JOURNAL OF COMBINATORIAL THEORY (A) 20, 385–389 (1976)

Note

All Right Triangles Are Ramsey in E^2!

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Communicated by the Managing Editors

Received May 20, 1975

A triangle T is said to "Ramsey" in E^2 if for each coloring of the plane with two colors there is a monochromatic triangle congruent to T. Many triangles are known to be Ramsey (see [1–4]). Only the family of equilateral triangles are known to be non-Ramsey. For right triangles, it is known that all right triangles with the ratio of two-sides rational, or the ratio of the legs the square root of a rational are Ramsey [3, 5]. The conjecture has been made [1] that all triangles other than the equilateral triangles are Ramsey.

In this paper we show

(1) All right triangles are Ramsey.

(2) For every parallelogram P there is a congruent parallelogram with three vertices of one color.

(3) All triangles $(a, b, (b^2 + 2a^2)^{1/2})$, $2b \geqslant a$, are Ramsey.

(4) All triangles $(a, b, (4b^2 - a^2)^{1/2})$, $(3/2)^{1/2} < a < (5/2)^{1/2}\, b$, are Ramsey.

The following Lemma is the key to these results.

LEMMA 1. *For any real number a and two-coloring of the plane, there is a monochromatic equilateral triangle of side ka, $k \in \{1, 3, 5, 7\}$. (Note: k need not be the same for each a.)*

Proof. It is sufficient to prove the lemma for $a = 1$. We note that the triangle with sides 3, 5 and included angle 120° has the third side 7 and that the triangle with sides 7, 15 and included angle 60° has the third side 13. By [2, Theorem 1] it is sufficient to show that either R, S, qR, or qS occur monochromatically, where qR is the triangle similar to R with sides q times as large, q an odd integer.

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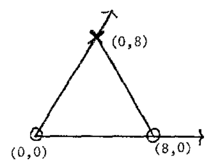

FIGURE 1

Since not all equilateral triangles of side 8 are monochromatic we can consider the general case given in Fig. 1. Let (0, 0), (8, 0) be red indicated by 0, and (0, 8) be blue indicated by x. Assume the theorem is false and proceed to color the lattice points with integer coordinates, avoiding monochromatic triangles with odd integer sides such as qR, qS above as well as equilateral triangles.

Hence, we can assume that (0, 3) is 0, which implies (3, 0) is x, and similarly (3, 5) is 0, (0, 5) is x, (5, 0) is 0, (5, 3) is x. We now consider four subcases each of which leads to a contradiction: Case 1, (0, 1) is 0, (0, 2) is x. Case 2, (0, 1) is 0, (0, 2) is 0. Case 3, (0, 1) is x, (0, 2) is 0. Case 4, (0, 1) is x, (0, 2) is x.

Case 1. (0, 1) is 0, (0, 2) is x.

Point	Color	Other points of the triangle
(1, 0)	x	(0, 0), (0, 1)
(−1, 1)	x	(0, 0), (0, 1)
(−3, 8)	0	(0, 5), (0, 8)
(−5, 5)	0	(0, 5), (0, 8)
(3, 2)	0	(0, 2), (0, 5)
(−3, 5)	0	(0, 2), (0, 5)
(2, 0)	x	(−3, 5), (−3, 8)
(2, −1)	0	(1, 0), (2, 0)
(3, −1)	0	(2, 0), (3, 0)
(−2, −1)	x	(3, −1), (3, 2)
(−2, 2)	x	(−5, 5), (3, 2)
(−1, 2)	0	(−1, 1), (−2, 2)
(−2, 1)	0	(−1, 1), (−2, 2)
(1, −1)	0	(−2, −1), (−2, 2)
(0, −1)	x	(0, 0), (1, −1)
(−1, −1)	x	(−1, 2), (2, −1)
(−1, 0)	0	(−1, −1), (0, −1)

Now (−2, 0) cannot be colored. Consider {(−2, 0), (−2, −1), (−1, −1)} and {(−2, 0), (−2, 1), (−1, 0)}.

Case 2. (0, 1) is 0 and (0, 2) is 0.

Point	Color	Other points of the triangle
(1, 0)	x	(0, 0), (0, 1)
(1, 1)	x	(0, 1), (0, 2)
(1, 2)	x	(0, 2), (0, 3)
(2, 0)	0	(1, 0), (1, 1)
(2, 1)	0	(1, 1), (1, 2)
(−1, 1)	x	(0, 0), (0, 1)
(−1, 2)	x	(0, 1), (0, 2)
(−2, 2)	0	(−1, 1), (−1, 2)
(3, 2)	x	(−2, 2), (3, 5)
(2, 3)	x	(2, 0), (5, 0)
(2, 2)	0	(3, 2), (2, 3)
(3, 1)	x	(2, 1), (2, 2)
(4, 0)	0	(3, 0), (3, 1)

Now (4, 1) cannot be colored. Consider {(4, 1), (4, 0), (5, 0) and (4, 1), (3, 1,) (3, 2)}.

Case 3. (0, 1) is x and (0, 2) is 0.

Point	Color	Other points of the triangle
(7, 1)	0	(0, 1), (0, 8)
(7, 0)	x	(7, 1), (8, 0)
(−1, 3)	x	(0, 2), (0, 3)
(−3, 3)	x	(0, 0), (0, 3)
(2, 0)	0	(−1, 3), (7, 0)
(2, 3)	x	(2, 0), (5, 0)
(2, 6)	0	(2, 3), (5, 3)
(2, 5)	x	(2, 6), (3, 5)
(2, −2)	0	(−3, 3), (2, 3)
(5, −3)	x	(2, 0), (5, 0)
(5, 2)	0	(2, 5), (5, −3)
(8, −1)	x	(0, 2), (5, 2)
(7, −1)	0	(7, 0), (8, −1)

Now (7, −2) cannot be colored. Consider {(7, −2), (2, −2), (7, 1)} and {(7, −2), (2, 3), (−1, 3)}.

Case 4. (0, 1) and (0, 2) are x.

Point	Color	Other points of the triangle
(1, 1)	0	(0, 1), (0, 2)
(3, 2)	0	(0, 2), (0, 5)
(−1, 2)	0	(0, 1), (0, 2)
(−2, 2)	x	(3, 2), (3, 5)
(−3, 5)	0	(0, 2), (0, 5)
(−3, 8)	0	(0, 5), (0, 8)
(2, 0)	x	(−3, 5), (−3, 8)
(2, 1)	0	(2, 0), (3, 0)
(1, 2)	x	(1, 1), (2, 1)
(1, −1)	0	(−2, 2), (1, 2)
(1, 0)	x	(1, −1), (0, 0)
(2, −1)	0	(1, 0), (2, 0)
(2, −2)	x	(−1, 2), (2, −1)
(1, 3)	0	(1, 2), (2, 2)
(0, 4)	x	(0, 3), (1, 3)
(1, 4)	0	(0, 4), (0, 5)
(2, 3)	x	(1, 3), (1, 4)
(−3, 3)	x	(0, 0), (0, 3)
(−2, 3)	0	(−3, 3), (−2, 2)

Now (−1, 3) cannot be colored. Consider {(−1, 3), (−2, 3), (−1, 2)} and {(−1, 3), (2, 0), (2, 3)}.

Since all cases contain a point which cannot be colored, the assumption is false and the theorem is proven.

We now can obtain several results.

THEOREM 2. *All right triangles are Ramsey.*

Proof. For any triangle $T(a, b, c)$, $a^2 + b^2 = c^2$, there is a monochromatic triangle ka, kb, kc, k odd. The "ladder" technique of [2] applies and yields the existence of a monochromatic triangle congruent to T.

THEOREM 3. *For every parallelogram P, there is a congruent parallelogram with three vertices of one color.*

Proof. Apply the ladder technique to the skew lattice determined by the parallelogram.

COROLLARY 4. *All triangles* $(a, b, (b^2 + 2a^2)^{1/2})$, $2b > a$, *are Ramsey.*

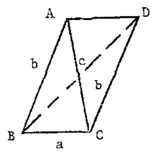

FIGURE 2

Proof. Apply Theorem 3 to the parallelogram *abab* with diagonal a as in Fig. 2. Note $c = (b^2 + 2a^2)^{1/2}$. Now if triangle ABC is monochromatic then triangle BDC must occur monochromatically.

COROLLARY 5. *All triangles* $(a, b, 4b^2 - a^2)$, $(3/2)^{1/2}\, b < a < (5/2)^{1/2}\, b$ *are Ramsey.*

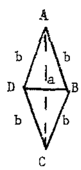

FIGURE 3

Proof. Apply Theorem 3 to the rhombus in Fig. 3. Note $AC = (4b^2 - a^2)^{1/2}$. If triangle ABD is monochromatic then triangle $(a, b, (4b^2 - a^2)^{1/2})$ occurs monochromatically.

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