

# Advertisement (Online) Auctions and Prophet-Inequality Setting

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# AdWord Auction

- Internet search companies such as Google, Yahoo, and MSN make billions of dollars out of it
- They decide what ads to display with each query to maximize revenue
- Users typing in query keywords, called **AdWords**
- Business place bids for individual AdWords together with a daily budget limit
- Search engines earn money from business when they show their ads in response to queries and charge them the second highest bid

# Google AdWords

[https://www.google.com/accounts/ServiceLogin?service=adwords&hl=en\\_US&ltmpl=regionalc&passive=false&ifr=false&alwf=true&continue=](https://www.google.com/accounts/ServiceLogin?service=adwords&hl=en_US&ltmpl=regionalc&passive=false&ifr=false&alwf=true&continue=)

Google AdWords

Change Language: English

## Advertise your business on Google

No matter what your budget, you can display your ads on Google and our advertising network. Pay only if people click your ads.

**AdWords** doubled my website traffic!

Sales at my shop are up another 10% this year. When you go online, you go Google.

Sponsored link  
**Romi Boutique**  
Designer apparel & gifts.  
Check out new arrivals!  
[www.shopromi.com](http://www.shopromi.com)

Romi, Owner — Romi Boutique

Start now

Want help creating a new account?

**Call 1-877-721-1738**

(9am - 9pm ET, Mon - Fri). [More](#)

Sign in with your  
**Google Account**

Email:

ex: pat@example.com

Password:

Stay signed in

[Can't access your account?](#)

### Benefits

[How it works](#)

[Reach more customers](#)

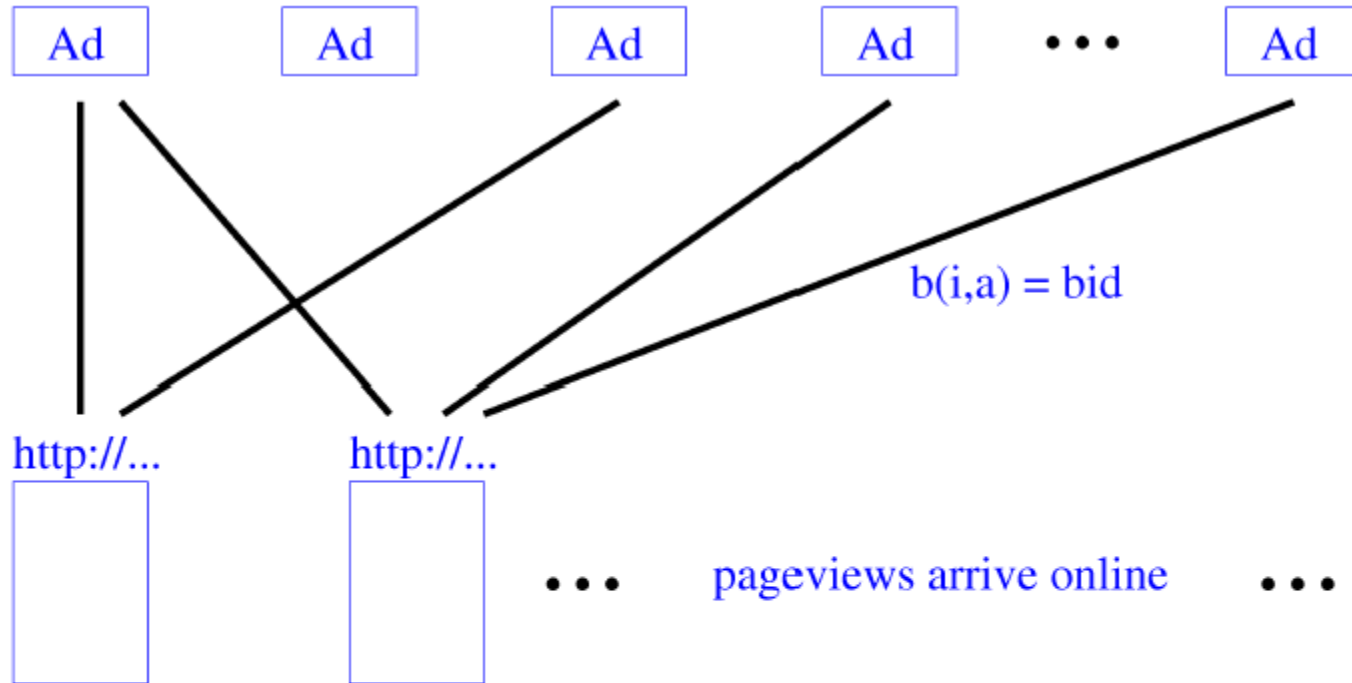
[Costs and payment](#)

[For local businesses](#)

### Benefits of AdWords

- ✓ **Effective:** Advertise on the most-used search engine worldwide.
- ✓ **Local:** Choose to show your ads in your region only.
- ✓ **Quick:** Set up your ad yourself or call us for free first-time set-up service.
- ✓ **Easy:** No expertise required.
- ✓ **Measurable:** Reports show you what you get for your money.
- ✓ **No risk:** You only pay when your advertising works.
- ✓ **Flexible:** Change, pause or end your campaign at any time.

# Online AdWord Auction



- ▶ When page arrives, assign an eligible ad.
  - ▶ revenue from assigning page  $i$  to ad  $a$ :  $b_{ia}$
- ▶ “AdWords” (AW) problem:
  - ▶ **Maximize revenue** of ads served:  $\max \sum_{i,a} b_{ia} x_{ia}$
  - ▶ **Budget** of ad  $a$ :  $\sum_{i \in A(a)} b_{ia} x_{ia} \leq B_a$

# Display Ad Auction

The screenshot shows the CNN website homepage. At the top, there's a navigation bar with 'EDITION: U.S.', 'INTERNATIONAL', and 'MÉXICO'. The CNN logo is prominently displayed in the center. To the right, there are links for 'Sign up' and 'Log in', and a search bar. Below the navigation bar, there's a horizontal menu with categories like 'Home', 'Video', 'NewsPulse', 'U.S.', 'World', 'Politics', 'Justice', 'Entertainment', 'Tech', 'Health', 'Living', 'Travel', 'Opinion', 'iReport', 'Money', and 'Sports'. The main content area is divided into several sections. On the left, there's a large image of a man shouting, with the headline 'Witnesses: Huge protests in Yemen over deal for president'. Below this, there's a 'Latest news' section with several links. In the center, there's a large image of a man in a white robe, with the headline 'Reticent believer 'gets the Holy Ghost''. Below this, there's a 'Featured' section with three small images and their respective headlines. On the right, there's a large advertisement for American Airlines with the text 'NOT ON ORBITZ. ALWAYS ON AA.COM.' and a 'Book Now' button. Below the ad, there's a 'POPULAR ON FACEBOOK' section and a 'Most popular stories right now' section with several links and progress indicators.

updated 10:51 a.m EDT, Sun April 24, 2011

Make CNN Your Homepage

### Witnesses: Huge protests in Yemen over deal for president

Witnesses report hundreds of thousands of demonstrators in Sanaa, Yemen, to protest a deal that would give the country's leader immunity from prosecution. [FULL STORY](#)

- Deal calls for president to step down

### Latest news

- China blocks Easter service in Beijing
- Libya rebels denounce 'dirty games'
- Flights arrive in St. Louis after twister
- Camera captures storm at airport
- New Yorkers hail cab for 3,000-mile trip
- Where did Easter Bunny come from?
- 'Bed Intruder' Internet star arrested
- Wife charged with stabbing NFL player
- Man killed near West Bank holy site

### Reticent believer 'gets the Holy Ghost'

CNN's John Blake had his first brush with the "Holy Ghost" when he was 9 years old, and he's still trying to digest what it meant more than 30 years later. This is his Easter story of love, faith and redemption. [FULL STORY](#)

### Featured

- Gallery: Easter around the world
- Ex-cons assist with WH Easter Egg Roll
- Rio's reimagined 'Passion Play'

### NOT ON ORBITZ. ALWAYS ON AA.COM.

[Book Now](#)

AmericanAirlines | AA.com

ADVERTISEMENT

Hi! Log in or sign up to personalize!

POPULAR ON FACEBOOK

NEWSPULSE

Most popular stories right now

- Viral video star arrested in Alabama
- New Yorkers take cross-country taxi ride
- Church: Police block Beijing Easter service
- My Faith: A reluctant churchgoer 'gets the Holy

# Display Ad Auction

- **Impression:** Display/Banner Ads, Video Ads, Text Links
- Cost-Per-Impression (CMI/CPM)
- Done through offline negotiations+ Online allocations
- Q1, 2010: One Trillion Display Ads in US, \$2.7 Billion
- Top publisher: Facebook, Yahoo and Microsoft sites
- Top Advertisers: AT&T, Verizon, Scottrade
- Ad Serving Systems e.g., Facebook, Google DoubleClick Ad Planner

# DoubleClick Ad Planner

https://www.google.com/accounts/ServiceLogin?service=branding&ltmpl=adplanner&continue=https%3A//www.google.com/adplanner/



Change Language:

[For publishers - Help](#)

## Which websites attract your target customers?

View a site listing:

[Ad Planner top 1,000 sites](#)

Refine your online advertising with DoubleClick Ad Planner, a free media planning tool that can help you:

### Identify websites your target customers are likely to visit

- Define audiences by demographics and interests.
- Search for websites relevant to your target audience.
- Access unique users, page views, and other data for millions of websites from over 40 countries.

### Easily build media plans for yourself or your clients

- Create lists of websites where you'd like to advertise.
- Generate aggregated website statistics for your media plan.

Sign in with your  
**Google Account**

Email: **hajiaghayi@gmail.com**

Password:

[Can't access your account?](#)

[Sign in as a different user](#)

Don't have a Google account?

### Are you a publisher?

Showcase your site with the [DoubleClick Ad Planner Publisher Center](#).

## Learn more about DoubleClick Ad Planner

How it works

[Tutorials](#)

[Features](#)

Define your audience



Find relevant sites for your ads



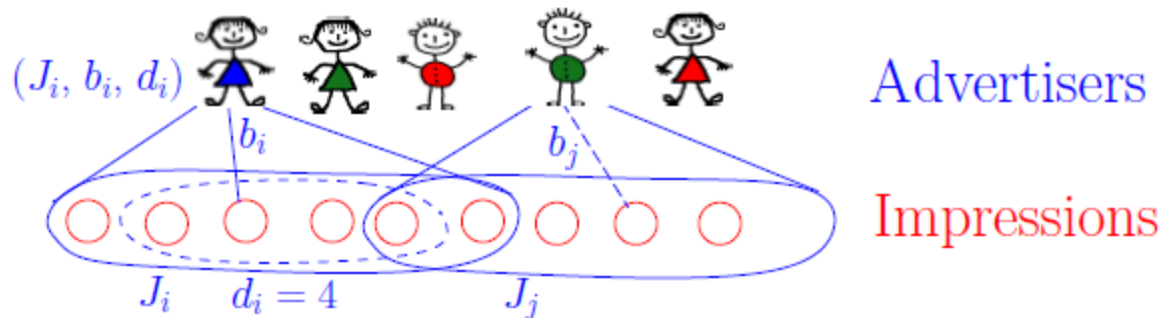
Build and analyze your media plan





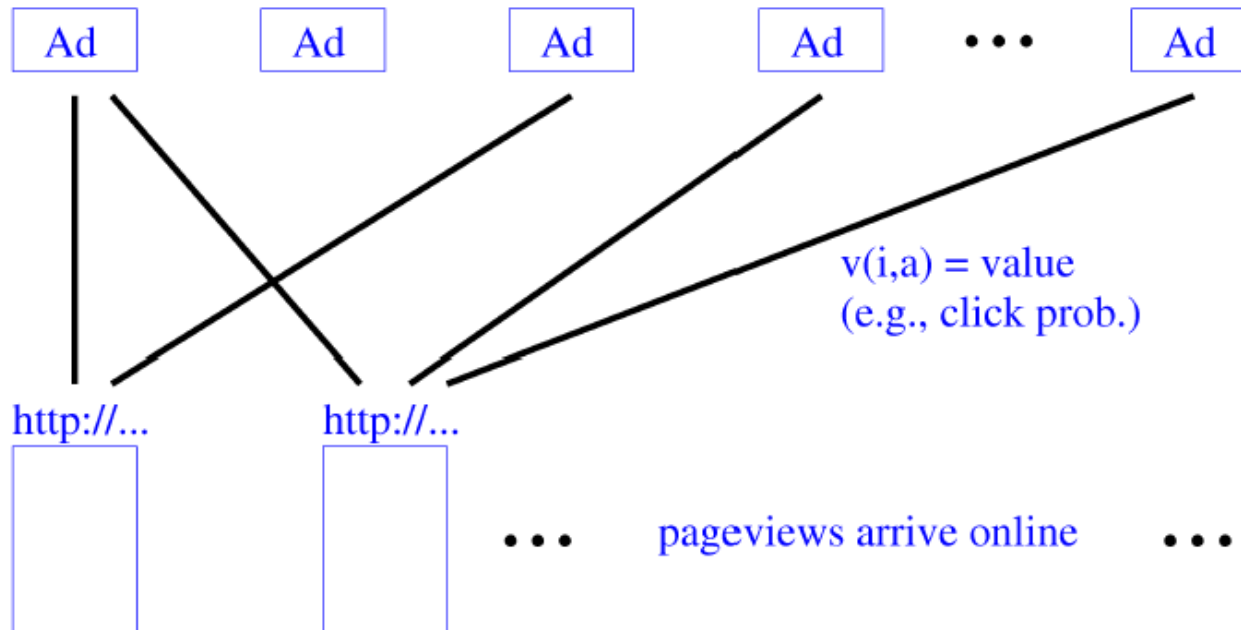
# Google Ad Planning

- ▶  $n$  advertisers, and set  $Y$  of impressions (items).
- ▶ Each advertiser  $i$ 
  - ▶ Interested in a set  $J_i$  of impressions, (e.g, young women in Seattle),
  - ▶ Needs  $d_i$  impressions (Demand),
  - ▶ Value  $v_{it}$  (or Bid  $b_i$ ) for each impression  $t$ ,



Efficiency (or Revenue) Maximization: Find an assignment with the maximum value.

# Online Display Ad



- ▶ When page arrives, assign an eligible ad.
  - ▶ value of assigning page  $i$  to ad  $a$ :  $v_{ia}$
- ▶ Display Ads (DA) problem:
  - ▶ **Maximize value** of ads served:  $\max \sum_{i,a} v_{ia} x_{ia}$
  - ▶ **Capacity** of ad  $a$ :  $\sum_{i \in A(a)} x_{ia} \leq C_a$

# AdCell Auction

shopalerts.att.com/sho/att/index.html?ref=portal



EXPLORE SHOP SUPPORT MY ACCOUNT Log In | Register

Wireless Internet Home Phone Digital TV Bundles Additional Services Special Offers

## ShopAlerts by AT&T

### Know What's Hot

Get text alerts about new products, special offers, and events from your favorite brands when you are nearby stores. It all comes included with America's best wireless service.

#### Get Started

\* Required Fields

Phone Number \*

Age Range\* (choose one)

E-mail

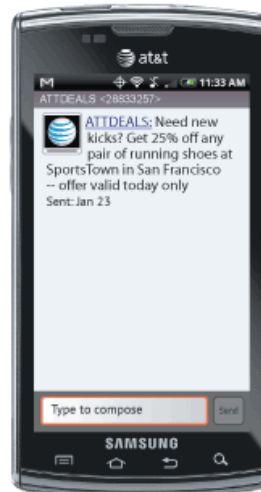
Zip Code

Gender (choose one)

I accept the [Terms and Conditions](#) and have read the [Privacy Policy](#)

Join Now

ShopAlerts is currently available in Chicago, Los Angeles, New York City, and San Francisco.



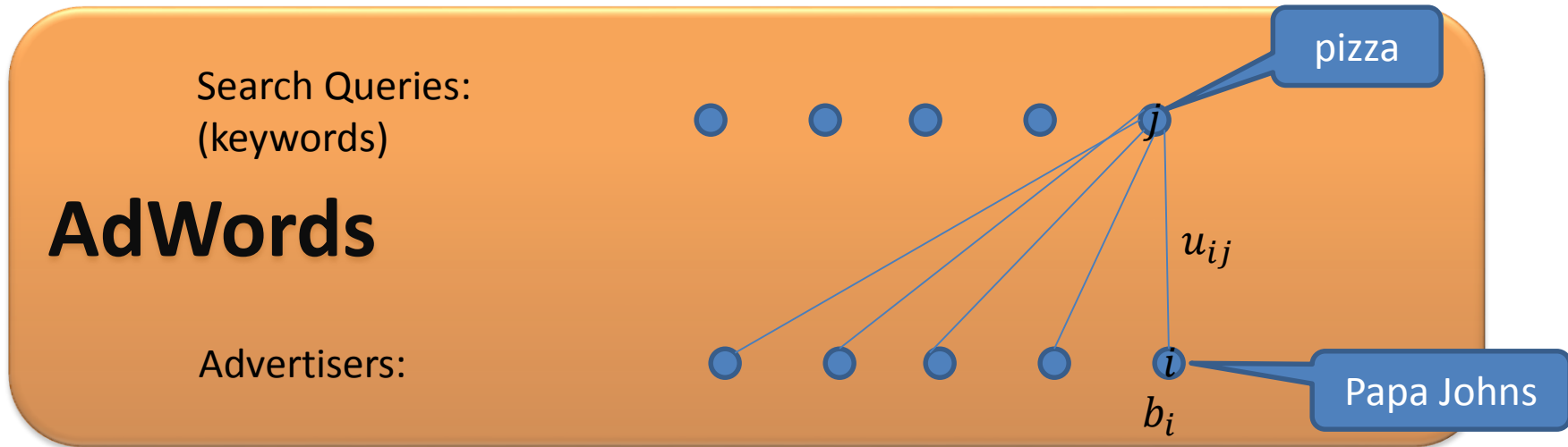
Deals From Your Favorite Brands

and more!

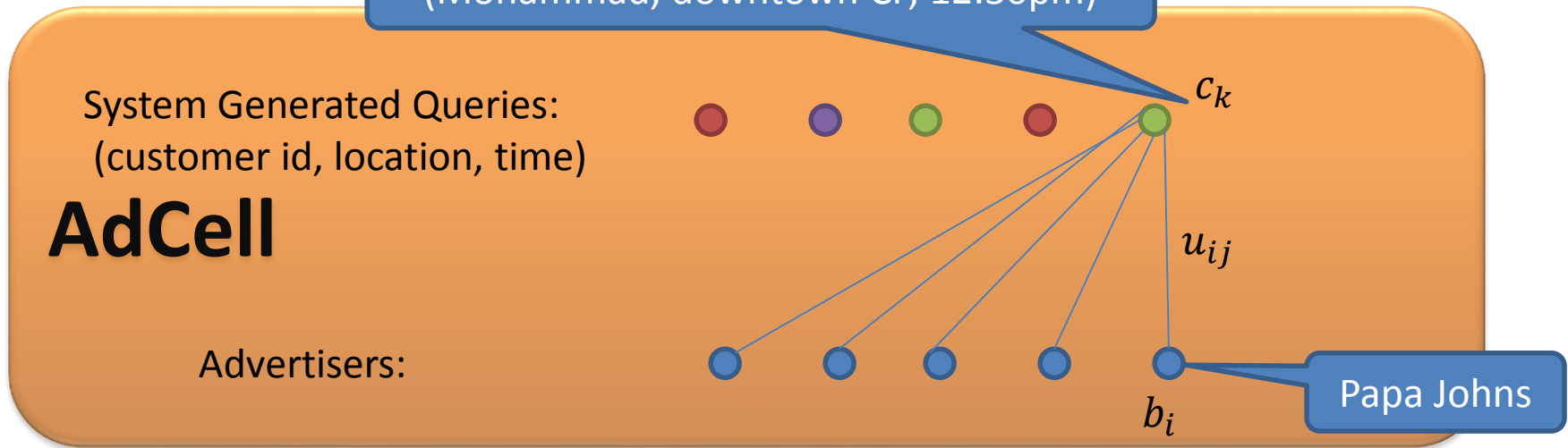
# AdCell Auction

- Online Advertising
  - Major source of revenue
- AdCell vs AdWords
  - Intrusive delivery (TEXT, MMS, etc)
  - Limited number of Ads per customer
  - System generated queries
- ShopAlerts by AT&T
- Formulated in [AHLPS'11, ESA'11]

# AdWords vs AdCell

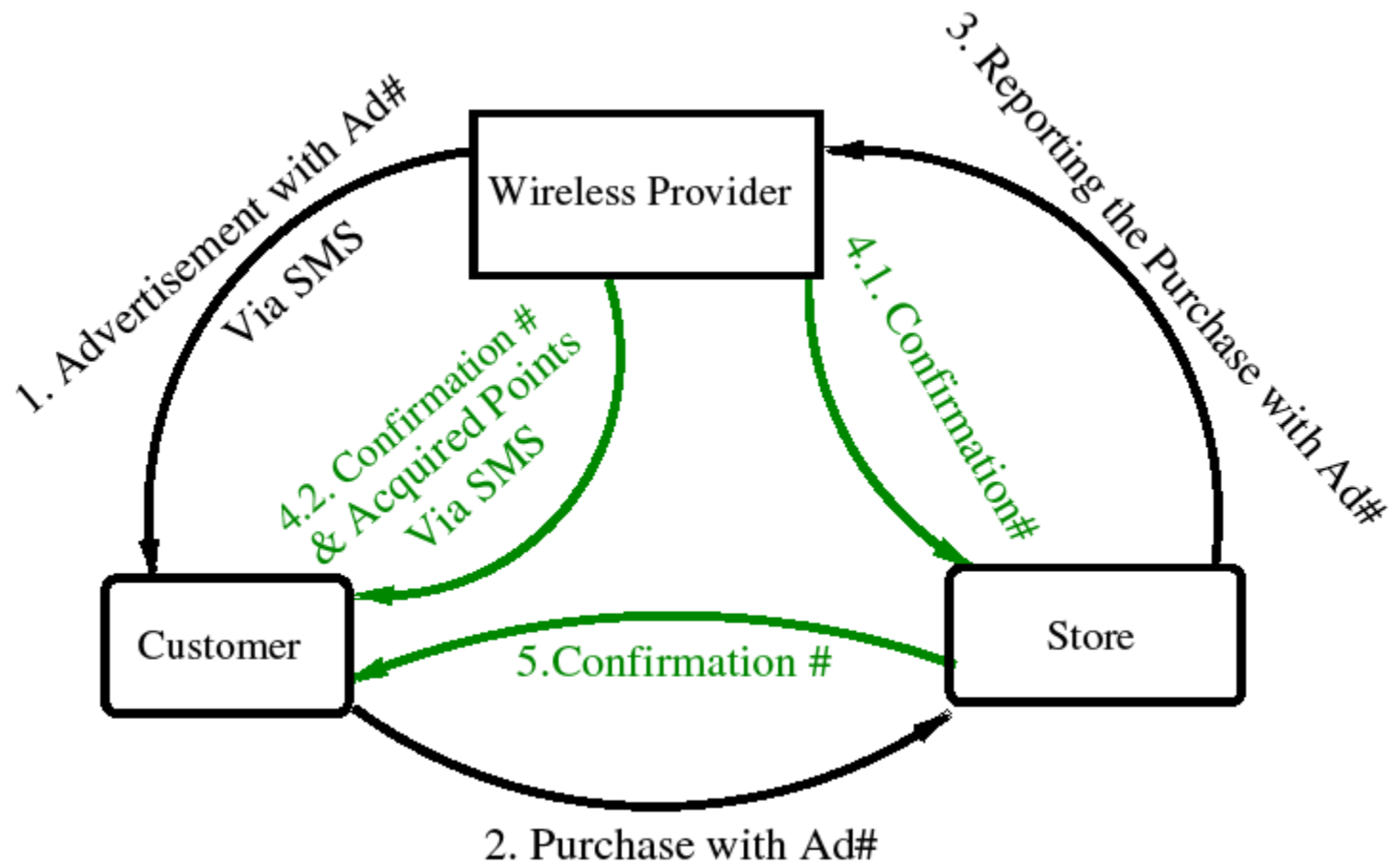


(Mohammad, downtown CP, 12:30pm)



# Customer Policy

- AdCell is intrusive
- Incentivizing customers
  - Discount on service plan if they opt in
  - Limited number of ads per month



# Online Bipartite Matching

- All these three problems are generalizations of Online Bipartite Matching:
- The input to the problem is:
  - bipartite graph  $G = (V_1 \cup V_2, E)$
  - $V_1$  is the set of advertisers and  $V_2$  is the set of keywords
  - the vertices in  $V_2$  (keywords) arrive in an on-line fashion
  - the edges incident to each vertex  $u$  in  $V_2$  are revealed when  $u$  arrives and determine the advertisers who want keyword  $u$ .
- When  $u$  arrives, the algorithm may match  $u$  to a previously unmatched adjacent vertex in  $V_1$ , if there is one.
- Such a decision, once made, is irrevocable.
- The objective is to maximize the size of the resulting matching.



# Online Bipartite Matching: Greedy

- Any **greedy** algorithm that always matches a vertex in  $V_2$  if a match is possible constructs a maximal matching, and therefore such an algorithm has a **competitive ratio** of  $\frac{1}{2} = \mathbf{0.50}$  (by a double counting argument).
- **Competitive ratio:** The ratio of our algorithm to the *best (optimum)* offline algorithm.
- On the other hand, given any deterministic algorithm, it is easy to construct an instance that forces that algorithm to find a matching of size no greater than half of the optimum.

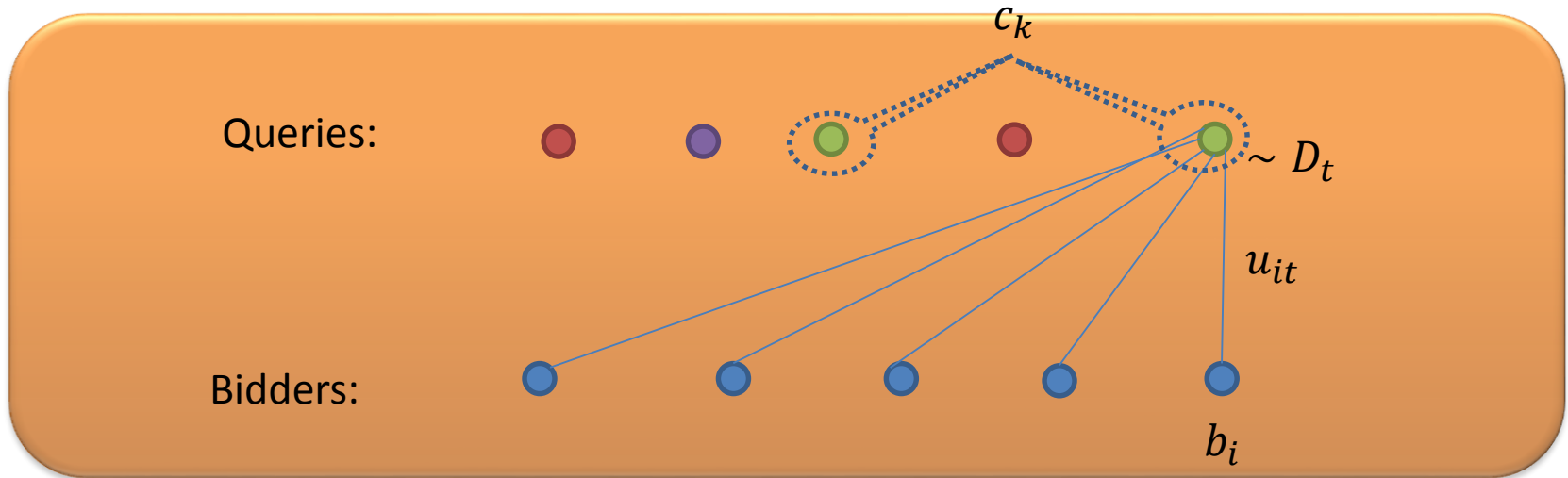
# Online Bipartite Matching: Randomized

- Any **randomized** algorithm that chooses a single random ranking on the vertices in  $V_1$
- When a vertex  $u$  in  $V_2$  arrives among its unmatched neighbors assign  $u$  to the one that has the highest ranking
- This produces a competitive ratio  $1-1/e \approx \mathbf{0.63}$
- This is the best that we can do in the online world.
- However, if we know stochastic information like distributions of the keywords (the probability that a node  $u$  in  $V_2$  arrives) and also the budget to the bid ratio is very large, we can obtain a competitive ratio very close to **1**.

# Motivating Problems

- Online Advertising
  - AdWord Auction
    - Budget constraints (advertisers)
  - Display Ad Auction
    - Capacity constraints (advertisers)
  - AdCell Auction
    - Budget constraints (advertisers) & capacity constraints (customers)
- The goal is maximizing revenue
  - We charge the bids themselves (first price-auction)

# Model



25

- $D_t$ :  $t^{\text{th}}$  query drawn from distribution  $D_t$
- $i$ : bidder, with budget  $b_i$  and bid  $u_{ij}$  for every query of type  $j$  in the support of  $D_t$
- $k$ : the set to which query  $x_t$  belongs.
- $c_k$ : capacity of set  $k$ .

# Use of History (model-based)

- Model
- 3 Variants of the problem
  - Only Budget Constraint
  - Only Capacity Constraint
  - Both Budget & Capacity Constraints
- **Prophet Inq. Setting**: Known but different distributions for different times, different customers, etc. (similarities to a Bayesian setting)
- As we have done empirical analysis using AT&T YellowPages data, **IN PRACTICE** we know probability distribution of 95% of queries (search keywords) [**AHHKLZ'12**].
- (Approximation) algorithms for **real-world** instead of the worst case

# Secretary setting vs. Prophet Inq. setting

- **Secretary problem** is a classic optimal stopping theory problem studied since 1963 [Dynkin] and more recently in auction design [HKP04, ACM EC'04] (many follow-up work)
- **Secretary setting:** given a sequence of random variables  $x_1, \dots, x_n$  drawn i.i.d. from the **same** distribution (or the random order model), an onlooker has to choose a certain number  $k$  of values and cannot choose a past value.
- **Prophet Inq.** is another well-studied optimal stopping theory problem since the 1970s [KS77, KS78, Ken87] and more recently in computer science [HKS07, AAI'07]
- **Prophet Inq. setting:** given possibly **different** distributions of random variables  $x_1, \dots, x_n$  an onlooker has to choose a certain number  $k$  of values and cannot choose a past value
- Called *Prophet Inq.* since a prophet with complete foresight has only a bounded advantage over an onlooker who observes the random variables one by one
- For  $k=1$ , the tight bound of  $\frac{1}{2}$  is known which we generalize.<sup>27</sup>

# IP (budgeted)

$$\text{maximize.} \quad \sum_i \min\left(\sum_t \sum_j \mathbf{x}_{ijt} u_{ij}, b_i\right) \quad (B)$$

$$\forall j \in [n], t \in [T] \quad \sum_i \mathbf{x}_{ijt} \leq \mathcal{R}_{jt} \quad (R)$$

$$\mathbf{x}_{ijt} \in \{0, 1\}$$

- $b_i$ : budget of buyer  $i$
- $u_{ij}$ : bid of buyer  $i$  for a query of type  $j$
- $\mathcal{R}_{jt}$ : Indicator random variable which is 1 iff the  $t^{\text{th}}$  query is of type  $j$
- $x_{ijt}$ : allocation variable

# IP (capacitated)

$$\begin{aligned} &\text{maximize.} && \sum_i \sum_t \sum_j x_{ijt} u_{ij} \\ &\forall j \in [n], t \in [T] && \sum_i x_{ijt} \leq R_{jt} && (R) \\ &\forall i \in [m] && \sum_t \sum_j x_{ijt} \leq c_i && (C) \\ &&& x_{ijt} \in \{0, 1\} \end{aligned}$$

- $c_i$ : capacity of advertiser  $i$
- $u_{ij}$ : bid of buyer  $i$  for a query of type  $j$
- $R_{jt}$ : Indicator random variable which is 1 iff the  $t^{\text{th}}$  query is of type  $j$
- $x_{ijt}$ : allocation variable



# Expectation LPs

maximize.

$$\sum_i \sum_t \sum_j \mathbf{x}_{ijt} u_{ij}$$

$\forall i \in [m]$

$$\sum_t \sum_j \mathbf{x}_{ijt} u_{ij} \leq b_i$$

$\forall j \in [n], t \in [T]$

$$\sum_i \mathbf{x}_{ijt} \leq p_t(j); \mathbf{x}_{ijt} \geq 0$$

$$\mathbf{x}_{ijt} \geq 0$$

maximize.

$$\sum_i \sum_t \sum_j \mathbf{x}_{ijt} u_{ij}$$

$\forall j \in [n], t \in [T]$

$$\sum_i \mathbf{x}_{ijt} \leq p_j(t)$$

$\forall i \in [m]$

$$\sum_t \sum_j \mathbf{x}_{ijt} \leq c_i$$

$$\mathbf{x}_{ijt} \geq 0$$

# Online Algorithm (budgeted)

maximize.

$$\sum_i \sum_t \sum_j \mathbf{x}_{ijt} u_{ij}$$

$\forall i \in [m]$

$$\sum_t \sum_j \mathbf{x}_{ijt} u_{ij} \leq b_i$$

$\forall j \in [n], t \in [T]$

$$\sum_i \mathbf{x}_{ijt} \leq p_t(j); \mathbf{x}_{ijt} \geq 0$$

$$\mathbf{x}_{ijt} \geq 0$$

- Algorithm *Alg<sub>B</sub>*:
  - Solve this LP and let  $\mathbf{x}_{ijt}^*$  denote the optimal assignment.
  - Upon arrival of query  $t$  of type  $j$  allocate it to buyer  $i$  w.p.  $\frac{\mathbf{x}_{ijt}^*}{p_{tj}}$

$$\approx 1 - \frac{1}{\sqrt{2\pi k}}$$

- Claim:  $Alg_B$  obtains at least  $(1 - \frac{k^k}{e^k k!})$  of  $LP_B$  (in expectation)  
 , where  $k$  is the minimum of the budget to the bid ratios
- $X_{ijt} = \begin{cases} u_{ij} & w.p. \quad x_{ijt}^* \\ 0 & w.p. \quad 1 - x_{ijt}^* \end{cases}$
- Expected revenue of  $Alg_B$  is:  $E[\sum_i \min(\sum_{j,t} X_{ijt}, b_i)]$
- Objective value of  $LP_B$  is:  $\sum_i \sum_{j,t} E[X_{ijt}]$
- We show that for each  $i$  :  

$$E[\min(\sum_{j,t} X_{ijt}, b_i)] \geq (1 - \frac{k^k}{e^k k!}) \min(\sum_{j,t} E[X_{ijt}], b_i)$$

# Expectation of Truncated Sum (ETS) vs Truncated Sum of Expectations (TSE)

- Theorem (ETS vs TSE):
  - If  $X_1, \dots, X_n$  are independent random variables,  
 $k \in \mathbb{N}$ ,  $b \in \mathbb{R}_+$  and all  $X_j \in [0, \frac{b}{k}]$  then the  
following holds:

- $$E[\min(\sum_j X_j, b)] \geq (1 - \frac{k^k}{e^{k k!}}) \min(\sum_j E[X_j], b)$$
$$\geq (1 - \frac{1}{\sqrt{2\pi k}}) \min(\sum_j E[X_j], b)$$

# Proof of ETS-vs-TSE

- Create  $Y_1, \dots, Y_n$  such that  $Y_i \leq C$  w.p.  $E[X_i]/C$  where  $C=b/k$ 
  - $E[Y_i] = E[X_i]$
  - $Y_i \in \{0, \frac{b}{k}\}$
- Then
  - $E[\min(\sum_j Y_j, b)] \geq E[\min(\sum_j X_j, b)]$
- So it is enough to show that
  - $E[\min(\sum_j Y_j, b)] \geq \frac{\min(\sum_j E[Y_j], b) \cdot k^k}{e^k k!}$

# Proof of ETS-vs-TSE (cont'd)

- Take any  $Y_{j_1}$  and  $Y_{j_2}$
- Replace them with RVs  $Y'_{j_1}$  and  $Y'_{j_2}$  such that
  - $Y'_{j_1}, Y'_{j_2} \in \{0, \frac{b}{k}\}$
  - $E[Y'_{j_1}] = E[Y'_{j_2}] = E[Y_{j_1} + Y_{j_2}]/2$
- Then
  - $E[\min(\sum_j Y_j, b)] \geq E[\min(Y'_{j_1} + Y'_{j_2} + \sum_{j \neq j_1, j_2} Y_j, b)]$
- So it is enough to prove the theorem for the case where  $E[Y_{j_1}] = E[Y_{j_2}]$  for all  $j_1$  and  $j_2$ .
- The final step involves simple algebraic manipulations which we omit.

# A Less Simple Model: Capacity Only

- Consider one bidder with two query types  $q_1$  and  $q_2$
- Bid for  $q_1$  is 1 and for  $q_2$  is  $(1-\epsilon)/\epsilon$
- The first query is of type  $q_1$  with Prob. 1 and the second query is of type  $q_2$  with Prob.  $\epsilon$
- The expected revenue of any online (randomized) algorithm is  $\max\{1, \epsilon(\frac{1-\epsilon}{\epsilon})\} = 1$
- However optimum offline would allocate the second query if it arrives; otherwise allocate the first query. Its expected revenue is  $(1 - \epsilon) \times 1 + \epsilon(\frac{1-\epsilon}{\epsilon}) \approx 2$

# A Less Simple Model: Capacity Only

- This is the lower bound  $\frac{1}{2}$  also for Prophet Inq. known from the 1970s for which we generalize the upper bound.
- This is in contrast to factors  $1 - 1/e$  and better for the same distribution case, e.g., [MSVV'05, FMMM'09]
- Note that without stochastic information we cannot get any approximation factor (generalizes the secretary problem in this setting).



# The Capacitated Case

maximize.

$$\sum_i \sum_t \sum_j x_{ijt} u_{ij}$$

$\forall j \in [n], t \in [T]$

$$\sum_i x_{ijt} \leq p_j(t)$$

$\forall i \in [m]$

$$\sum_t \sum_j x_{ijt} \leq c_i$$

$$x_{ijt} \geq 0$$

- Algorithm *Alg<sub>C</sub>*:

- Solve the LP and let  $x_{ijt}^*$  denote the optimal assignment.
- Run a dynamic program for each advertiser:  $E_{i,t}^r$
- Upon arrival of query  $t$  of type  $j$ , select advertiser  $i$  w.p.  $\frac{x_{ijt}^*}{p_{tj}}$  then allocate it to  $i$  iff  $u_{ij} + E_{i,t+1}^{r-1} \geq E_{i,t+1}^r$

# Dynamic Program of $DP_i$

- For each advertiser  $i$  we can compute  $E_{i,t+1}^r$  using the following dynamic program:

$$\mathcal{E}_{i,t}^r = \sum_j \mathbf{x}_{ijt}^* \max\{u_{ij} + \mathcal{E}_{i,t+1}^{r-1}, \mathcal{E}_{i,t+1}^r\} + (1 - \sum_j \mathbf{x}_{ijt}^*) \mathcal{E}_{i,t+1}^r$$

, where the quantity is the expected benefit we get from bidder  $i$  with only  $r$  remaining capacity at or after time  $t$ .

- Claim:  $DP_i$  obtains at least  $\frac{1}{2}$  of  $LP_C$  for each advertiser  $i$  (in expectation)

- We show that for each advertiser  $i$  :

$$E_{i,1}^{Ck} \geq \frac{1}{2} \sum_t \sum_t u_{ij} x_{ijt}^*$$

# Stochastic Uniform Knapsack (SUK)

- A knapsack of capacity  $c$
- Item  $t$  arrives with prob  $p_t$  and has value  $v_t$  at time  $t$
- Dynamic Program (DP):
  - At time  $t$ , if item  $t$  arrives: put it in the knapsack iff  $v_j + E_{t+1}^{r-1} \geq E_{t+1}^r$

$$E_t^r = \max(p_t v_t + (1 - p_t) E_{t+1}^{r-1}, E_{t+1}^r)$$

# Dynamic Program for SUK

- Dynamic Program (DP):

- At time  $t$ , if item  $t$  arrives: put it in the knapsack iff  $v_j$   
 $+t E_{k,t+1}^{r-1} \geq E_{k,t+1}^r$

- $E_t^r = \max(p_t v_t + (1 - p_t) E_{t+1}^{r-1}, E_{t+1}^r)$

- Theorem (SUT-DP):

- Let  $O_{DP} = E_1^c$  and  $O^* = \sum_j p_j v_j$ . If  $\sum p_j \leq c$  then

$$O_{DP} \geq \frac{1}{2} O^*$$

# Proof of Theorem (SUT-DP)

- $O_{DP} = E_1^c$
- $O^* = \sum_t p_t v_t$
- $E_t^r = \max(p_t v_t + (1 - p_t)E_{t+1}^{r-1}, E_{t+1}^r)$

- Lemma:

–  $E_t^r$  has decreasing marginal value in  $r$ ,

i.e.:  $E_t^r - E_t^{r-1} \leq E_t^{r-1} - E_t^{r-2}$ ,

Therefore:  $E_t^{r-1} \geq \frac{r-1}{r} E_t^r$

- Lemma:

–  $E_t^r \geq \max(p_t v_t + \left(1 - \frac{p_t}{r}\right) E_{t+1}^r, E_{t+1}^r)$

# Proof of Theorem (SUT-DP)

- $O_{DP} = E_1^C$
- $O^* = \sum_t p_t v_t$
- $E_t^C \geq \max(p_t v_t + (1 - \frac{p_t}{r}) E_{t+1}^C, E_{t+1}^C)$
- Lemma:
  - WLOG, we may assume  $O^*=1$
  - then  $O_{DP}/O^*$  is at least:

minimize.

$$E_1^C$$

$\forall t \in [u - 1] :$

$$E_t^C - p_t v_t - (1 - \frac{p_t}{r}) E_{t+1}^C \geq 0 \quad (\alpha_t)$$

$$E_u^C - p_u v_u \geq 0 \quad (\alpha_u)$$

$\forall t \in [u - 1] :$

$$E_t^C - E_{t+1}^C \geq 0 \quad (\beta_t)$$

$$\sum_{t=1}^u p_t v_t \geq 1 \quad (\gamma)$$

$$v_t \geq 0, \quad E_t^C \geq 0$$



$$u = t_{max}$$

# Proof of Theorem (SUT-DP)

minimize.

$$\mathbf{E}_1^C$$

$$\forall t \in [u - 1] : \quad \mathbf{E}_t^C - p_t \mathbf{v}_t - \left(1 - \frac{p_t}{r}\right) \mathbf{E}_{t+1}^r \geq 0 \quad (\alpha_t)$$

$$\mathbf{E}_u^C - p_u v_u \geq 0 \quad (\alpha_u)$$

$$\forall t \in [u - 1] : \quad \mathbf{E}_t^C - \mathbf{E}_{t+1}^C \geq 0 \quad (\beta_t)$$

$$\sum_{t=1}^u p_t \mathbf{v}_t \geq 1 \quad (\gamma)$$

$$\mathbf{v}_t \geq 0, \quad \mathbf{E}_t^C \geq 0$$

maximize.

$$\gamma$$

$$\forall t \in [u] : \quad \gamma p_t - \alpha_t p_t \leq 0 \quad (\mathbf{v}_t)$$

$$\alpha_1 + \beta_1 \leq 1 \quad (\mathbf{E}_1^C)$$

$$\forall t \in [2 \cdots u - 1] : \quad \alpha_t + \beta_t - \left(1 - \frac{p_{t-1}}{C}\right) \alpha_{t-1} - \beta_{t-1} \leq 0 \quad (\mathbf{E}_t^C)$$

$$- \left(1 - \frac{p_{u-1}}{C}\right) \alpha_{u-1} - \beta_{u-1} \leq 0 \quad (\mathbf{E}_u^C)$$

$$\alpha_t \geq 0, \quad \beta_t \geq 0, \quad \gamma \geq 0$$



# Proof of Theorem (SUT-DP)

maximize.

$$\gamma$$

$\forall t \in [u] :$

$$\gamma \leq \alpha_t$$

$(\mathbf{v}_t)$

$$\alpha_1 + \beta_1 \leq 1$$

$(\mathbf{E}_1^C)$

$\forall t \in [2 \cdots u - 1] :$

$$\alpha_t + \beta_t \leq \left(1 - \frac{p_{t-1}}{C}\right) \alpha_{t-1} + \beta_{t-1}$$

$(\mathbf{E}_t^C)$

$$0 \leq \left(1 - \frac{p_{u-1}}{C}\right) \alpha_{u-1} + \beta_{u-1}$$

$(\mathbf{E}_u^C)$

$$\alpha_t \geq 0, \quad \beta_t \geq 0, \quad \gamma \geq 0$$

- We construct a feasible dual that obtains  $\frac{1}{2}$ :
  - Setting  $\alpha_t = \gamma$  and  $\beta_t = \beta_{t-1} - \frac{p_{t-1}}{C} \gamma$  and  $\beta_1 = 1 - \gamma$
  - Then  $\beta_t = 1 - \gamma - \gamma \sum_{k=1}^{t-1} \frac{p_k}{C} \geq 1 - 2\gamma$
  - So we can set  $\gamma = 1/2$

# Capacity at least k

- Theorem:

If the capacity of each customer is at least  $k$ , then the same algorithm  $\text{Alg}_c$  obtains at least  $1 - \frac{1}{\sqrt{k+3}}$  fraction of the optimum solution in expectation.

- The analysis is more complicated and needs a process called **Sand/Barrier** to analyze it

[AHL'12, ACM EC'12].

- So if the capacities are large enough we obtain almost the optimum expectation.

# General Model (Capacity and Budget)

- This is the case of AdCell Auction
- We write the combined LP of the capacitated case and the budgeted case by writing both set of constraints
- The algorithm  $\text{Alg}_{BC}$  is the same as that of the capacitated case ( $\text{Alg}_C$ ) except we are using the new LP
- Theorem:
  - ✓  $\text{Alg}_{BC}$  obtains  $\left(1 - \frac{1}{\sqrt{k+3}}\right) \left(1 - \frac{1}{\sqrt{2\pi k}}\right) \approx 1$  fraction of the optimum in expectation (**VERY PRACTICAL**).
- The proof comes due to negative correlations.

# Online Stochastic Generalized Assignment Problem

- Generalization of all problems considered so far, as well as stochastic knapsack
- Items arrive **online**, each with a value and a size maybe dependent to bins
- Upon arrival of an item, it can be placed in a bin or discarded
- Distribution is available about size and value but the items are coming in an adversarial order
- **Objective** is to maximize the value of the placement

# Online Stochastic Generalized Assignment Problem

- Theorem: Under the assumption that no item takes more than  $1/k$  capacity of any bin of non-zero value, we can obtain a  $1 - \frac{1}{\sqrt{k}}$ -competitive algorithm.
- The algorithm is competitive even when the adversary can do fractional assignment
  - application to **banner advertisement** (the problem is independent-set hard otherwise)
- The bound is **tight** for  $k=1$

# Online Stochastic Generalized Assignment Problem

Ideas of the proof:

- First we write an Expected LP as before
- Upon arrival of an item, we decide the bin that we want to try based on  $x$  variables as before
- To decide whether we really want to assign the item to the selected bin we are running a continues version of sand/barrier process (a.k.a. the magician problem)
- Intuitively a probabilistic algorithm which derives a new bound like *Chernoff* but with zero violation prob.
- The proof is involved; but uses induction as a basis

# Questions?

Thank you!

感謝

GRACIAS!

謝謝

Thank You Well!

Dankes!

Thank you!

Merci!

謝謝

Thank you!

Gracias!

Today!

Merci!

どうもありがとうございました

Today!

Obrigado

تشكر

Thank you!

どうもありがとうございます

どうもありがとうございました

感謝

Gracias!

Dank Je Well!

謝謝

THANK YOU!

どうもありがとうございました

Merci!

謝謝

どうもありがとうございました

MERCI!

Thank You!

Thank You!

感謝

Gracias!

Gracias!

どうもありがとうございました

Merci!

Merci!

Today!

GRACIAS! Thank you!

Dankes!