# MEASURING $1^{ST}$ ORDER STRETCH WITH A SINGLE FILTER

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**ABSTRACT**

We analytically develop a filter that is able to measure the linear stretch of the transformation around a point, and present results of applying it to real signals. We show that this method is a real-time alternative solution for measuring local signal transformations. Experimentally, this method can accurately measure stretch, however, it is sensitive to shift.

***Index Terms*—** Frequency domain analysis, Image analysis, Signal analysis

## 1. INTRODUCTION

In many situations, one is given two signals, where one of them is a transformed version of the other, and the goal is to recover this transformation. Assuming one wants to estimate the zero (shift) and first order (stretch) component of the transformation, a general method is to use the log of the magnitude of the Fourier transform. This technique, which is known as *Cepstral analysis*, was first introduced by Bogert et al. [1] and was made widely known by Oppenheim and Schafer [2]. It is commonly used in speech processing [3] to separate different parts of the speech signal. Cepstral analysis requires an explicit FTT on both signals with complexity $O(N \log(N))$. Is this paper we present a more efficient alternative for estimating stretch, assuming known shift. This method only requires the application of a single filter at one point in each image with constant complexity. This computational advantage is offset to an increased sensitivity to errors in shift estimation.

### 1.1. Related Work

In computer vision, phase (frequency) based techniques have been mainly used to estimate the shift between two signals, as in the case of stereo [4],[5] or optical flow [6]. A more general approach for retrieving both the stretch and the shift of 2D signals (i.e. 4 parameters) was employed by Srinivasa [7] for image registration. Recently, the importance of stretch estimation in the case of stereo was recognized (e.g. [8]), but methods for directly measuring the stretch are generally not used.

Related to our approach are the “scale representation" by L. Cohen [9] and the Mellin transform. Both of these methods decompose the signal using a set of basis functions. The stretch is encoded as a phase shift in these representations. Conversely, the current method uses only a single filter to estimate the stretch.

## 2. GABOR FUNCTION AND NOTATION PRELIMINARIES

According to its definition, a Gabor filter consists of a gaussian function of spatial bandwidth $\sigma$, that modulates a complex sinusoid of frequency $\omega$.

$$G(x, \omega, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{2\pi i \omega x} \qquad (1)$$

We consider the spatial bandwidth ($\sigma$) to be fixed with respect to the frequency ($\omega$)

$$\sigma = \frac{c}{\omega}, \qquad (2)$$

where $c$ is a constant (e.g. Sanger uses $c = 1$ [4]) . As a consequence, the Gabor function only has two parameters, namely $x$ and $\omega$.

Denote with calligraphic font the Fourier transform ($\mathcal{F}_\omega$) of a signal (or a filter). In order to avoid any confusion, we denote with a subscript the integration variable when needed.

## 3. ESTIMATING THE STRETCH

Suppose that one is given two signals $I_1(x)$ and $I_2(x)$, where $I_2$ is a “stretched version of $I_1$.

$$\forall x \in \mathbb{R},\ I_2(x) = I_1(\alpha x) \qquad (3)$$

In this paper we describe a way to estimate the unknown stretch parameter $\alpha$. Our approach is based on two observations:

- Convolving the first signal ($I_1$) with a Gabor filter of frequency $\omega$ is equivalent to convolving the second signal ($I_2$) with a Gabor filter of frequency $\alpha\omega$ (Theorem 3.1).
- Considering the log-frequency domain of the Gabor filters the multiplication is transformed into addition (i.e., stretch is transformed into shift) and thus can be estimated using the phase shift property of the Fourier transform (Theorem 3.2).

In the remaining section we formally present our approach in incremental steps using two theorems. We note that the final result is a *single filter on the spatial domain*, even thought we are using the frequency domain in our proofs.

**Theorem 3.1.** *If the two stretched signals ($I_1, I_2$) are as in Eq. 3, then*

$$\forall\omega,\ [I_1(x)\star G(x,\omega)](0) = [I_2(x)\star G(x,\alpha\omega)](0) \tag{4}$$

*Proof.* According to the definition of a Gabor filter (Eq. 1) and its standard deviation (Eq. 2) we get

$$G(x,\alpha\omega) = \frac{1}{\sqrt{2\pi}\sigma'}e^{-\frac{x^2}{2\sigma'^2}}e^{2\pi i\alpha\omega x},$$

$$\sigma' = \sigma_{\alpha\omega} = \frac{c}{\alpha\omega} = \frac{\sigma_\omega}{\alpha}$$

Thus,

$$G(x,\alpha\omega) = \frac{\alpha}{\sqrt{2\pi}\sigma}e^{-\frac{x^2\alpha^2}{2\sigma^2}}e^{2\pi i\omega\alpha x} = \alpha G(\alpha x,\omega). \tag{5}$$

From the definition of convolution we have

$$[I_1(x)\star G(x,\omega)](0) = \int_x I_1(x)G(-x,\omega)dx. \tag{6}$$

Similarly,

$$\begin{aligned}[I_2(x)\star G(x,\alpha\omega)](0) &= \int_x I_2(x)G(-x,\alpha\omega)dx \\ &= \int_x I_1(\alpha x)G(-x,\alpha\omega)dx\end{aligned} \tag{7}$$

Setting $y=\alpha x$, then $dy = \alpha dx$,

$$[I_2(x)\star G(x,\alpha\omega)](0) = \int_y I_1(y)G(-\frac{y}{\alpha},\alpha\omega)\frac{dy}{\alpha}.$$

Using Eq.5 we have

$$\begin{aligned}[I_2(x)\star G(x,\alpha\omega)](0) &= \int_y I_1(y)G(-y,\omega)dy \\ &= [I_1(x)\star G(x,\omega)](0).\end{aligned}$$

□

Based on Theorem 3.1 the response of the convolution of $I_1, I_2$ with the gabor filter is a function of the frequency $\omega$, that is

$$\begin{aligned}R_1(\omega) = [I_1(x)\star G(x,\omega)](0) = \\ [I_2(x)\star G(x,\alpha\omega)](0) = R_2(\alpha\omega).\end{aligned} \tag{8}$$

If we consider the log frequency $\psi$ instead of the frequency $\omega$

$$\psi = e^\omega \Leftrightarrow \omega = \log\psi, \tag{9}$$

then Eq. 8 is transformed to

$$R_1(\psi) = R_2(\psi + \log\alpha). \tag{10}$$

In principle, we could estimate the shift (in the log-frequency domain $\psi$) by transforming it into a phase shift using the Fourier transform

$$\mathcal{R}_1(u) = \mathcal{F}_\psi\{R_1\} = e^{2\pi i\log\alpha u}\mathcal{R}_2(u) \tag{11}$$

and measuring the difference in the phase of $\mathcal{R}_1$ and $\mathcal{R}_2$ for any specific frequency $u$[1]. While this is a valid approach, it is rather computationally expensive. For every point of the two signals one has to compute the frequency response $R_1, R_2$ (by convolving with Gabor filters of different frequencies) and then take the Fourier transform of those responses. The following theorem provides an alternative solution that amounts to convolving the two signals with a single filter.

**Theorem 3.2.** *There exists a filter $H(x,u)$ whose convolution with $I_1, I_2$ directly encodes the stretch as*

$$[I_1(x)\star H(x,u)](0) = e^{2\pi i\log\alpha u}[I_2(x)\star H(x,u)](0). \tag{12}$$

*Specifically, the filter has the analytic form*

$$\boxed{H(x,u) = \int_\omega G(x,e^\omega)e^{-2\pi i\omega u}d\omega.} \tag{13}$$

*Proof.* From Eqs. 8,9 and 10 we have

$$R_1(e^\omega) = R_2(e^\omega + \log\alpha)$$

If we consider the Fourier transform of $R_1(e^\omega)$ with respect to $\omega$, then

$$\begin{aligned}\mathcal{R}_1(u) &= \int_\omega e^{-2\pi i\omega u}R_1(e^\omega)d\omega \\ &= \int_\omega e^{-2\pi i\omega u}[\int_x I_1(x)G(-x,e^\omega)dx]d\omega \\ &= \int_x I_1(x)[\int_\omega G(-x,e^\omega)e^{-2\pi i\omega u}d\omega]dx \\ &= \int_x I_1(x)H(-x,u)dx \\ &= [I_1(x)\star H(x,u)](0).\end{aligned}$$

[1]We have noticed that the following issue is often at first confusing to readers. We use two different frequency domains. Symbols $\omega$ (and $\psi$) denote the frequency in the "traditional" sense, while symbol $u$ denotes the Fourier transform of $\psi$, so in some sense is the *"frequency of the frequency domain"*.

**Algorithm 1** Estimating the Stretch

| | | |
|---|---|---|
| **Input** | : | |
| $I_1, I_2$ | : | Input Signals |
| $x_0$ | : | A single point along the X-axis |
| **Output** | : | |
| $\alpha$ | : | The stretch between the two signals around point $x_0$ |
| **Algorithm** | : | |

Create the filter $H(x,u) = \int_{\omega_1}^{\omega_2} G(x, e^{\omega}) e^{-2\pi i \omega u} d\omega$
Convolve the two signals ($I_1, I_2$) with $H(x,u)$ around $x_0$
Compute the difference in phase of the two measurements ($\Delta\theta$)
Compute the "log-frequency shift" $\Delta\psi = \frac{\Delta\theta}{2\pi u}$
Compute the stretch $\alpha = e^{\Delta\psi}$

Similarly for $R_2(e^{\omega})$ we get

$$\mathcal{R}_2(u) = [I_2(x) \star H(x,u)](0).$$

From the phase shift property of the Fourier transform we get

$$\mathcal{R}_1(u) = e^{2\pi i \log \alpha u} \mathcal{R}_2(u)$$

and thus

$$[I_1(x) \star H(x,u)](0) = e^{2\pi i \log \alpha u} [I_2(x) \star H(x,u)](0).$$

□

The algorithm is a straightforward implementation of the theory and is presented in Alg. 1.

## 4. EXPERIMENTS

### 4.1. Stretch without Shift Experiments

On this first set of experiments, the original signal ($I_1$) is the horizontal lines of various textures [10] (Fig. 1). We randomly selected 200 scanlines and stretched each one of them around its center in order to produce a second signal (Fig. 2, first and second row). Then we convolved both signals with a *single filter* of frequency $u = 0.25$ as shown in Fig. 2 (third row). Following the steps described in Alg. 1 we estimated the stretch. We experimentally found that frequencies in the range $u = [0.25 \ldots 0.5]$ worked well. The higher the frequency, the better the results were for stretches closer to $1$ and the worse for stretches closer to $0$. For the lower and upper bounds of integral $H$ (Eq. 13) we used the values $-3.5$ and $-1$, respectively.

In Fig. 3 we present the results as a function of the stretch $\alpha$. Each graph corresponds to an image from Fig. 1. For each stretch value we pick 200 random points and synthetically stretch the signal about each. We plot both the *median* value and the $99\%$ *confidence interval* for the estimated stretches. The results are good considering the following facts. First,

**Fig. 1**. Texture images

we are using a single filter to estimate the stretch. Second, the size of the filter is $\sim 20$ pixels. Third, we have discrete signals, thus for a stretch of $\alpha = 0.5$ only $10$ pixels are common in the original and the stretched image. Fourth, for practical purposes, we are usually interested in stretches close to one (e.g. $\alpha = [0.9 \ldots 1.1]$) in which case the estimated stretch is quite accurate. Thus, in Fig. 4 we display the results on that range of stretches. In all cases the estimated stretch is very close to the real stretch between the two signals.

### 4.2. Stretch Estimation in the Presence of Translation

In real applications, the most common case is for the two signals to be both shifted and stretched i.e., $i_2(x) = i_1(\alpha x + \beta)$. In such cases, estimation of the stretch ($\alpha$) is affected by the shift ($\beta$) and vice versa. In the following experiments, we empirically investigate the sensitivity of the stretch estimation in the presence of translation between the two signals. Fig. 5 we display the error in the stretch estimates, when the two signals are stretched and shifted, as a function of the shift. As expected (due to the small size of the filters), this approach is very sensitive to shifts. Furthermore, the error in the stretch estimation increases with the shift.

## 5. CONCLUSIONS

In this paper we presented a filter that retrieves the local stretch of two signals. We also presented experiments that indicate that this approach produces very good results, but is also very sensitive to the shift between the two signals. From Fig. 5, it appears that there is an approximate linear dependence between the error in stretch estimation and the original translation of the two signals. Future work could address this dependence theoretically. The comparison of this method with traditional stretch estimation techniques based

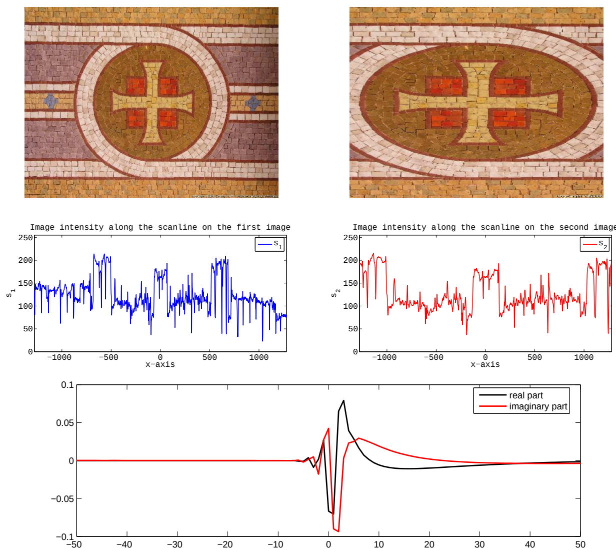


**Fig. 2**. First Row: Original and stretched image ($\alpha = 0.5$). Second Row: The intensities along a single scanline on both images. Third Row: The stretch filter $H$ we are using.

on phase correlation is also left as future work.

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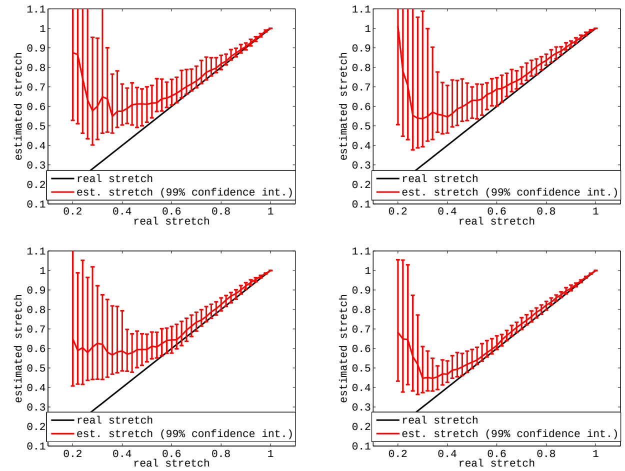


**Fig. 3**. Results for various stretches $\alpha$

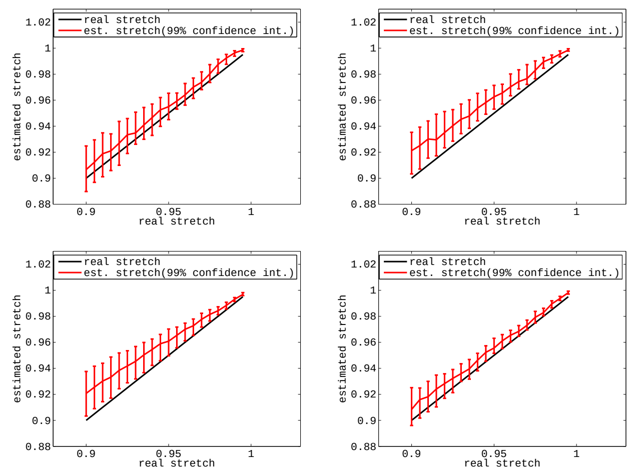


**Fig. 4**. Results when the stretch is close to one.

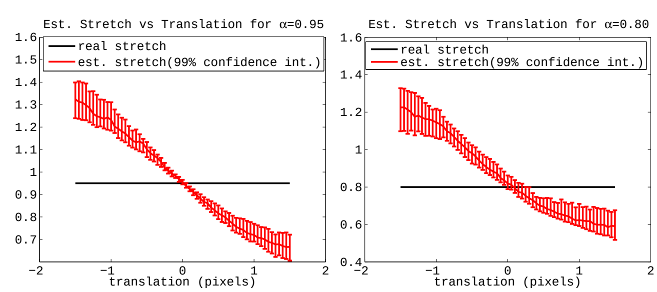


**Fig. 5**. Stretch estimation vs. translation error