Joint Review by

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of

Ramsey Theory (2nd Ed.) by Graham,Rothchild,Spencer
Wiley, 1991, 208 pages, Softcover, $96

Ramsey Theory for Discrete Structures by Prömel
Springer, 2013, 225 pages, Hardcover, $84.00

Rudiments of Ramsey Theory (2nd Ed.) by Graham,Butler
AMS, 2015, 83 pages, Softcover, $63

An Intro. to Ramsey Theory: Fast Functions, ∞, and Metamathematics
by M. Katz,Reimann. AMS, 2018, 207 pages, Softcover, $52

Ramsey Theory Ov the Integers (2nd Ed.) by Landmann,Robertson
AMS, 2021, 384 pages, Softcover, $61

Fundamentals of Ramsey Theory by Robertson
CRC, 2021, 207 pages, Softcover, $63

Basics of Ramsey Theory by Jungic
CRC, 2023, 238 pages, Hardcover, $120

1 Introduction

Some people collect Bibles. Some people collect First Editions. Some people collect First Editions of Bibles. I collect books on Ramsey Theory. I did not intend to do this. However, since I work in the field and I was the SIGACT News book Review Editor, such books just happened to show up on my doorstep.

The best way to straighten out in my own head what each book covers is to do a joint review of all 20 of them. *That Would Be Insane!* In this column I review those that mostly use techniques from combinatorics and are mostly elementary. I may, in later columns, review other sets of books.

We will need the following notation.

Def 1.1

1. N is the naturals, $\{1, 2, 3, \ldots\}$.

2. If $n \in \mathsf{N}$ then $[n] = \{1, \ldots, n\}$.

3. If A is a set and $a \in \mathsf{N}$ then $\binom{A}{a}$ is the set of a-sized subsets of A. Note that $\binom{n}{a}$ is the edges of the complete a-hypergraph.

4. Let $A \subseteq \mathsf{N}$ (it will either be $[n]$ or N). Let $a, c\mathsf{N}$ and $\mathrm{COL}\binom{A}{a} \to [c]$. Let $H \subseteq A$. A is *homogenous* (henceforth *homog*) if COL restricted to $\binom{H}{a}$ is constant.

We describe three theorems from Ramsey Theory, and their extensions.

1. *Ramsey's Theorem:* For all k, c there exists $n = R_c(k)$ such that for all $\mathrm{COL}\colon \binom{[n]}{a}$ there exist a homog set of size k. There has been much work on upper and lower bounds on $R_c(k)$ especially $R_2(k) = R(k)$. The lower bound $R(k) \geq 2^{k/2}$ is done by the probablistic method. That last sentence is true but odd. Parul Erdös invented the probablitic method in order to get this lower bound on $R(k)$.

2. *Van Der Warden's Theorem (henceforth VDW's theorem:* For all k, c there exists $W = W(k, c)$ such that for all $\mathrm{COL}\colon [W] \to [c]$ there is a monochromatic arithmetic sequence of length k (henceforth a k-AP). The Gallai-Witt theorem is a generalization to more dimensions. The Hales-Jewitt theorem is a further generalization. The initial proof yieled enormous bounds on $W(k, c)$ that were Ackerman-like. Shelah obtained bounds that were primitive recursive, though still large. His proof is elementary (though difficult).

   The folloiwng are also known but not covered in all of the books: Szemeredi's theorms is a density version of VDW. Gowers obrained more reasonable bounds on $W(k, c)$ by using the density approach. His proof uses advanced math.

3. *Rado's Theorem* Let $a_1, \ldots, a_n \in \mathsf{Z}$. The following are equivalent:

   - For all finite colorings of N there exists $x_1, \ldots, x_n \in \mathsf{N}$ that are the same color such that $a_1 x_1 + \cdots + a_n x_n = 0$.
   - Some subset of $\{a_1, \ldots, a_n\}$ sums to 0.

There is a generalization of this theorem to systems of linear equations. Some nonlinear equations have also been studied.

All of these theorems are covered very well in all of the books. (except Szemeredi's theorem and Gower's theorem). Hence when I review the books I will concentrate on what else they bring to the table.

2 An Introduction to Ramsey Theory: Fast Functions, Infinity, and Metamathematics by Matthew Katz and Jan Reimann

This book has sections on infinite Ramsey theory, Ramsey Theory on large cardinals, and the Paris-Harrington Theorem. We describe the later.

Recall Gödel's incompleteness theorem which we summarize as:

> There are statements S such that S is true of the natural numbers but cannot be proven in Peano Arithmetic.

This is a very important theorem since it shows that Peano Arithmetic cannot do everything in Number Theory. However, the statement S is not natural. Paris and Harrington came up with a natural statement in Ramsey theory that is not provable in Peano Arithmetic.

Def 2.1

1. PA is Peano Arithmetic. It is a system of axioms and rules of inference. Most theorems in number theory can be proven in it. Almost all (all but a finite number :-)) interesting theorems in number theory can be proven in it.

2. If ϕ is a statement that cannot be proven in PA then we write $PA \vdash \phi$. Note that ϕ must be written in the language of PA.

3. If ϕ is a statement and it cannot be proven in PA then we write $PA \not\vdash \phi$. Note that ϕ must be written in the language of PA. If ϕ was a theorem in analysis (e.g., the intermediate value theorem) then technically $PA \not\vdash \phi$ is true but one would never write that.

4. PH is the Paris-Harrington statement which I will define later in this review.

Theorem 2.2 *Ramsey's Theorem for Infinite Hypergraphs: For all c, k, for all c-colorings of $\binom{\mathsf{N}}{k}$, there is an infinite homogenous set.*

Def 2.3 A *large set* is a finite set of N where the size of the set is bigger than the smallest element.

One might wonder if you can get a homogenous set that's large. That's not quite right since $\{1, 2\}$ is a large homog set. Hence we look at the following:

Theorem 2.4 PH Ramsey's Theorem for Hypergraphs: *For all a, k, c, there exists $n = PH(a, k, c)$ such that for all* COL: $\binom{\{k,k+1,\ldots,k+n\}}{a} \to [c]$. *a large homog set.*

One way to prove this is from the Infinite Ramsey Theorem. This proof does not give any bounds on n. Is there a proof that gives bounds on n? NO! They prove that $PA \not\vdash PH$. One consequence of this is that there is no proof that gives bounds on $PH(a, k, c)$.

There are two ways to prove $PA \not\vdash PH$. One way is to show that the PH function grows so quickly that it can't be proven to exist in PA. The other way is to use non-standard models of PA, indiscernibles, and, ironically, Ramsey Theory! That is the way the authors proved it. Their presentation is self-contained and can be followed by a non-logician. It will take some time to get through but is well worth it.

Note that this yields a natural theorem that is not provable in PA.

3 Needed Background

Def 3.1 Let $k, n \in \mathsf{N}$.

1. $[n]$ is the set $\{1, \ldots, n\}$.
2. If A is a set then $\binom{A}{k}$ is the set of all k-sized subsetes of A.

3. Let COL: $\binom{A}{k} \to [c]$. A set $H \subseteq A$ is *homogenous (henceforth homog)* if COL restricted to $\binom{H}{k}$ is constant.

The following theorems are the first and most basic theorems in Ramsey Theory.

1. (Ramsey's Theorem [**?**]) For all a, c, k there exists $n = n(a, c, k)$ such that for all, COL: $\binom{[n]}{a} \to [c]$, there exists homog $H \subseteq [n]$ such that $|H| = k$.
2. For all a, c, for all COL: $\binom{\mathsf{N}}{a} \to [c]$ there exists an infinite homog $H \subseteq \mathsf{N}$. such that $|H| = k$.
3. (van Der Waerden's Theorem [**?**]) For all c, k there $W = W(k, c)$ such that, for all COL: $[W] \to [k]$, there exists a monochromatic arithmetic sequence of length k.

4 Rudiments of Ramsey Theory, Second Edition, Graham and Butler. 1979, 2015

5 Ramsey Theory by Grahamm, Rothchild, Spencer. 1980, 1990

(First edition 1980, Second edition 1990)

6 Ramsey Theory on the Integers by Landman and Robertson. 2014

7 An Introduction to Ramsey Theory: Fast Functions, Infinity, And Metamathmatics by M. Katz and Reimann. 2018

8 Fundamentals of Ramsey Theory by Aaron Robertson. 2021

9 Basics of Ramsey Theory by Jungic. 2023