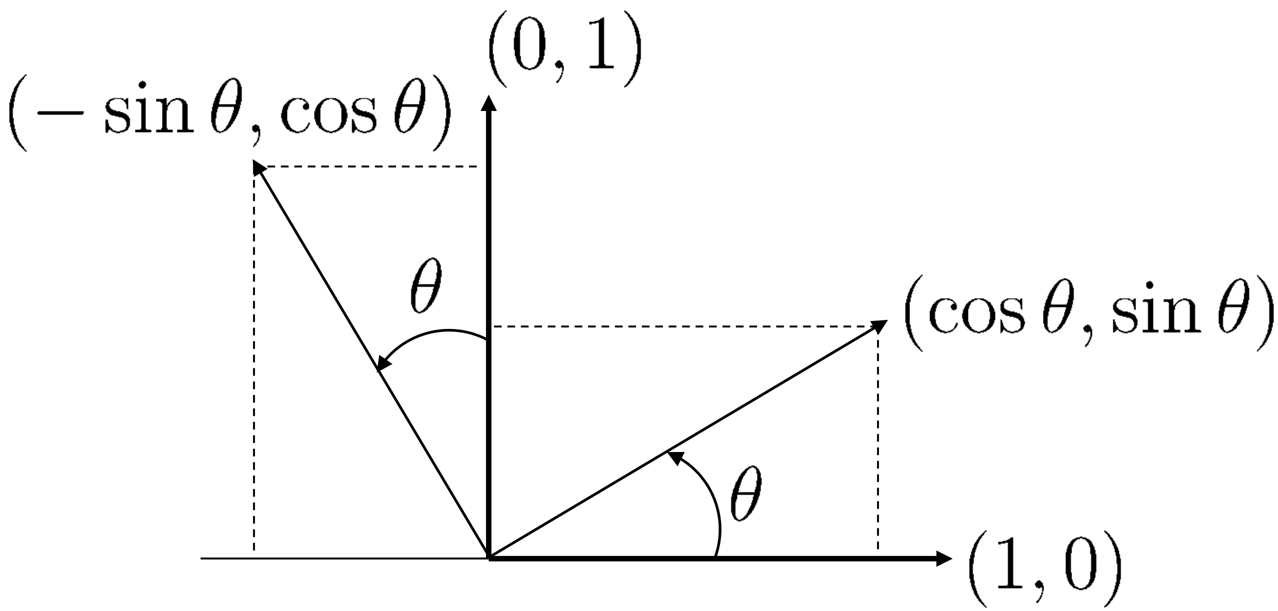
Rotating a vector in 2D



$$\mathbf{R}(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\mathbf{R}(\theta)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Rotating a vector in 2D



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$$\mathbf{R}(\theta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation in 3D

Rotation around z-axis

- z-coordinate does not change

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{v}' = \mathbf{R}_z(\theta)\mathbf{v}$$

$$\mathbf{R}_z(\theta)\mathbf{v} = \begin{bmatrix} \cos(\theta)v_x - \sin(\theta)v_y \\ \sin(\theta)v_x + \cos(\theta)v_y \\ v_z \\ 1 \end{bmatrix}$$

- What is the matrix for $\theta = 0, \theta = 90, \theta = 180$?

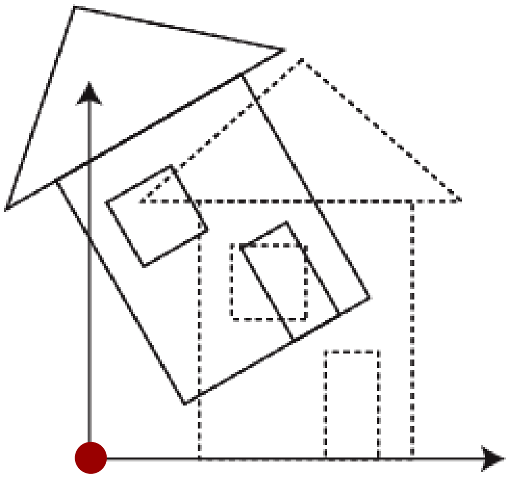
Other coordinate axes

- Same matrix to rotate points and vectors
- Points are rotated around origin

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

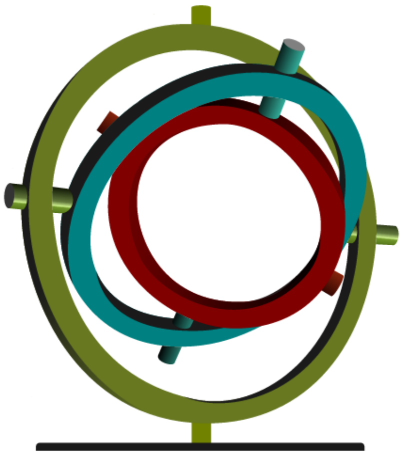
$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Rotation in 3D

- Concatenate rotations around *x,y,z* axes to obtain rotation around arbitrary axes through origin

$$\mathbf{R}_{x,y,z}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)$$



Gimbal

https://en.wikipedia.org/wiki/Gimbal

- $\theta_x, \theta_y, \theta_z$ are called Euler angles

  http://en.wikipedia.org/wiki/Euler_angles

- Disadvantage: result depends on order!

$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) \neq \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x)$$

Rotation around arbitrary axis

- Still: origin does not change
- Counterclockwise rotation
- Angle θ, unit axis $\mathbf{a}$
- $c_\theta = \cos\theta, s_\theta = \sin\theta$

$$\mathbf{R}(\mathbf{a},\theta) = \begin{bmatrix} a_x^2 + c_\theta(1-a_x^2) & a_x a_y(1-c_\theta) - a_z s_\theta & a_x a_z(1-c_\theta) + a_y s_\theta & 0 \\ a_x a_y(1-c_\theta) + a_z s_\theta & a_y^2 + c_\theta(1-a_y^2) & a_y a_z(1-c_\theta) - a_x s_\theta & 0 \\ a_x a_z(1-c_\theta) - a_y s_\theta & a_y a_z(1-c_\theta) + a_x s_\theta & a_z^2 + c_\theta(1-a_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Intuitive derivation see http://mathworld.wolfram.com/RotationFormula.html

Summary

- Different ways to describe rotations mathematically
  - Sequence of rotations around three axes (Euler angles)
  - Rotation around arbitrary angles (axis-angle representation)
  - Other options exist (quaternions, etc.)
- Rotations preserve
  - Angles
  - Lengths
  - Handedness of coordinate system
- Rigid transforms
  - Rotations and translations

Rotation matrices

- Orthonormal
  - Rows, columns are unit length and orthogonal
- Inverse of rotation matrix?

Rotation matrices

- Orthonormal
  - Rows, columns are unit length and orthogonal
- Inverse of rotation matrix?
  - Its transpose

$$\mathbf{R}(\mathbf{a}, \theta)^{-1} = \mathbf{R}(\mathbf{a}, \theta)^T$$

Rotations

- Given a rotation matrix $\mathbf{R}(\mathbf{a}, \theta)$
- How do we obtain $\mathbf{R}(\mathbf{a}, -\theta)$?

Rotations

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Rotations

- Given a rotation matrix $\mathbf{R}(\mathbf{a}, \theta)$
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$$\mathbf{R}(\mathbf{a}, -\theta) = \mathbf{R}(\mathbf{a}, \theta)^{-1} = \mathbf{R}(\mathbf{a}, \theta)^T$$

- How do we obtain $\mathbf{R}(\mathbf{a}, 2\theta), \mathbf{R}(\mathbf{a}, 3\theta)$ ...?

Rotations

- Given a rotation matrix $\mathbf{R}(\mathbf{a},\theta)$
- How do we obtain $\mathbf{R}(\mathbf{a},-\theta)$?

$$\mathbf{R}(\mathbf{a},-\theta) = \mathbf{R}(\mathbf{a},\theta)^{-1} = \mathbf{R}(\mathbf{a},\theta)^T$$

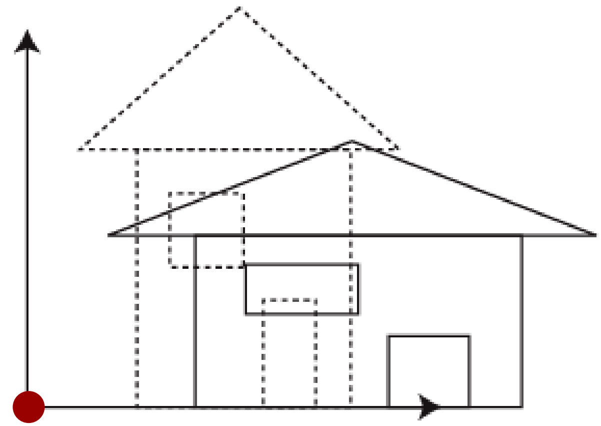
- How do we obtain $\mathbf{R}(\mathbf{a},2\theta), \mathbf{R}(\mathbf{a},3\theta)$ …?

$$\mathbf{R}(\mathbf{a},2\theta) = \mathbf{R}(\mathbf{a},\theta)^2 = \mathbf{R}(\mathbf{a},\theta)\mathbf{R}(\mathbf{a},\theta)$$

$$\mathbf{R}(\mathbf{a},3\theta) = \mathbf{R}(\mathbf{a},\theta)^3 = \mathbf{R}(\mathbf{a},\theta)\mathbf{R}(\mathbf{a},\theta)\mathbf{R}(\mathbf{a},\theta)$$

Scaling

- Origin does not change



$$\mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- Inverse scaling?

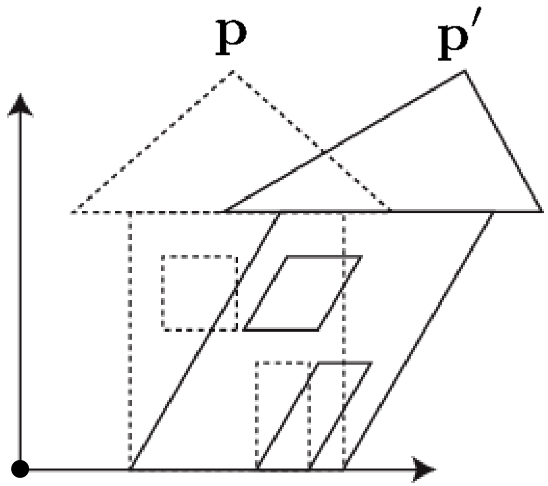
$$\mathbf{S}(s_x, s_y, s_z)^{-1} =$$

Scaling

- Inverse scaling?

$$\mathbf{S}(s_x, s_y, s_z)^{-1} = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$$

Shear



$$\mathbf{p}' = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \mathbf{p}$$

$$\mathbf{Z}(z_1 \dots z_6) = \begin{bmatrix} 1 & z_1 & z_2 & 0 \\ z_3 & 1 & z_4 & 0 \\ z_5 & z_6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Pure shear if only one parameter is non-zero
- Cartoon-like effects

Summary affine transformations

- Linear transformations (rotation, scale, shear, reflection) + translation

Vector space,
http://en.wikipedia.org/wiki/Vector_space

- vectors as [xyz] coordinates
- represents vectors
- linear transformations

Affine space
http://en.wikipedia.org/wiki/Affine_space

- points and vectors as [xyz1], [xyz0] homogeneous coordinates
- distinguishes points and vectors
- linear trafos. and translation

Summary affine transformations

- Implemented using 4x4 matrices, homogeneous coordinates
  - Last row of 4x4 matrix is always $[0\ 0\ 0\ 1]$
- Any such matrix represents an affine transformation in 3D
- Factorization into scale, shear, rotation, etc. is always possible, but non-trivial
  - Polar decomposition
    http://en.wikipedia.org/wiki/Polar_decomposition

Today

Transformations & matrices

- Introduction
- Matrices
- Homogeneous coordinates
- Affine transformations
- Concatenating transformations
- Change of coordinates
- Common coordinate systems

Concatenating transformations

- Build "chains" of transformations

$$\mathbf{M}_3, \mathbf{M}_2, \mathbf{M}_1 \in \mathbf{R}^{4\times 4}$$

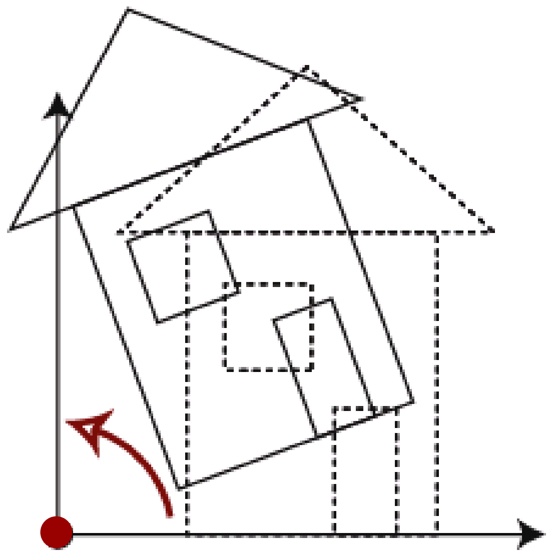
- Apply $\mathbf{M}_1$ followed by $\mathbf{M}_2$ followed by $\mathbf{M}_3$
- Overall transformation $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$ is an affine transformation

$$\mathbf{p}' = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p} = \mathbf{M}\mathbf{p}$$

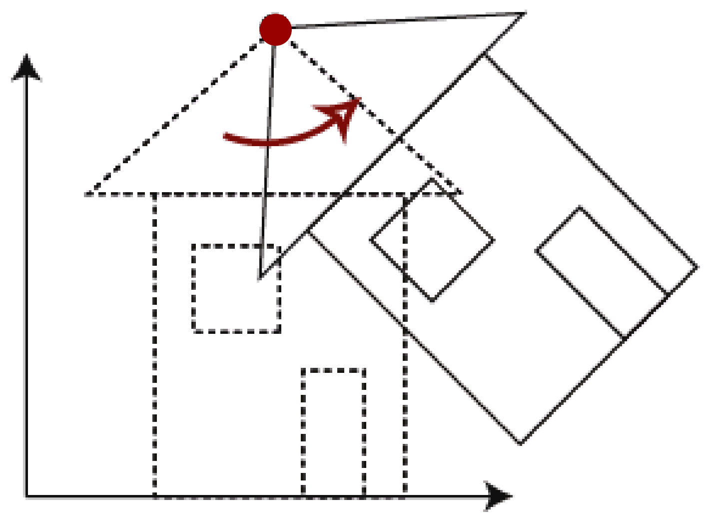
Concatenating transformations

- Result depends on order because matrix multiplication not commutative
- Thought experiment
  - Translation followed by rotation vs. rotation followed by translation

Rotating with pivot

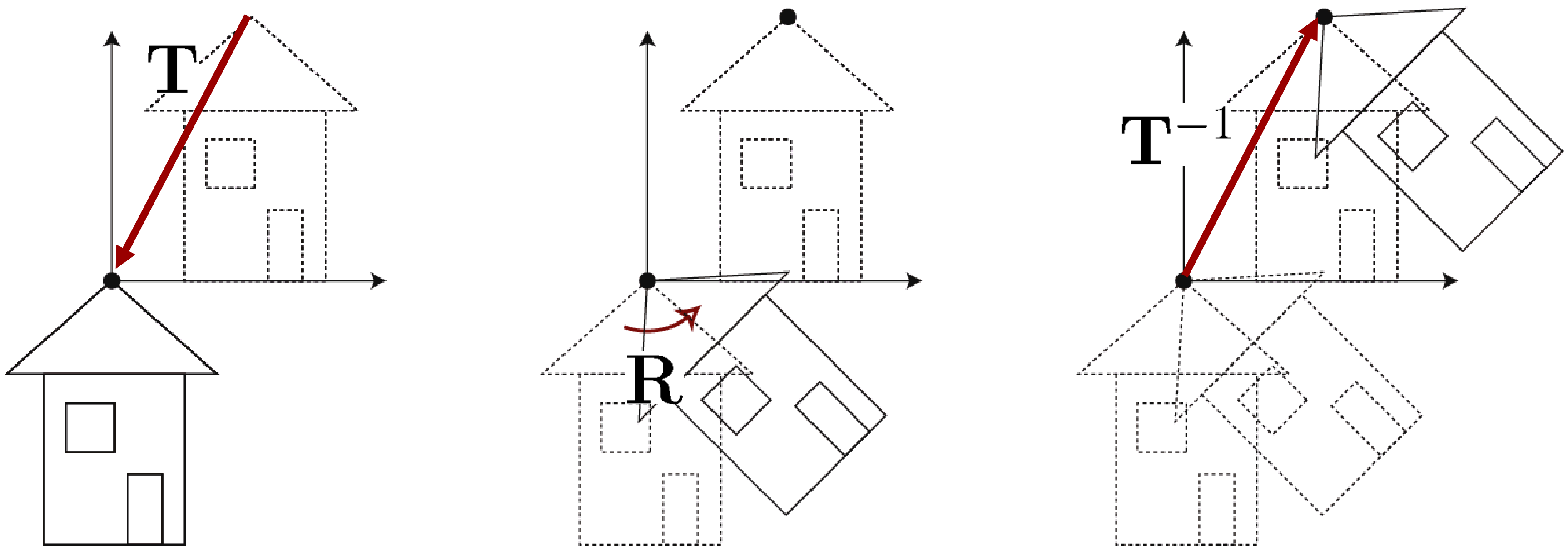


Rotation around origin



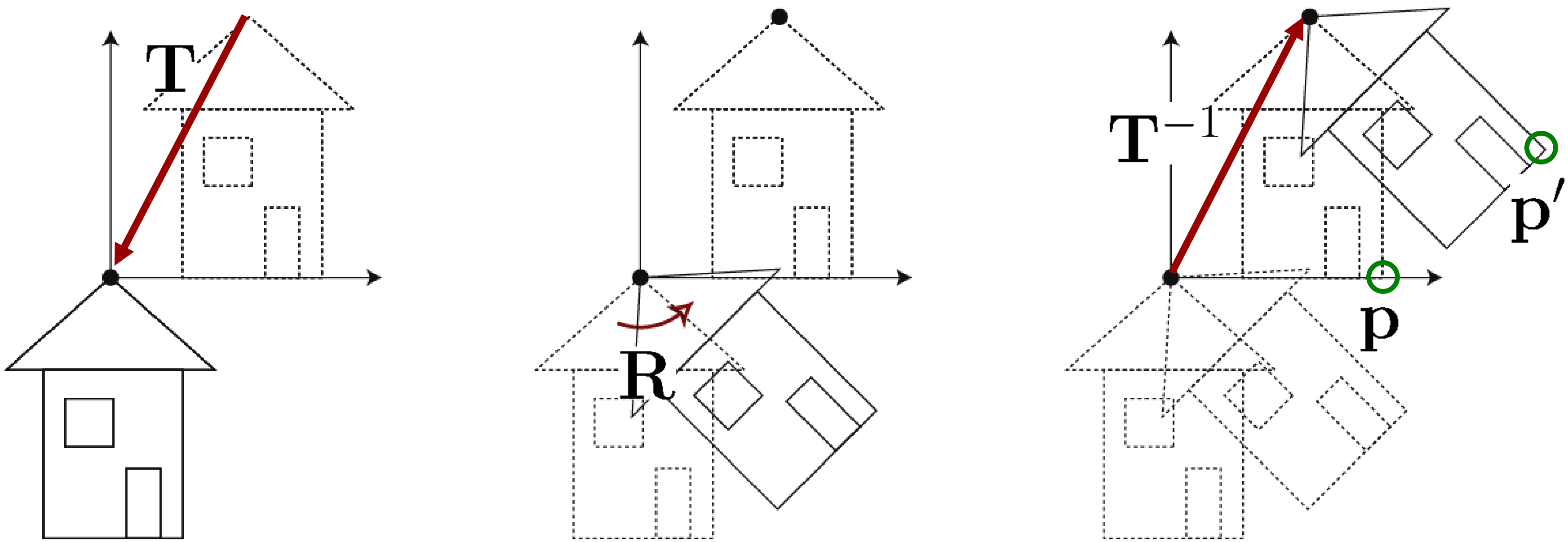
Rotation with pivot

Rotating with pivot



1. Translation $\mathbf{T}$ 2. Rotation $\mathbf{R}$ 3. Translation $\mathbf{T}^{-1}$

Rotating with pivot



1. Translation $\mathbf{T}$ 2. Rotation $\mathbf{R}$ 3. Translation $\mathbf{T}^{-1}$

$$\mathbf{p}' = \mathbf{T}^{-1}\mathbf{R}\mathbf{T}\mathbf{p}$$

Concatenating transformations

- Arbitrary sequence of transformations

$$\mathbf{p}' = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p}$$

$$\mathbf{M}_{total} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

$$\mathbf{p}' = \mathbf{M}_{total}\mathbf{p}$$

- Note: associativity

$$\mathbf{M}_{total} = (\mathbf{M}_3\mathbf{M}_2)\mathbf{M}_1 = \mathbf{M}_3(\mathbf{M}_2\mathbf{M}_1)$$

T=M3.multiply(M2); Mtotal=T.multiply(M1)
T=M2.multiply(M1); Mtotal=M3.multiply(T)

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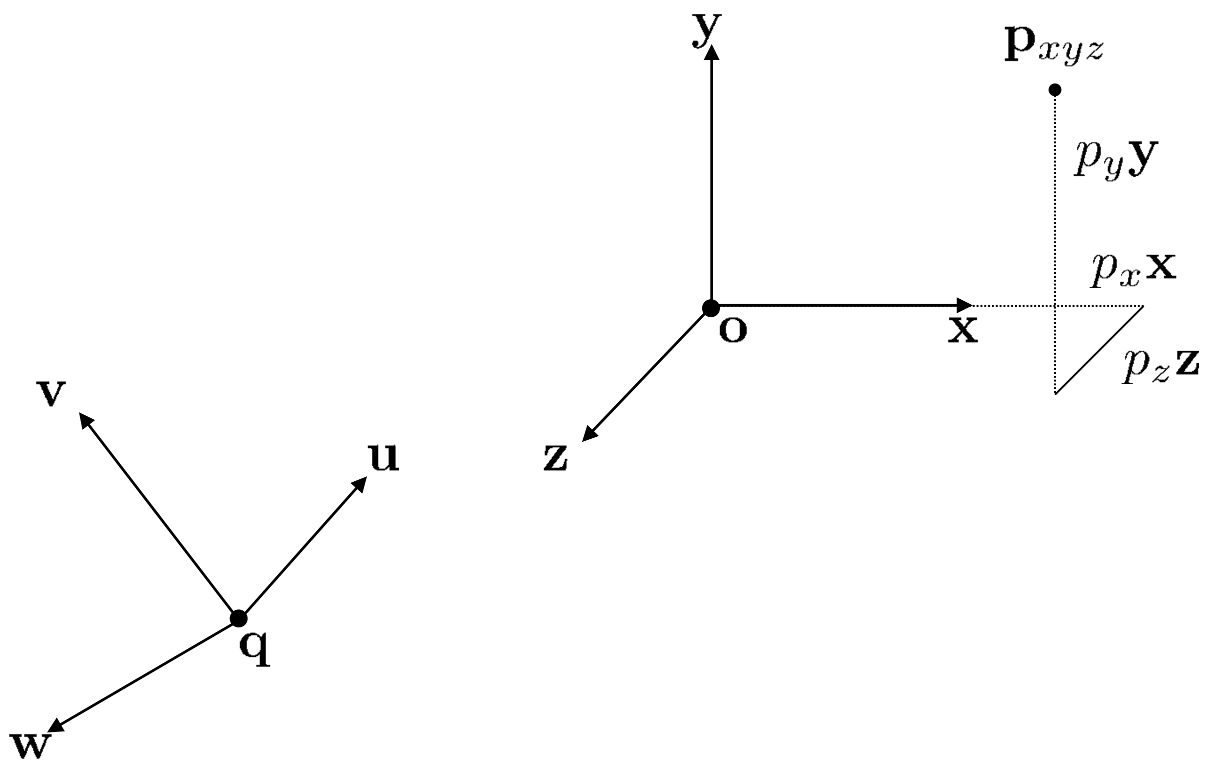
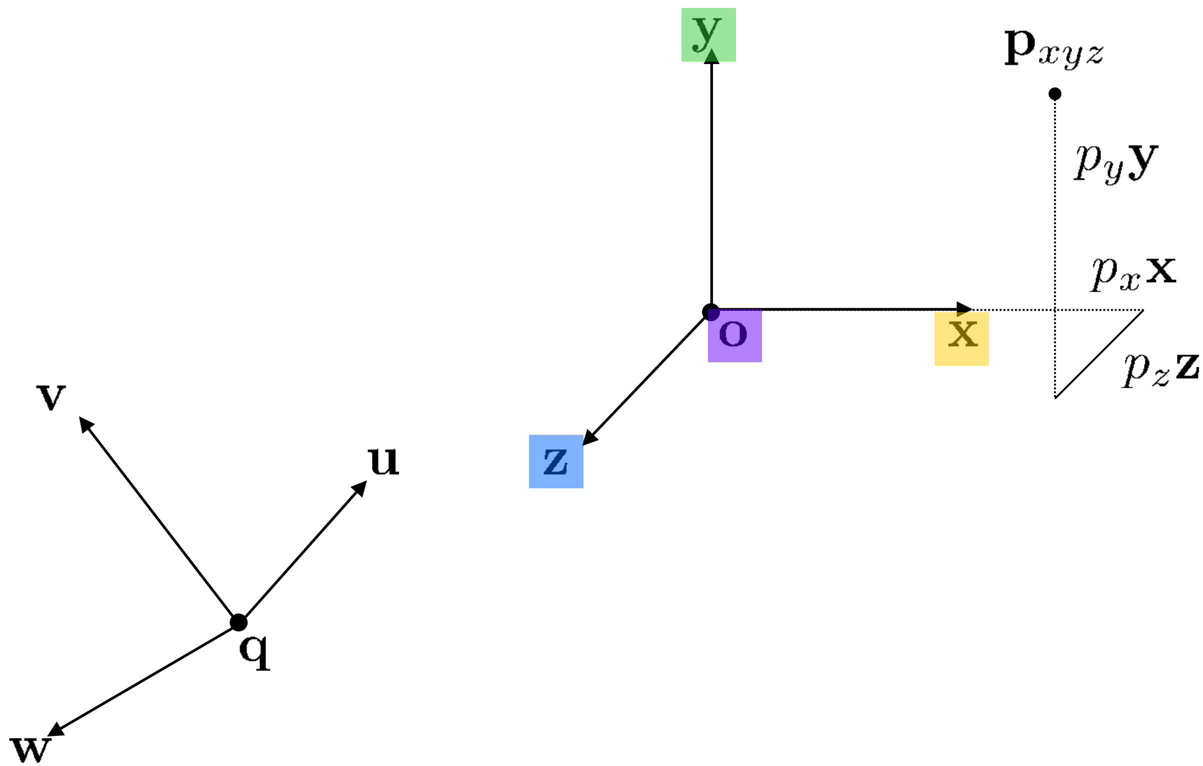
Change of coordinates



New $\mathbf{uvwq}$
coordinate system

Goal: Find coordinates of $\mathbf{p}_{xyz}$ with respect to new $\mathbf{uvwq}$ coordinate system

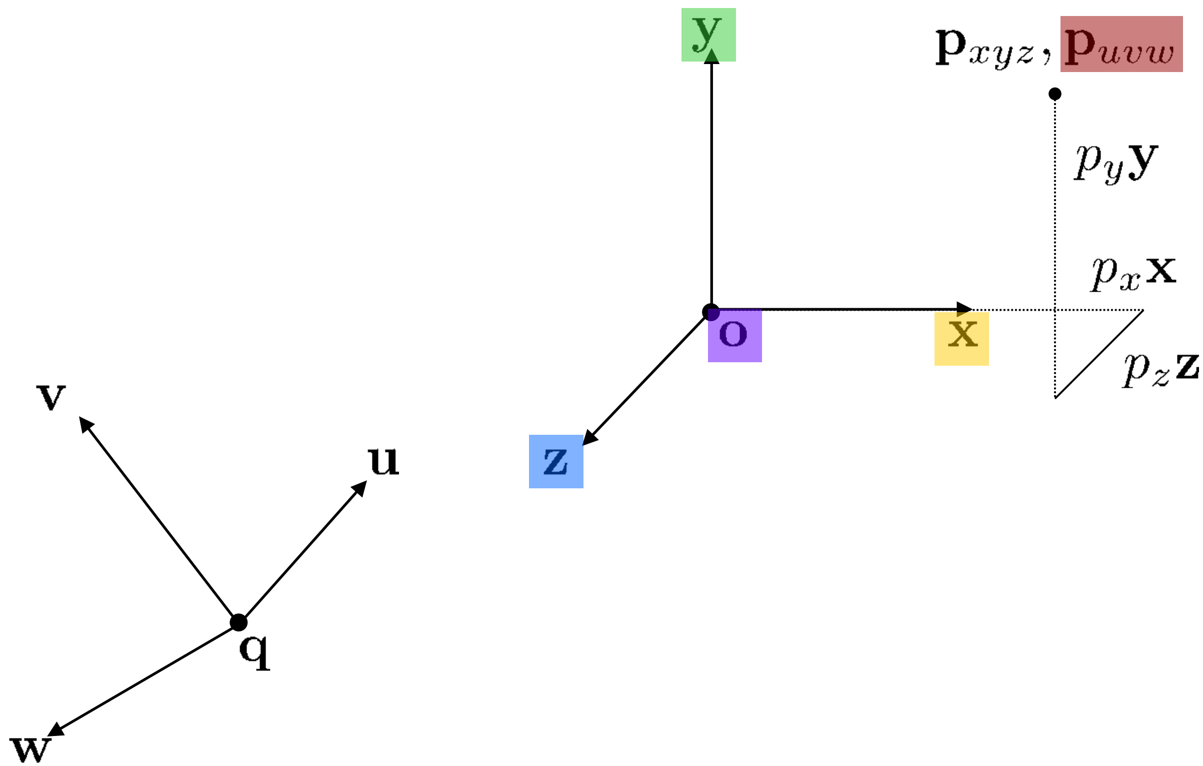
Change of coordinates



Coordinates of $\mathbf{xyzo}$ frame w.r.t. $\mathbf{uvwq}$ frame

$$\mathbf{x} = \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} \qquad \mathbf{o} = \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

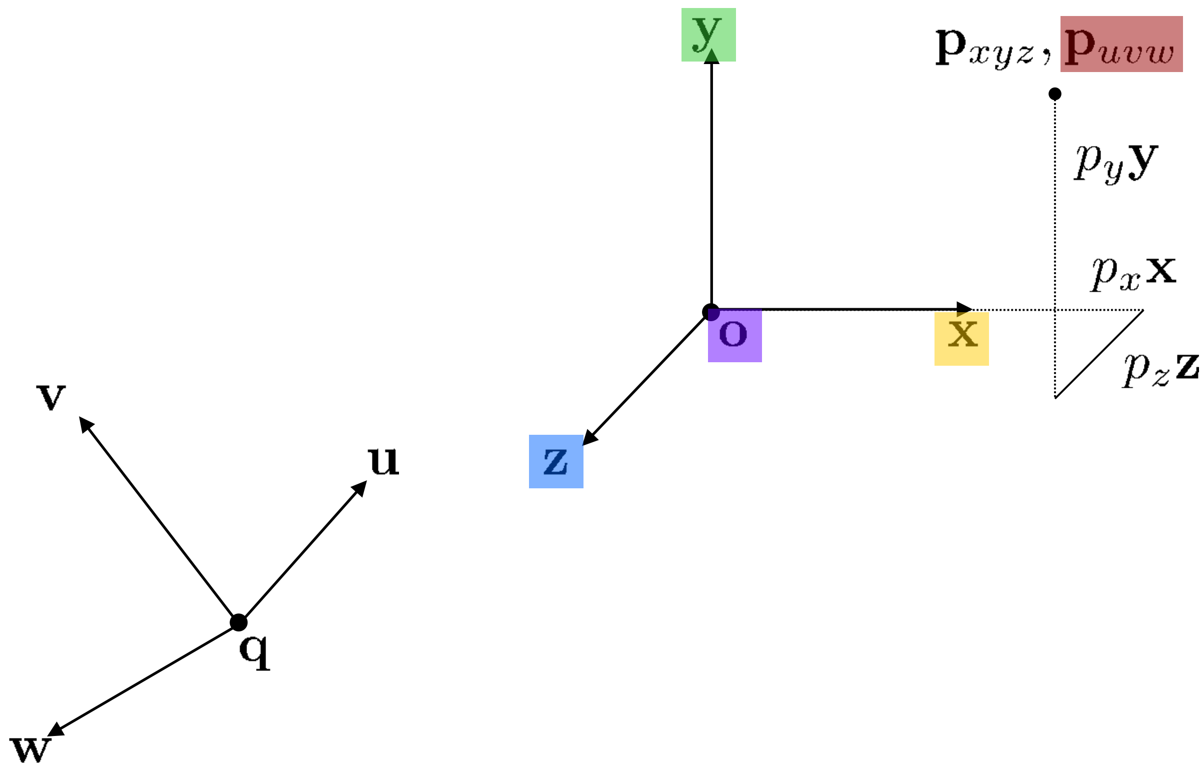
Change of coordinates



Same point $\mathbf{p}$ in 3D, expressed in new $\mathbf{uvwq}$ frame

$$\mathbf{p}_{uvw} = p_x \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} + p_y \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} + p_z \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} + \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

Change of coordinates



$$\mathbf{p}_{uvw} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{o} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Change of coordinates

- Given coordinates

$$\mathbf{x} = \begin{bmatrix} x_u \\ x_v \\ x_w \\ 0 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_u \\ y_v \\ y_w \\ 0 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} z_u \\ z_v \\ z_w \\ 0 \end{bmatrix} \qquad \mathbf{o} = \begin{bmatrix} o_u \\ o_v \\ o_w \\ 1 \end{bmatrix}$$

  of basis **xyzo** with respect to new frame **uvwq**

- Coordinates of any point $\mathbf{p}_{xyz}$ with respect to new frame **uvwq** are

$$\mathbf{p}_{uvw} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{o} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

- Matrix contains old basis vectors (x,y,z,o) in new coordinates (u,v,w,q)

Change of coordinates

Inverse transformation

- Given point $\mathbf{p}_{uvw}$ w.r.t. frame $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{q}$
- Want coordinates $\mathbf{p}_{xyz}$ w.r.t. frame $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{o}$

$$\mathbf{p}_{xyz} = \begin{bmatrix} x_u & y_u & z_u & o_u \\ x_v & y_v & z_v & o_v \\ x_w & y_w & z_w & o_w \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix}$$

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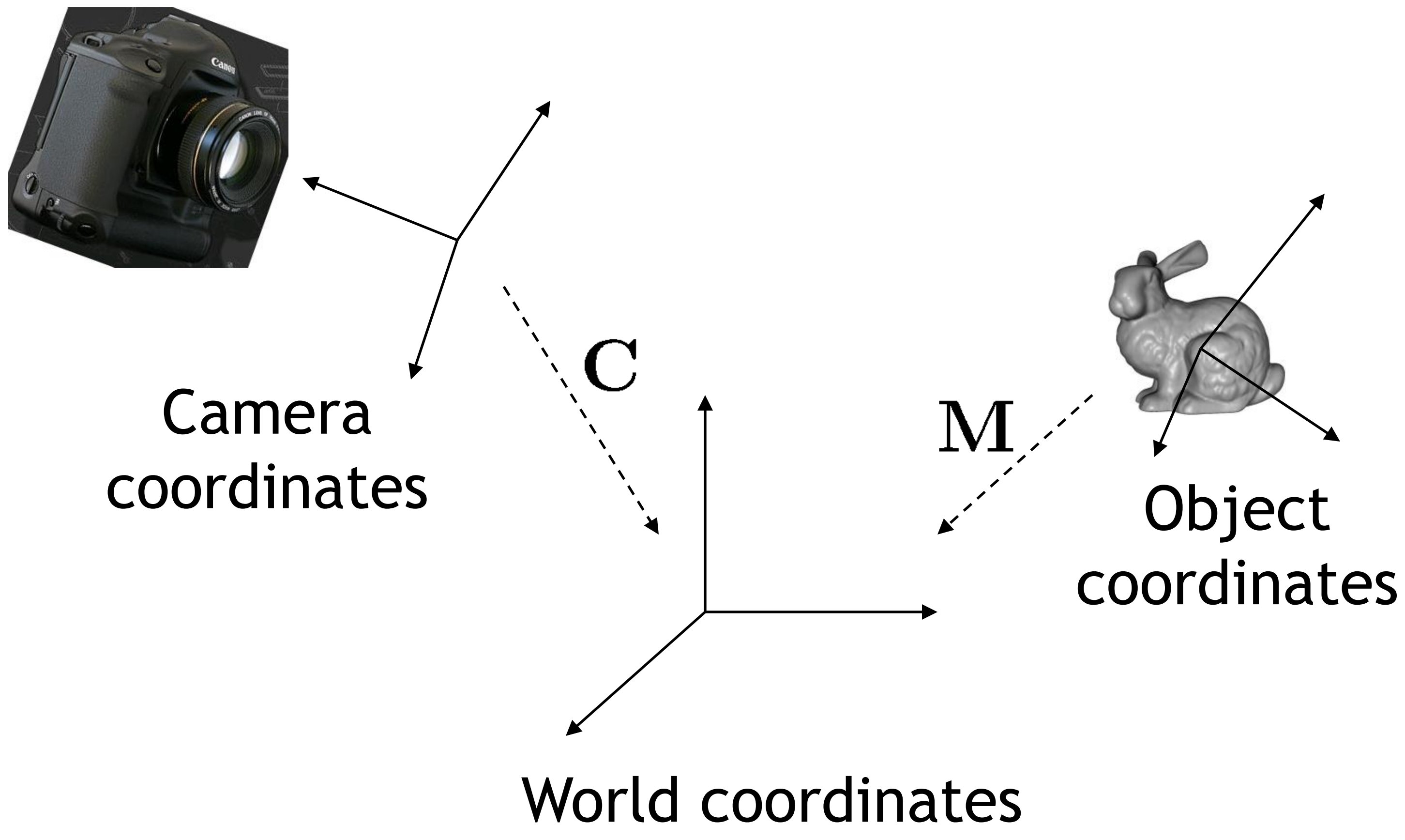
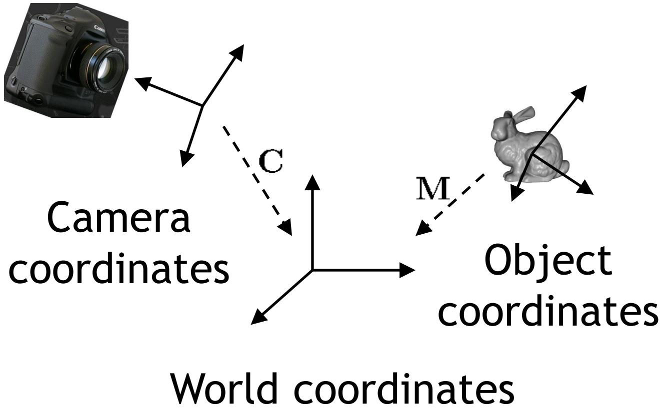
Common coordinate systems

- Camera, world, and object coordinates
- Matrices for change of coordinates $\mathbf{C}, \mathbf{M}$



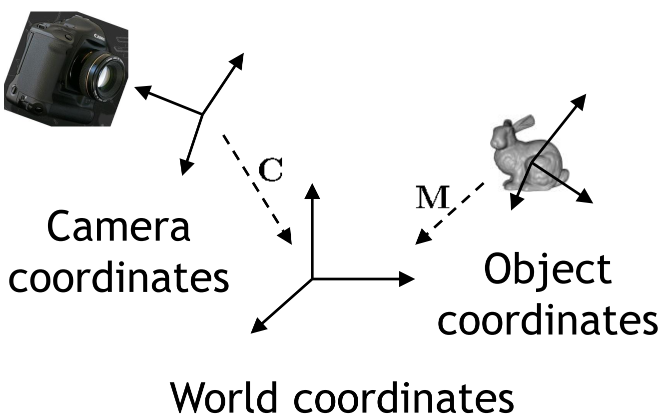
Object coordinates

- Coordinates the object is defined with
- Often origin is in middle, base, or corner of object
- No right answer, whatever was convenient for the creator of the object



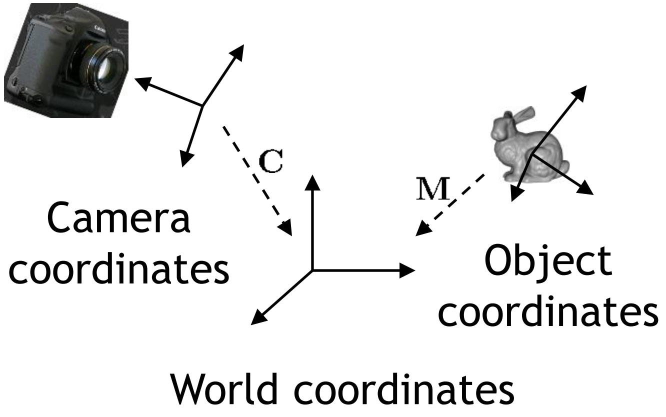
World coordinates

- "World space"
- Common reference frame for all objects in the scene
- Chosen for convenience, no right answer
  - If there is a ground plane, usually x-y is horizontal and z points up



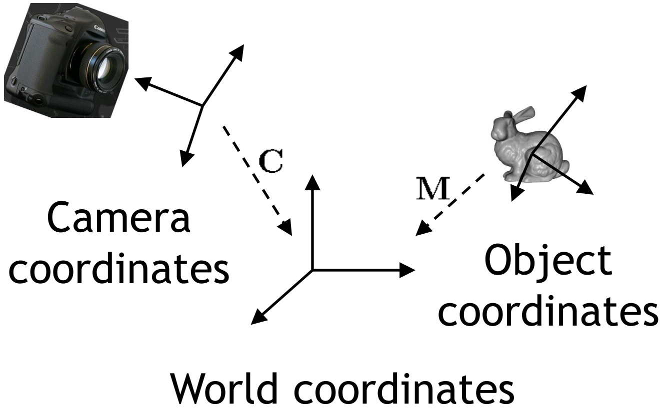
World coordinates

- Transformation from object to world space is different for each object
- Defines placement of object in scene
- Given by "model matrix" (model-to-world transform) $\mathbf{M}$



Camera coordinate system

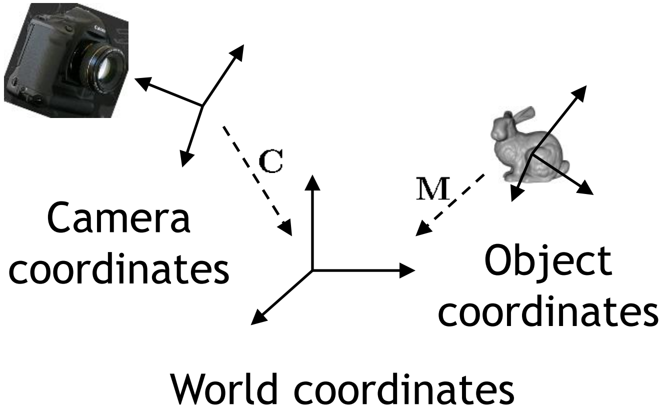
- "Camera space"
- Origin defines center of projection of camera
- Common convention in 3D graphics
  - x-y plane is parallel to image plane
  - z-axis is perpendicular to image plane



Camera coordinate system

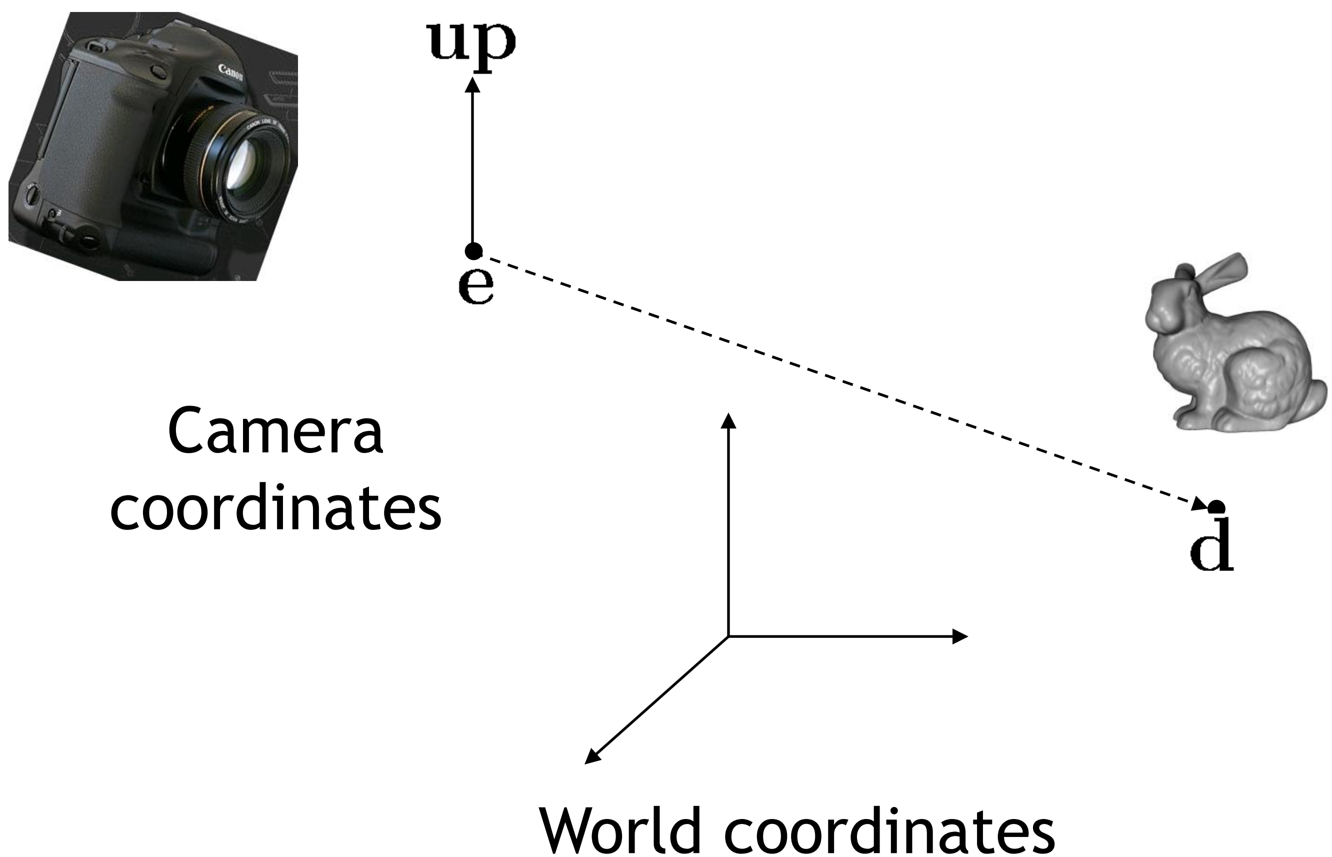
- "Camera matrix" defines transformation from camera to world coordinates
  - Placement of camera in world
- Transformation from object to camera coordinates

$$\mathbf{p}_{camera} = \mathbf{C}^{-1}\mathbf{M}\mathbf{p}_{object}$$



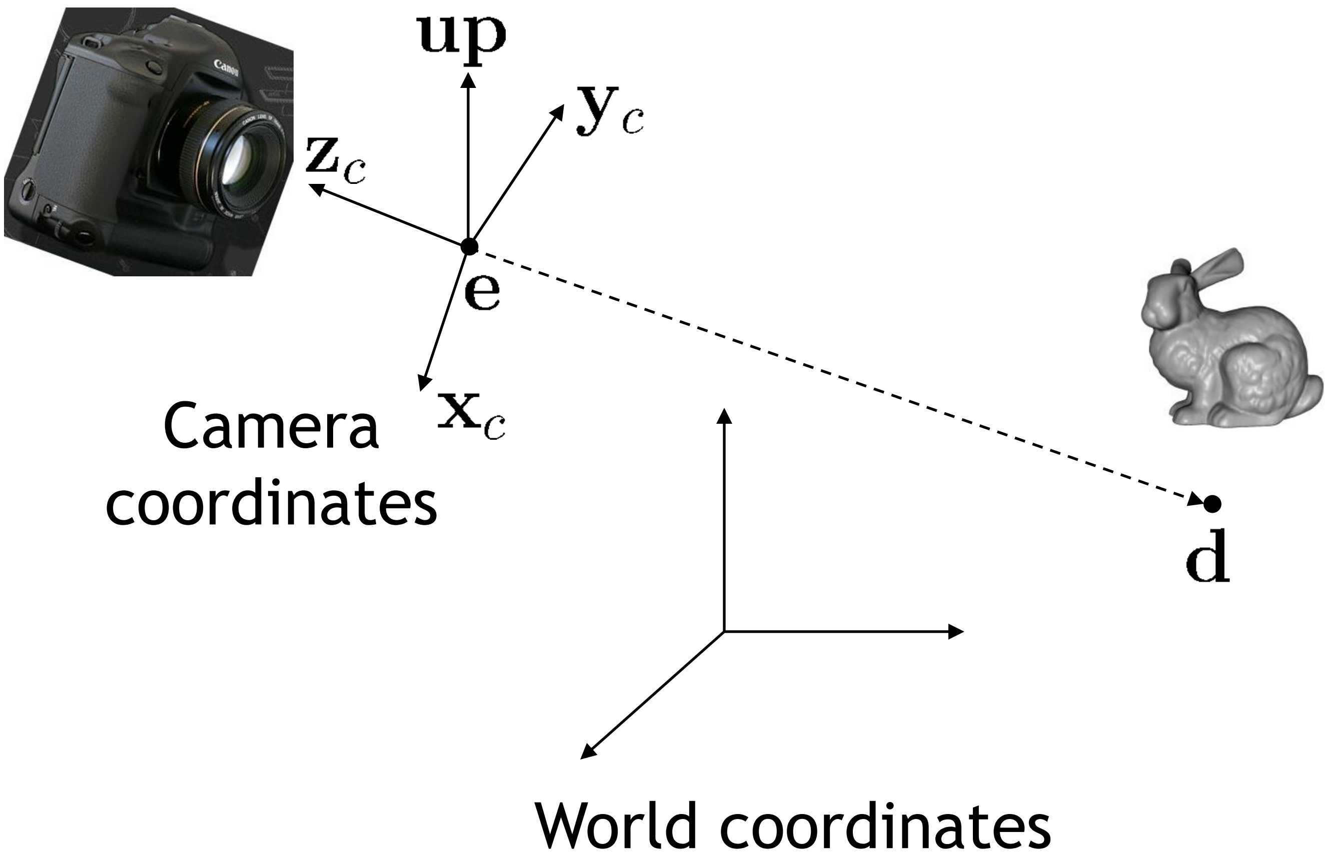
Camera matrix

- Construct from center of projection $\mathbf{e}$, look at $\mathbf{d}$, up-vector $\mathbf{up}$ (given in world coords.)



Camera matrix

- Construct from center of projection $\mathbf{e}$, look at $\mathbf{d}$, up-vector $\mathbf{up}$ (given in world coords.)



Camera matrix

- z-axis

$$\mathbf{z}_c = \frac{\mathbf{e} - \mathbf{d}}{\|\mathbf{e} - \mathbf{d}\|}$$

- x-axis

$$\mathbf{x}_c = \frac{\mathbf{up} \times \mathbf{z}_c}{\|\mathbf{up} \times \mathbf{z}_c\|}$$

- y-axis

Camera matrix

- z-axis

$$\mathbf{z}_c = \frac{\mathbf{e} - \mathbf{d}}{\|\mathbf{e} - \mathbf{d}\|}$$

- x-axis

$$\mathbf{x}_c = \frac{\mathbf{up} \times \mathbf{z}_c}{\|\mathbf{up} \times \mathbf{z}_c\|}$$

- y-axis

$$\mathbf{y}_c = \mathbf{z_c} \times \mathbf{x}_c$$

Camera matrix

- Camera to world transformation

$$\mathbf{C} = \begin{bmatrix} \mathbf{x_c} & \mathbf{y_c} & \mathbf{z_c} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Think about: What does it mean to compute

$$\mathbf{p}' = \mathbf{C}\mathbf{p}$$

$$\mathbf{q}' = \mathbf{C}^{-1}\mathbf{q}$$

Coming up

Project 1

- Due Thursday Oct. 6
- Everybody: turn-in via Ilias before Thursday 10:00
- Turn-in
  - In person
  - Sign up for a time slot on Ilias

Next lecture

- From 3D to 2D: projections