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Ramsey Type Theorems in the Plane

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By a Euclidean Ramsey theorem an assertion of the following form is meant: let $\mathscr{K}_1, \ldots, \mathscr{K}_r$ be configurations in the n-dimensional Euclidean space E^n, then for any coloring of E^n in r colors there is some i and some $\mathscr{K}_i'$ consisting only of points of the i-th color such that $\mathscr{K}_i'$ is congruent to $\mathscr{K}_i$. Here we shall deal with the case $n = r = 2$. A detailed account on Euclidean Ramsey theorems can be found in [1, 2]; for a further result, see [3].

Erdös *et al.* asked in [2, p. 535] whether the following proposition is valid: if the points of E^2 are 2-colored (red and blue) so that no two blue points are at distance 1 apart then there will be four red points forming a unit square (for E^3, the analogous statement is obvious). In this paper we show, that the answer is affirmative even if we say "arbitrary four-point configuration" instead of "unit square" (Theorem 1). A supplementary result (Theorem 2) shows that Theorem 1 becomes certainly false if we replace "four" by "twelve" in it. The interval $4 < n < 12$ is no man's land at present.

Further on, by a coloring we shall mean always a 2-coloring of E^2 in red and blue.

First we introduce some notions. Given an arbitrary coloring, a circle will be called *t-alternating*, if any two points on it having distance t are of different color (we exclude the degenerate case when t exceeds the diameter of the circle). We say that the circle $\gamma_P(r)$ of center P and radius r and circle $\gamma_Q(r)$ form a *complementary pair*, if for any red point on $\gamma_P(r)$ (or on $\gamma_Q(r)$), the corresponding point on the other circle is blue ($X \in \gamma_P(r)$ and $X' \in \gamma_Q(r)$ are corresponding points if XX' is a translate of PQ).

Concerning these notions we prove two simple lemmas which will be very useful in the sequel.

LEMMA 1. *Given any coloring without blue points of distance t, both members of a complementary pair of radius r ($r \geqslant t/2$) are t-alternating.*

Proof. Let $\gamma_P(r)$ and $\gamma_Q(r)$ be a complementary pair. If for $X, Y \in \gamma_P(r)$

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$d(X, Y)$ (the distance of X and Y) equals t, then X and Y cannot both be blue at the same time. Moreover, they may not be red simultaneously, since then X' and Y' would be blue.

LEMMA 2. *Given any coloring without blue points of distance t, if the circle $\gamma_0(r)$ is t-alternating, then the circle $\gamma_0(r_1)$ ($r_1 = (\sqrt{4r^2 - t^2} + t\sqrt{3})/2$) consists of red points only.*

Proof. Any point on $\gamma_0(r_1)$ is a vertex of a suitable regular triangle whose opposite side is a chord of $\gamma_0(r)$ of length t.

Before stating the next lemma let us agree to call a four-point configuration determining a rhombus with angle 60° and side t shortly a *regular t-rhombus.*

LEMMA 3. *Given any coloring without blue points of distance t there exists a red regular t-rhombus.*

Proof. First we suppose that there exist two blue points, say A and B, with $d(A, B) = t\sqrt{3}$. Then the circles $\gamma_A(t)$ and $\gamma_B(t)$ are red. Let C and D denote their common points. A, B, C and D form a regular t-rhombus. Consider all translates $\{A', B', C', D'\}$ of this rhombus, such that $A' \in \gamma_A(t)$. We have also $B' \in \gamma_B(t)$, $C' \in \gamma_C(t)$ and $D' \in \gamma_D(t)$. A' and B' are red every time. If C' and D' are both red then we have the red configuration as desired. In the contrary case if C' is red then D' is blue. Thus, by Lemma 1, both $\gamma_C(t)$ and $\gamma_D(t)$ are t-alternating. Hence it follows that the diametrally opposite points of $\gamma_C(t)$ (as well as $\gamma_D(t)$) are of different color; furthermore, by Lemma 2, the circles $\gamma_C(t\sqrt{3})$ and $\gamma_D(t\sqrt{3})$ are red.

Now let us consider the point lattice generated by A, B, C, D (Fig. 1). Suppose that our lemma is false. Since A and B are blue, E, F, H, K, O have to be red, hence G and L are blue. As L is blue, N must be red; as O, K, N are red, P must be blue. The points A and G have distance $t\sqrt{3}$, so the circle $\gamma_E(t\sqrt{3})$ is red; similarly L and P are blue and $d(L, P) = t\sqrt{3}$, whence

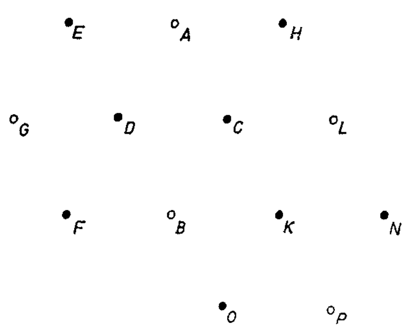

FIGURE 1

$\gamma_N(t\sqrt{3})$ is red, too. As E and N are centrally symmetric with respect to C, one of the points of $\gamma_N(t\sqrt{3}) \cap \gamma_C(t)$ and one of $\gamma_E(t\sqrt{3}) \cap \gamma_C(t)$ are diametrally opposite on $\gamma_C(t)$ and they are red, which contradicts our assertion above. Thus, in the case when there exist blue points of distance $t\sqrt{3}$, the assertion of the lemma is proved.

In the other case, let O be blue. Then $\gamma_0(t)$ and $\gamma_0(t\sqrt{3})$ are red. If the circle $\gamma_0(2t)$ has a red point P, then P, P_1, P_2, and P_3 ($\{P_1, P_2\} = \gamma_0(t\sqrt{3}) \cap \gamma_P(t)$, $\{P_3\} = \gamma_0(t) \cap \gamma_P(t)$) form a red regular t-rhombus. Otherwise, all points of $\gamma_0(2t)$ are blue and so we have blue points of distance t, again a contradiction.

LEMMA 4. *Let $\{A, B, C, D\}$ be a configuration having two points with distance a. Given any coloring without blue points of distance* 1, *if there exists a red configuration $\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$ such that $\{P_1, P_2, P_3\}$ is a regular triangle with unit side and $\{Q_1, Q_2, Q_3\}$ arises from $\{P_1, P_2, P_3\}$ by a translation by distance a, then we can find a red configuration congruent to $\{A, B, C, D\}$.*

Proof. Let $\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$ a configuration which fulfils the condition of lemma, and let $\{P_1, Q_1, R_1, S_1\}$ be congruent to $\{A, B, C, D\}$. Denote by S_i and R_i the images of S_1 and R_1, respectively, under the translation moving P_1 into P_i ($i = 2, 3$). Then $\{R_1, R_2, R_3\}$ as well as $\{S_1, S_2, S_3\}$ are regular triangles with unit side. Thus at least two points from R_1, R_2, R_3 —say R_1 and R_2—are red. At the same time, at least one point from S_1, S_2 —say S_2—is red. Then $\{P_2, Q_2, R_2, S_2\}$ is a red configuration which is congruent to $\{A, B, C, D\}$.

Now we are ready to prove the promised result.

THEOREM 1. *Let $\{A, B, C, D\}$ be an arbitrary configuration. Given any coloring without two blue points at distance* 1, *there exists a red configuration which is congruent to $\{A, B, C, D\}$.*

Proof. First we observe that for any coloring and any $\{A, B, C, D\}$ at least one of the following three assertions is valid:

(1) A, B, C, and D are the four vertices of a parallelogram with side length a and b, and there exist no blue points at distance a or b.

(2) No distance of blue points is equal to the distance of two points from A, B, C, and D.

(3) Among the points A, B, C, D there exist two points, say A and B, such that AB and CD do not bisect each other (i.e., if A, B, C, D are vertices of a parallelogram, then AB is a side—and not a diagonal—of it), and there exist blue points whose distance equals to $d(A, B)$.

We shall prove the statement of the theorem for these three cases separately.

(1) In this case Lemma 3 guarantees the existence of a red regular a-rhombus $\{P, Q, R, S\}$ (Fig. 2). Let $\{P, Q, Y, X\}$ be congruent to $\{A, B, C, D\}$. If X and Y are red, we are finished. In the contrary case one of them is blue. If Y is blue, then let us consider the translation moving P into S. It also moves

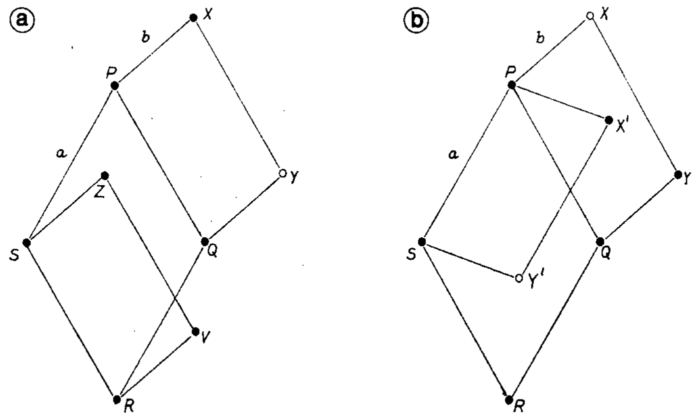

FIGURE 2

Q into R, X into Z, and Y into V (Fig. 2a). Taking into account $d(Y, Z) = d(Y, V) = a$, we have Z and V red, i.e., $\{S, R, V, Z\}$ is a red configuration congruent to $\{A, B, C, D\}$. Now suppose that X is blue. First consider the clockwise 60° rotation around P (Fig. 2b), which rotates X into X'. As $d(X, X') = b$ and X is blue, X' must be red. If the image Y' of Y under the considered rotation is red then the proof is complete. If Y' is blue then we are back at the first case.

(2) In this case let $d(A, B) = a$. By Lemma 3, there is a red regular a-rhombus $\{P, Q, R, S\}$ (Fig. 3). Let $\{P, Q, Y, X\}$ be congruent to $\{A, B, C, D\}$. If X and Y are red, we are finished. In the contrary case, one of them is red, and the other is blue. Then consider the counter-clockwise 60° rotation around Q. It moves X into X', and Y into Y'. If X' and Y' are red, the proof is complete. If not, then one of them is red, and the other is blue, moreover, X and X' as well as Y and Y' are of distinct colors. Now take the clockwise 60° rotation around S. It moves X' into X^*, and Y' into Y^*. If at least one from X^* and Y^* is not red, then as before we get that X' and X^* as well as Y' and Y^* are of distinct colors. Since one from X and Y, say Y, is blue, Y^* is blue too. As the product of the two above rotations equals to a translation by distance a, we see that $d(Y, Y^*) = a$ and they are blue, contrary to the assumption (2).

(3) Let P_0 and Q_0 blue points such that $d(P_0, Q_0) = d(A, B)$. Further-

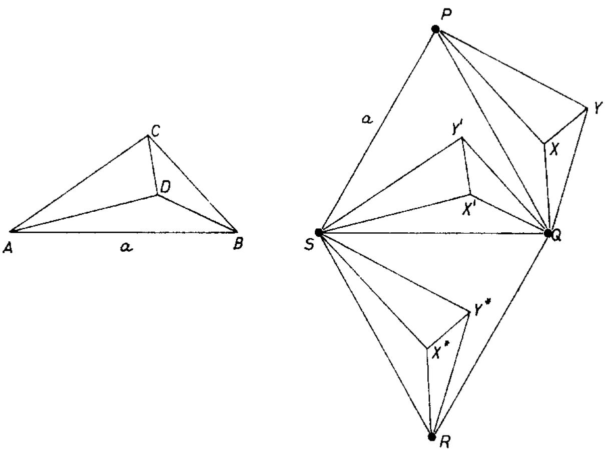

FIGURE 3

more, let $\{P_0, Q_0, R_0, S_0\}$ be congruent to $\{A, B, C, D\}$ (Fig. 4). Clearly, the circles $\gamma_{P_0}(1)$ and $\gamma_{Q_0}(1)$ are red. If one of the points R_0, S_0—say R_0—is blue, then $\gamma_{R_0}(1)$ is red, too. When we move the configuration $\{P_0, Q_0, R_0, S_0\}$ so that P_0, Q_0, and R_0 run on the circles $\gamma_{P_0}(1)$, $\gamma_{Q_0}(1)$, and $\gamma_{R_0}(1)$, respectively, then S_0 is moving on $\gamma_{S_0}(1)$. If in the course of this motion we never get a red configuration, then all points of $\gamma_{S_0}(1)$ are blue, whence the existence of blue points at distance 1 follows, a contradiction. Thus we have to consider only that case when R_0 and S_0 are red. If in the course of the above motion there is no situation in which the moved con-

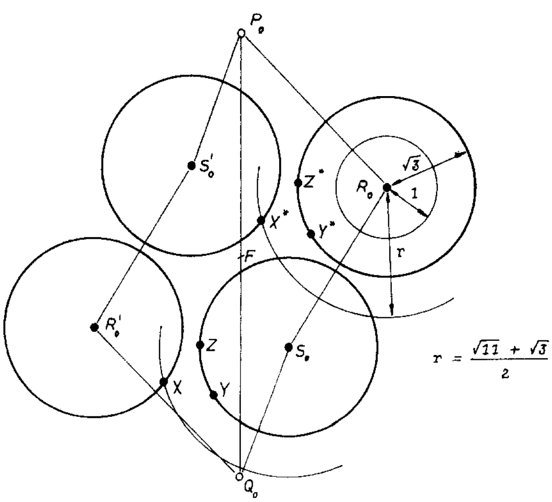

FIGURE 4

figuration is red (the images of P_0 and Q_0 are red every time!) then, by Lemma 1, $\gamma_{S_0'}(1)$ and $\gamma_{R_0}(1)$ are 1-alternating, whence, using Lemma 2, it turns out that $\gamma_{S_0}(\sqrt{3})$ and, $\gamma_{R_0}(\sqrt{3})$ are red. Analogous considerations are valid for the image of $\{P_0, Q_0, R_0, S_0\}$ under reflection in the midpoint F of P_0Q_0, and yield that $\gamma_{S_0'}(1)$ and $\gamma_{R_0'}(1)$ are 1-alternating, while $\gamma_{S_0'}(\sqrt{3})$ and $\gamma_{R_0'}(\sqrt{3})$ are red.

Consider those regular triangles with side 1 whose bases are chords of $\gamma_{S_0}(\sqrt{3})$ and whose third vertices are outside this circle. These third vertices form another circle $\gamma_{S_0}((\sqrt{11} + \sqrt{3})/2)$. Analogously, take the circle $\gamma_{R_0}((\sqrt{11} + \sqrt{3})/2)$. It is easy to see that for

$$X \in \gamma_{S_0}\left(\frac{\sqrt{11} + \sqrt{3}}{2}\right) \cap \gamma_{R_0'}(\sqrt{3}), \qquad \{Y, Z\} = \gamma_X(1) \cap \gamma_{S_0}(\sqrt{3})$$

and

$$X^* \in \gamma_{R_0}\left(\frac{\sqrt{11} + \sqrt{3}}{2}\right) \cap \gamma_{S_0'}(\sqrt{3}), \quad \{Y^*, Z^*\} = \gamma_{X^*}(1) \cap \gamma_{R_0}(\sqrt{3}),$$

all points X, Y, Z, X^*, Y^*, Z^* are red, and the regular triangle $\{X^*, Y^*, Z^*\}$ of side 1 is a translate of $\{X, Y, Z\}$ under that translation which moves S_0 into R_0. As $d(S_0, R_0) = d(C, D)$, Lemma 4 guarantees the existence of a red configuration congruent to $\{A, B, C, D\}$.

Of course, it may happen that $\gamma_{S_0}((\sqrt{11} + \sqrt{3})/2)$ and $\gamma_{R_0'}(\sqrt{3})$ do not intersect. This is the case if

$$d(S_0, R_0') > \frac{\sqrt{11} + 3\sqrt{3}}{2}.$$

Then we can proceed as follows.

For any circle $\gamma_0(r)$ ($r \geqslant \frac{1}{2}$), any point of $\gamma_0((\sqrt{4r^2 - 1} + \sqrt{3})/2)$ is the third vertex of a suitable regular triangle of unit side, whose base is a chord of $\gamma_0(r)$. This remark suggests the definition of the sequence r_i as follows: let

$$r_1 = 1 \qquad \text{and} \qquad r_n = \frac{\sqrt{4r_{n-1}^2 - 1} + \sqrt{3}}{2}.$$

Observe that if there exist no two red regular triangles with unit side such that one is translated into the other by the translation moving S_0 into R_0, and for n, the circles $\gamma_{R_0}(r_n)$ and $\gamma_{S_0}(r_n)$ are red, then $\gamma_{R_0}(r_{n+1})$ and $\gamma_{S_0}(r_{n+1})$ are 1-alternating, implying that $\gamma_{R_0}(r_{n+2})$ and $\gamma_{S_0}(r_{n+2})$ are red. Considering that $r_2 = \sqrt{3}$ and $\gamma_{R_0}(\sqrt{3})$, $\gamma_{S_0}(\sqrt{3})$ are red, we get that $\gamma_{R_0}(r_{2k})$ and $\gamma_{S_0}(r_{2k})$ ($k = 1, 2, \ldots$) are always red. Note that the sequence r_{2k} diverges; furthermore, $r_{2k+1} - r_{2k-1} < \sqrt{3}$ for any natural number k. Thus, among the circles $\gamma_{S_0}(r_3)$, $\gamma_{S_0}(r_5)$, etc. we can find one, say $\gamma_{S_0}(r_{2k+1})$, such that it intersects the red circle $\gamma_{R_0'}(\sqrt{3})$. If $X \in \gamma_{S_0}(r_{2k+1}) \cap \gamma_{R_0'}(\sqrt{3})$, $\{Y, Z\} = Y_X(1) \cap \gamma_{S_0}(r_{2k})$ and $X^* \in \gamma_{R_0}(r_{2k+1}) \cap \gamma_{S_0'}(\sqrt{3})$ so that $d(X, X^*) = d(S_0, R_0)$, $\{Y^*, Z^*\} = \gamma_{X^*}(1) \cap \gamma_{R_0}(r_{2k})$, then all points X, Y, Z, X^*, Y^*, Z^* are red, and the regular triangle $\{X^*, Y^*, Z^*\}$ of unit side is a translate of $\{X, Y, Z\}$ under the translation moving S_0 into R_0.

The other case in which the circles $\gamma_{S_0}((\sqrt{11} + \sqrt{3})/2)$ and $\gamma_{R_0'}(\sqrt{3})$ do not intersect is when

$$d(S_0, R_0') < \frac{\sqrt{11} - \sqrt{3}}{2} \left(< \frac{\sqrt{3}}{2}\right).$$

Now let M be the midpoint of R_0S_0 and M' the midpoint of $R_0'S_0'$. Consider the translates of $\{P_0, R_0, S_0, Q_0\}$ by all vectors $kM'M$ with k integer (Fig. 5). If for a non-negative integer n the circles $\gamma_{P_n}(r_{2n+1})$ and $\gamma_{Q_n}(r_{2n+1})$ are red, then moving $\{P_n, R_n, S_n, Q_n\}$ so that P_n runs on $\gamma_{P_n}(r_{2n+1})$ and Q_n runs on $\gamma_{Q_n}(r_{2n+1})$ the circles $\gamma_{R_n}(r_{2n+1})$ and $\gamma_{S_n}(r_{2n+1})$ turn out to be 1-alternating whence, by Lemma 2, $\gamma_{R_n}(r_{2(n+1)})$ and $\gamma_{S_n}(r_{2(n+1)})$ are red. Now moving $\{Q_{n+1}, S_n, R_n, P_{n+1}\}$ so that R_n runs on $\gamma_{R_n}(r_{2(n+1)})$ and S_n runs on $\gamma_{S_n}(r_{2(n+1)})$ we can see that $\gamma_{P_{n+1}}(r_{2(n+1)})$ and $\gamma_{Q_{n+1}}(r_{2(n+1)})$ are 1-alternating whence $\gamma_{P_{n+1}}(r_{2(n+1)+1})$ and $\gamma_{Q_{n+1}}(r_{2(n+1)+1})$ are red. As $\gamma_{P_0}(1)$ and $\gamma_{Q_0}(1)$ are red, we have that $\gamma_{P_k}(r_{2k+1})$ and $\gamma_{Q_k}(r_{2k+1})$ must be red for any non-negative integer k. By the same reasoning, $\gamma_{P_{-k}}(r_{2k+1})$ and $\gamma_{Q_{-k}}(r_{2k+1})$ are red, too. But if k is big enough then $d(P_k, P_{-k}) \geqslant \sqrt{3}/2$, whence the existence of the desired pair of red regular triangles of unit side follows. Thus, we can apply Lemma 4. The proof is complete.

Consider now any coloring of the plane without blue points being at distance 1 apart. Encouraged by the preceding result we might hope that for any finite configuration there must be a red configuration congruent to it. However, in [2, pp. 534–535], a coloring and a configuration $\mathcal{K}$ consisting of 10^{12} points is given so that any configuration which is congruent to $\mathcal{K}$ contains necessarily a blue point. As the following theorem shows, the number of points in $\mathcal{K}$ may be reduced considerably.

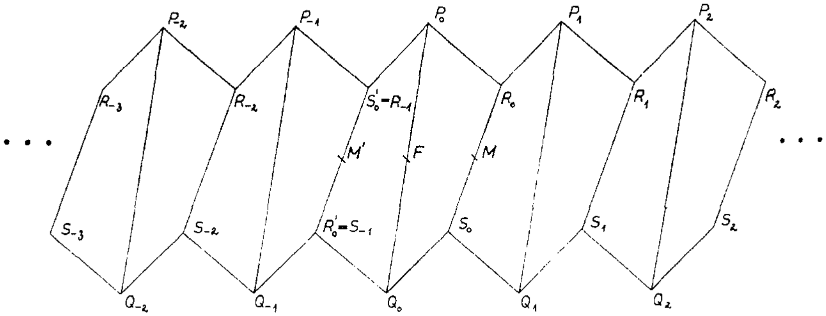

FIGURE 5

THEOREM 2. *There exist a coloring of the plane without blue points at distance* 1 *and a configuration* $\mathscr{K}$ *consisting of* 12 *points such that any configuration which is congruent to* $\mathscr{K}$ *contains necessarily a blue point.*

Proof. Consider in the plane a point lattice whose base parallelogram is a regular 2-rhombus. Let every lattice point be the center of a blue open disc of radius 1/2. For every disc the boundary points under the horizontal diameter as well as the left endpoint of this diameter are blue, too. The remaining points are red (Fig. 6a). Obviously, under this coloring no blue points are at distance 1 apart.

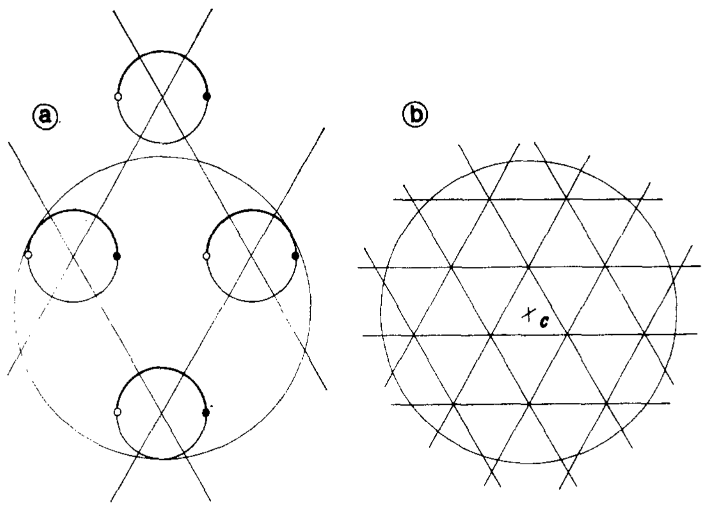

FIGURE 6

Now observe that on the plane colored in this way any closed disc of radius $2\sqrt{3}/3 + \frac{1}{2}$ must contain at least one of the blue discs (together with boundary points). On the other hand, for any point lattice generated by a regular $\sqrt{3}/2$-rhombus each blue disc (considered together with its blue boundary points) contains at least one lattice point. Let C be the center of a base triangle of a such lattice $\mathscr{L}$ and take the circle $\gamma_C(2\sqrt{3}/3 + \frac{1}{2})$. Inside this circle there are exactly 12 lattice points from $\mathscr{L}$. The configuration $\mathscr{K}$ consisting of these 12 points (Fig. 6b) fulfils the requirement of the theorem as it can be seen easily from the preceding observations.

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