An Improved LP-based Approximation for Steiner Tree

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

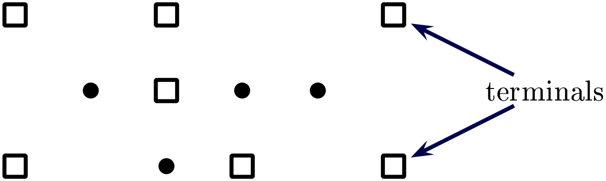
Steiner Tree

Given:

- undirected, metric graph $G = (V, E)$
- cost $c : E \to \mathbb{Q}_+$
- terminals $R \subseteq V$

Find:

$$opt := \min\{c(T) \mid T \text{ spans } R\}$$



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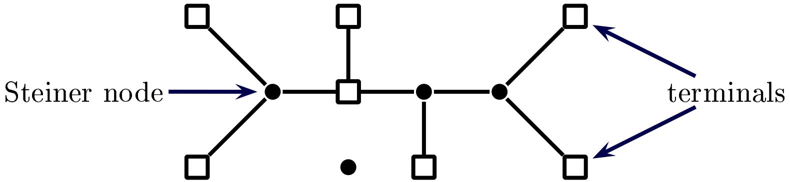

Known results:

Hardness:

- **NP**-hard even if edge costs $\in \{1, 2\}$ [Bern & Plassmann '89]
- no $< \frac{96}{95}$-apx unless $\mathbf{NP} = \mathbf{P}$ [Chlebik & Chlebikova '02]

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Approximations:

- 2-apx (*minimum spanning tree heuristic*)
- 1.83-apx [Zelikovsky '93]
- 1.667-apx [Prömel & Steger '97]
- 1.644-apx [Karpinski & Zelikovsky '97]
- 1.598-apx [Hougardy & Prömel '99]
- 1.55-apx [Robins & Zelikovsky '00]
- PTAS for $\mathbb{R}^d$ (d fixed) [Arora '97]
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Integrality gap:

- Integrality gap: ≤ 2 [Goemans & Williamson '95, Jain '98]

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Theorem

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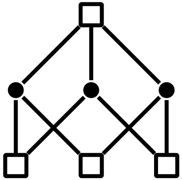
There is an LP-relaxation with an integrality gap of at most 1.7.

- Can be improved to 1.55 (more involving)

Bi-directed cut relaxation



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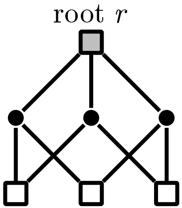

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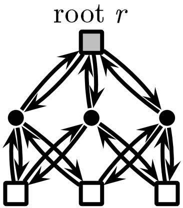

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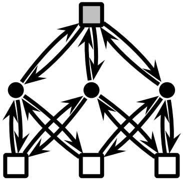
$$\min \sum_{e \in E} c(e) z_e \qquad \text{(BCR)}$$

$$\sum_{e \in \delta(S)} z_e \geq 1 \qquad \forall S \subseteq V \setminus \{r\} : S \cap R \neq \emptyset$$

$$z_e \geq 0 \qquad \forall e \in E.$$



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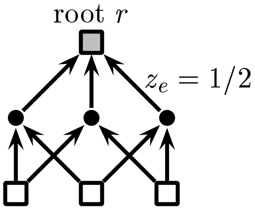

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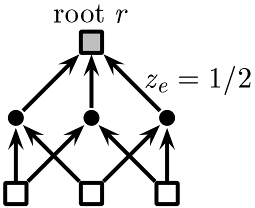

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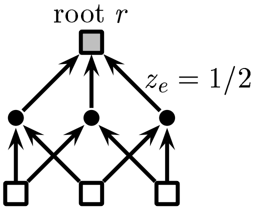

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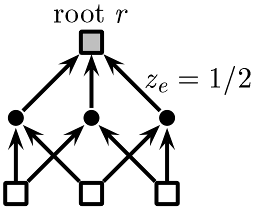

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Theorem (Edmonds '67)

$R = V \Rightarrow BCR$ integral

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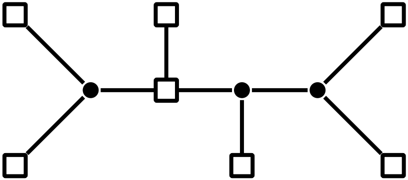

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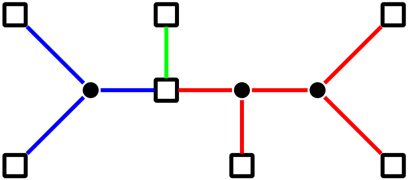

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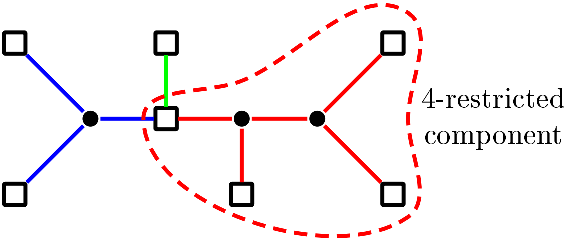

- component k-**restricted**: $\leq k$ terminals

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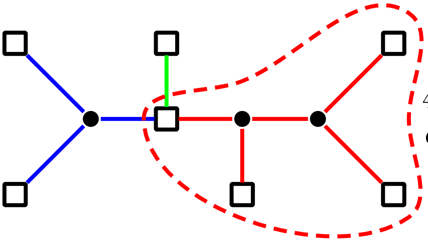

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$$opt_k = \min\left\{c(T) \mid T \text{ spans } R \;\&\; T \text{ is } k\text{-restricted}\right\}$$

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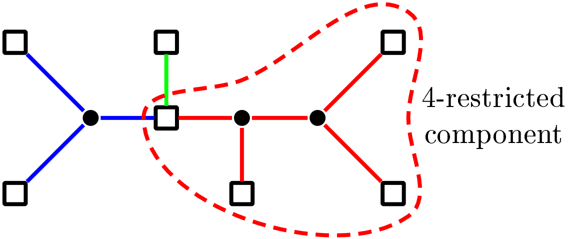

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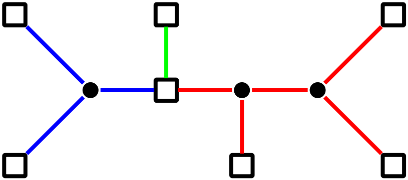
Theorem (Borchers & Du '97)

$$opt_k \leq \left(1 + \frac{1}{\lfloor \log_2 k \rfloor}\right) \cdot opt$$

Directing components



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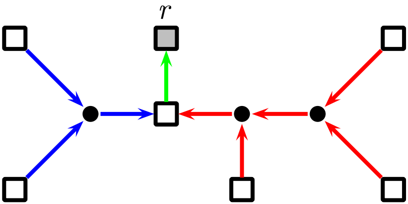

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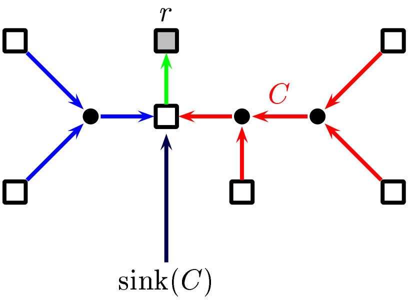

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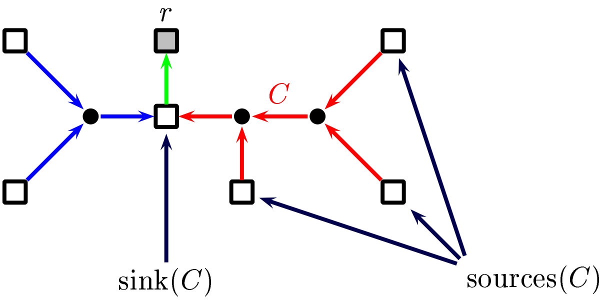

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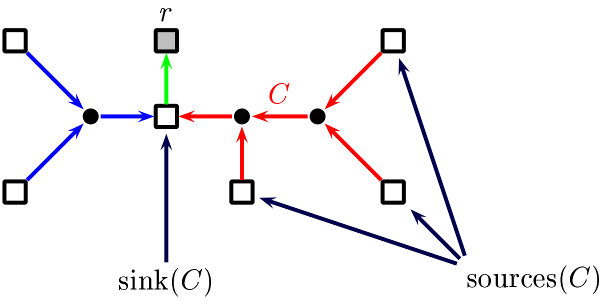

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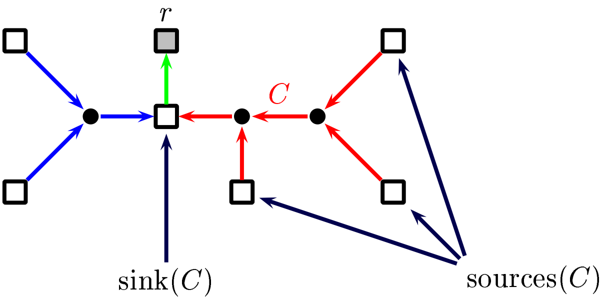


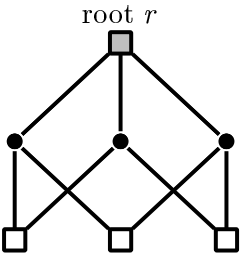
- Compute list $C_1, \ldots, C_h$ of potential k-restricted components
- $h = poly(n)$ (for fixed k)

Directed component cut relaxation

$$opt_k^f := \min \sum_j c(C_j) x_j \qquad (k\text{-DCR})$$

$$\sum_{\substack{j\,:\,\mathrm{sources}(C_j)\cap S \neq \emptyset,\\ \mathrm{sink}(C_j)\notin S}} x_j \geq 1 \quad \forall \emptyset \subset S \subseteq R \setminus \{r\}$$

$$x_j \geq 0 \quad \forall j = 1, \ldots, h.$$



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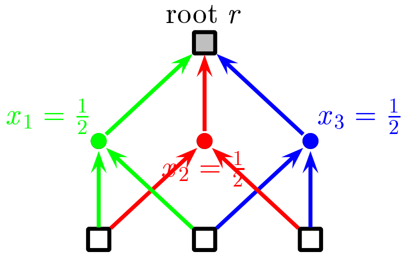

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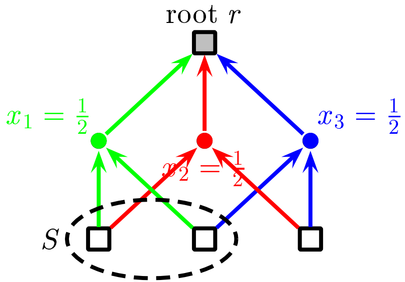

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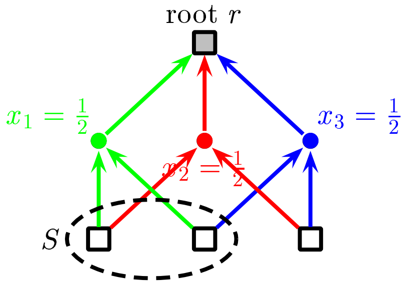

Lemma

k-DCR can be solved in poly-time using the Ellipsoid method.

The algorithm

(1) FOR $t = 1, \dots, \mu$ DO
  (2) Solve k-DCR $\rightarrow x^t$
  (3) Sample a component C^t from x^t and contract it.
(4) Compute a terminal spanning tree T^μ in the remaining instance
(5) Output $T^\mu \cup \bigcup_{t=1}^{\mu} C^t$.

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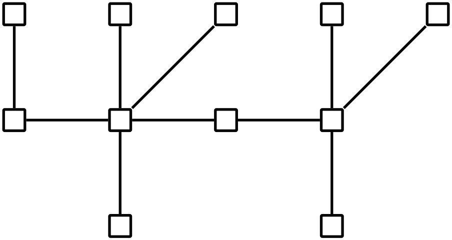
- W.l.o.g. $\Sigma := \sum_j x_j^t \ \forall t$

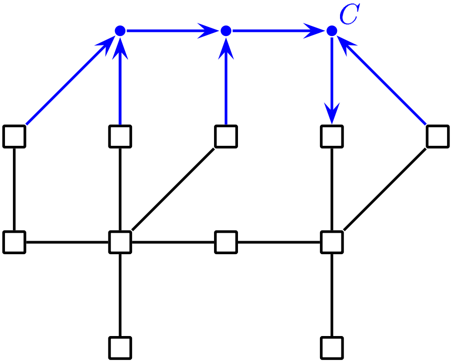
Bridges

- Let T terminal spanning tree



Bridges

- Let T terminal spanning tree, C some component used in the fractional solution



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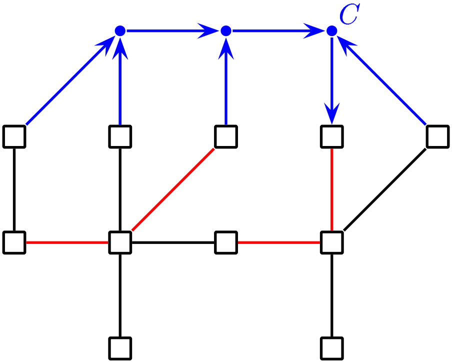

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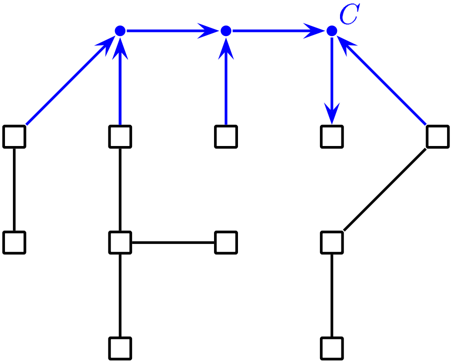

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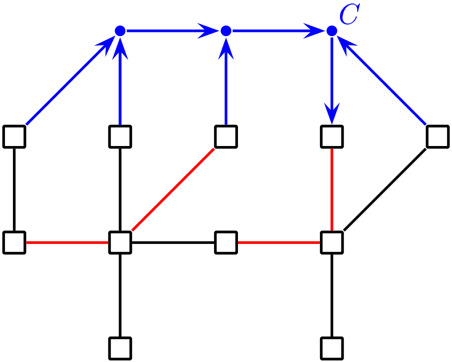

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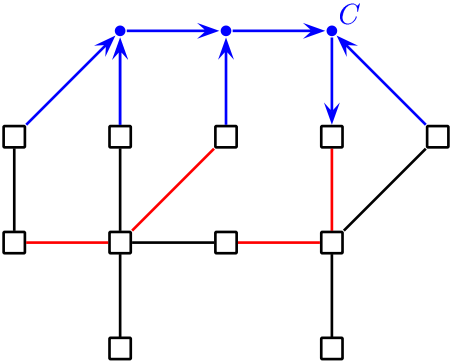

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- Edges $Br_T(C) := B$ attaining this max. are called **bridges**

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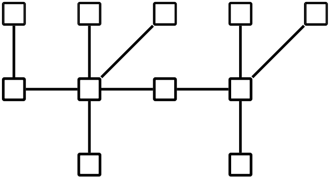
Lemma (Bridge Lemma)

For T terminal spanning tree, x k-DCR solution:

$$\sum_j x_j \cdot br_T(C_j) \geq c(T)$$



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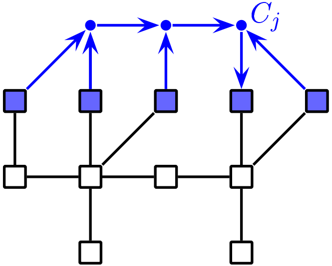

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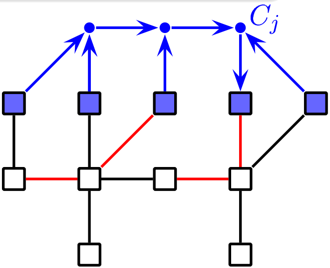

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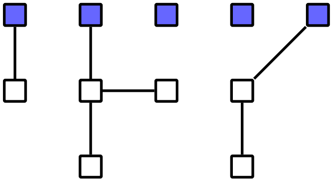

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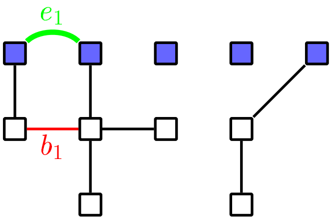

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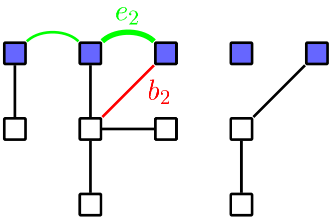

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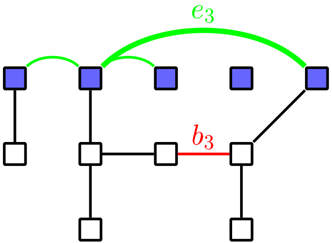

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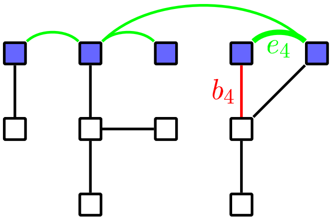

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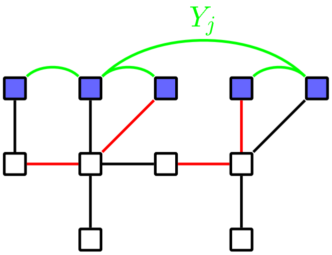

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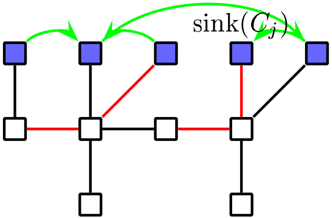

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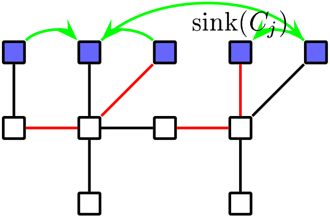

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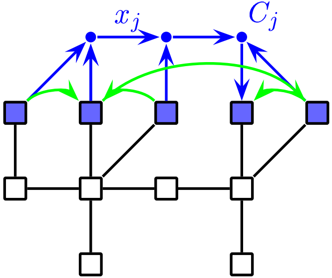
- y is feasible solution for (BCR)



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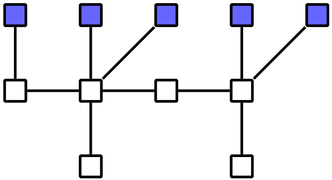

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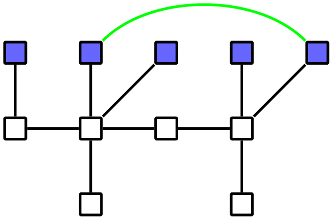

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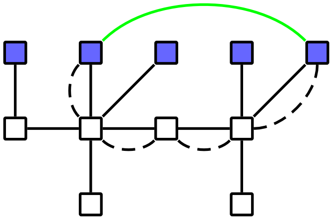

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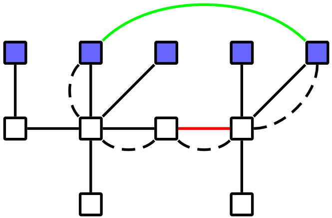

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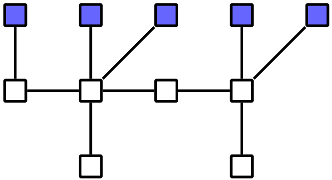

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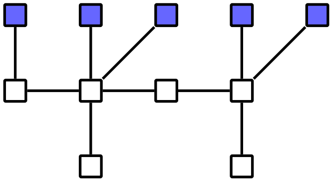

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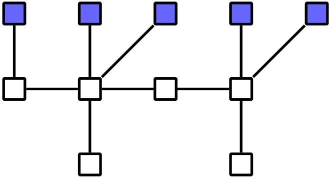

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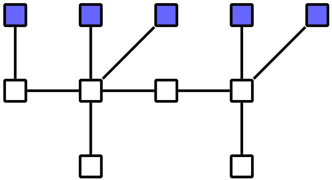

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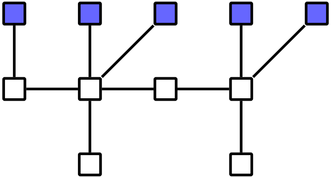

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$$(2e^{-\delta} + \delta) \cdot opt_k^f \overset{\delta := \ln(2) \approx 0.69}{=} \underbrace{(1 + \ln(2))}_{<1.7} \cdot opt_k^f \quad \square$$

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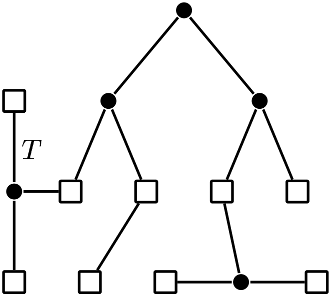

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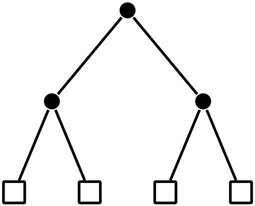

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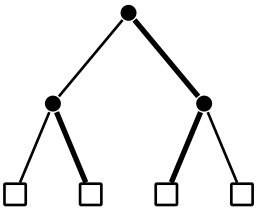

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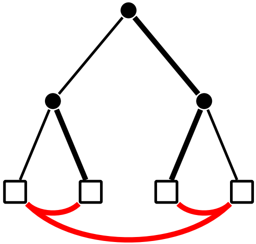

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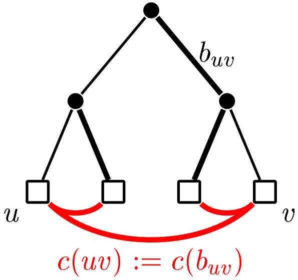

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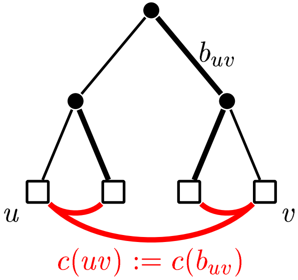

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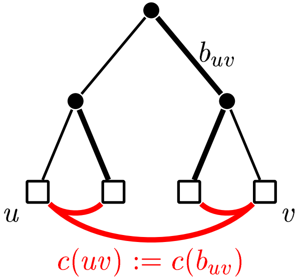

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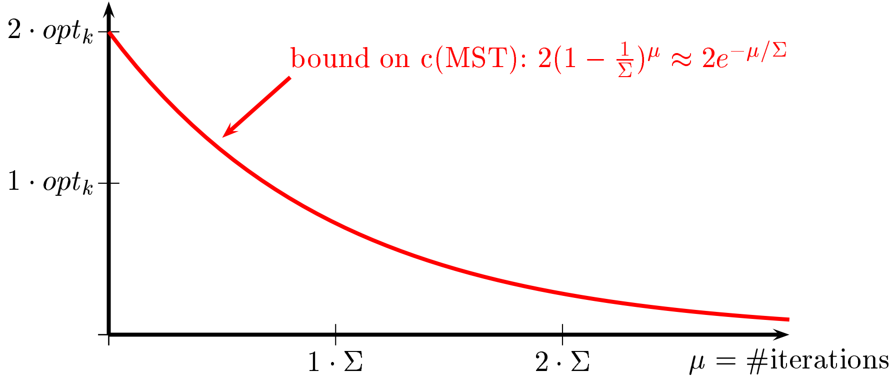

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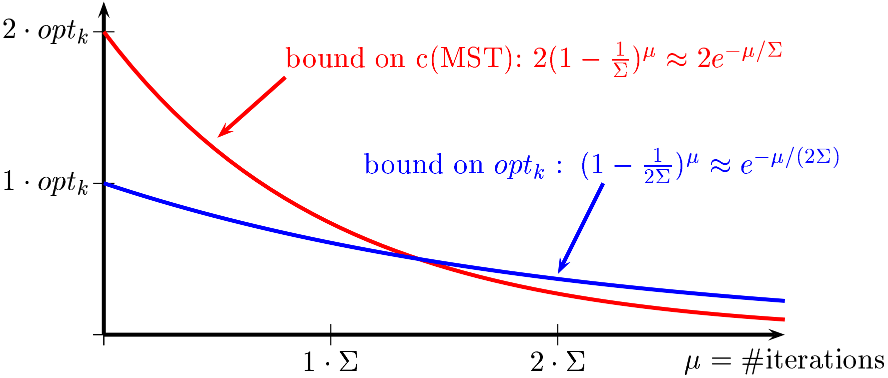

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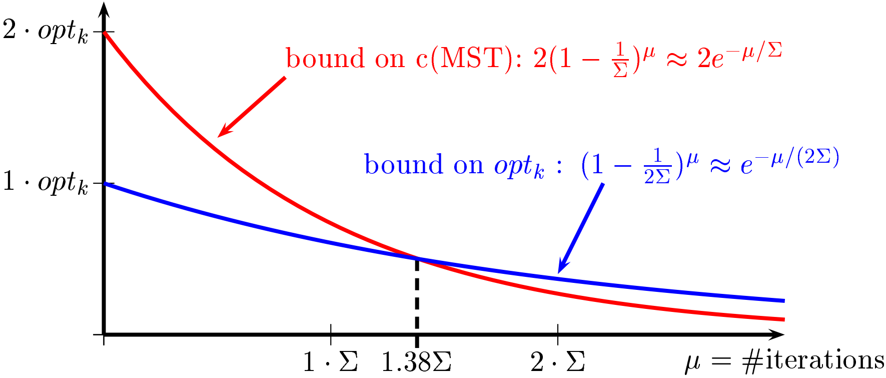
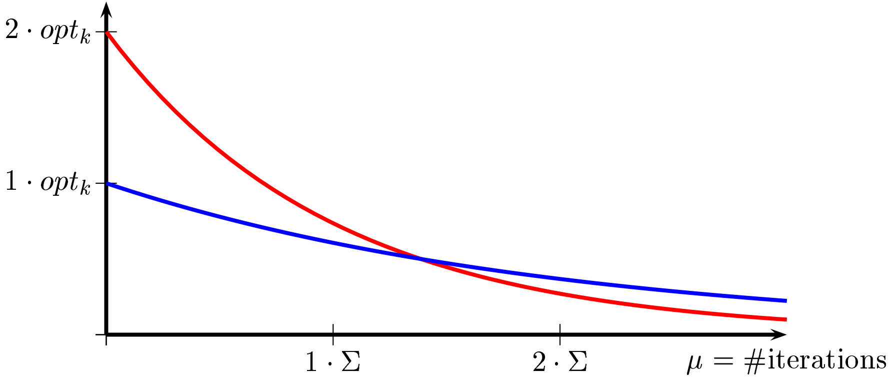


Theorem

For $\mu := \ln(4) \cdot \Sigma$ *one has:* $cost \leq \frac{3}{2} opt_k$.

An even better bound



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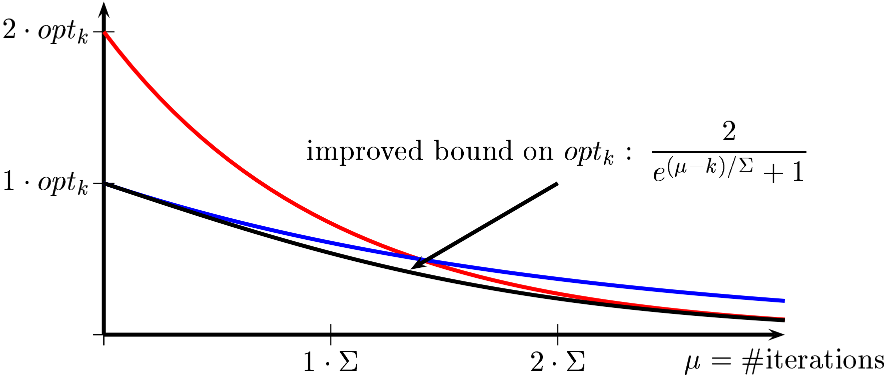

Theorem

For $\mu := \infty$ one has: cost $\leq 1.39 \cdot opt_k$

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