SIAM J. DISCRETE MATH.
Vol. 25, No. 1, pp. 394–400

MORE CONSTRUCTIVE LOWER BOUNDS ON CLASSICAL RAMSEY NUMBERS*

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Abstract. We present several new constructive lower bounds for classical Ramsey numbers. In particular, the inequality $R(k,s+1) \geq R(k,s)+2k-2$ is proved for $k \geq 5$. The general construction permits us to prove that, for all integers k, l, with $k \geq 5$ and $l \geq 3$, the connectivity of any Ramsey-critical (k,l)-graph is at least k, and if $k \geq l-1 \geq 1$, $k \geq 3$ and $(k,l) \neq (3,2)$, then such graphs are Hamiltonian. New concrete lower bounds for Ramsey numbers are obtained, some with the help of computer algorithms, including: $R(5,17) \geq 388$, $R(5,19) \geq 411$, $R(5,20) \geq 424$, $R(6,8) \geq 132$, $R(6,12) \geq 263$, $R(7,8) \geq 217$, $R(7,9) \geq 241$, $R(7,12) \geq 417$, $R(8,17) \geq 961$, $R(9,10) \geq 581$, $R(12,12) \geq 1639$, and also one three-color case $R(8,8,8) \geq 6079$.

Key words. Ramsey numbers, connectivity

AMS subject classification. 05C55

DOI. 10.1137/10080868X

1. Introduction.

For positive integers $k_1,\ldots,k_m$, $m \geq 2$, the *classical Ramsey number* $R(k_1,\ldots,k_m)$ is the smallest integer n meeting the following condition: in any edge coloring of K_n with m colors, for some $i \in \{1,2,\ldots,m\}$, there exists a subgraph K_{k_i} of K_n whose all edges are colored with color i. Although the existence of Ramsey numbers $R(k_1,\ldots,k_m)$ was proved long ago [7], it is still notoriously difficult to find their exact values in nontrivial cases, or even just to obtain good bounds.

The sets of vertices and edges of a graph G and their cardinalities will be denoted by VG, EG, $n(G)$, and $e(G)$, respectively. For $X \subset VG$, we will denote by $G[X]$ the subgraph induced in G by X. Any partition (coloring) of the edges of K_n into m classes avoiding K_{k_i} in color i will be called a $(k_1,\ldots,k_m;n)$-coloring. For two colors, $m=2$, edge colorings can be considered as graphs where the second color corresponds to nonedges. We will refer to (k,l)-graph as a graph without K_k and without independent sets of order l. Hence, the construction of any (k,l)-graph on n vertices, i.e., a $(k,l;n)$-coloring, will prove that $n < R(k,l)$. Any $(k,l;R(k,l)-1)$-graph will be called *Ramsey-critical* (k,l)*-graph.*

Known lower and upper bounds for various types of Ramsey numbers are gathered in the dynamic survey [6] by the third author. Many constructive lower bounds for classical cases are presented in [11] and [10], and some of them are enhanced in this paper.

After reviewing previous work in section 2, the difference between similar Ramsey numbers is studied in section 3, where some constructive lower bounds generalizing those in [11] and [10] are presented. In particular, we show that $R(k,s+1) \geq R(k,s)+$

*Received by the editors September 15, 2010; accepted for publication (in revised form) January 18, 2011; published electronically March 24, 2011.
http://www.siam.org/journals/sidma/25-1/80868.html

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$2k-2$ for $k \geq 5$. Next, several new lower bounds for some concrete small cases are obtained directly from the general construction and others with the help of computer algorithms. The new bound $R(8,8,8) \geq 6079$ is derived in section 4. Finally, in section 5 we prove that, for all integers k, l, with $k \geq 5$ and $l \geq 3$, the connectivity of any (k,l)-graph is no less than k, and that if $k \geq l-1 \geq 1$, $k \geq 3$ and $(k,l) \neq (3,2)$, then such graphs are Hamiltonian.

2. Previous work.

The following theorem and corollary were proved by constructive methods as parts of Theorems 2 and 3 in [11].

THEOREM 1 (see [11]). *Given a (k,s)-graph G and a (k,t)-graph H, for some $k \geq 3$ and $s,t \geq 2$, if both G and H contain an induced subgraph isomorphic to some K_{k-1}-free graph M, then*

$$R(k, s+t-1) \geq n(G) + n(H) + n(M) + 1.$$

COROLLARY 1 (see [11]). *If $2 \leq s \leq t$ and $k \geq 3$, then*

$$R(k, s+t-1) \geq R(k,s) + R(k,t) + \begin{cases} k-3, & if\ s = 2; \\ k-2, & if\ s \geq 3. \end{cases}$$

The first inequality of Corollary 1 for $s = 2$, $R(k,t+1) \geq R(k,t) + 2k - 3$ was proved by Burr et al. in 1989 [2]. Here, we will improve it for $k \geq 5$ in Corollary 3 in the next section. The following Theorem 2 and its generalization in Corollary 2 were proved in [10].

THEOREM 2 (see [10]). *If $k \geq 2, s \geq 5$, then $R(2k-1, s) \geq 4R(k, s-1) - 3$.*

COROLLARY 2 (see [10]). *For $k_1 \geq 5$ and $k_i \geq 2, i \geq 2$, we have*

$$R(k_1, 2k_2 - 1, k_3, \ldots, k_r) \geq 4R(k_1 - 1, k_3, \ldots, k_r) - 3.$$

In 1980, Paul Erdős wrote in [4, p. 11] (using r for our R): "Faudree, Schelp, Rousseau and I needed recently a lemma stating

$$\text{(a)} \qquad \lim_{n\to\infty} \frac{r(n+1,n) - r(n,n)}{n} = \infty$$

We could prove (a) without much difficulty, but could not prove that $r(n+1,n) - r(n,n)$ increases faster than any polynomial of n. We of course expect

$$\text{(b)} \qquad \lim_{n\to\infty} \frac{r(n+1,n)}{r(n,n)} = C^{\frac{1}{2}},$$

where $C = \lim_{n\to\infty} r(n,n)^{1/n}$." Based on work reported in this paper, the best known lower bound estimate for the difference in (a) seems to be barely $\Omega(n)$. Between 2007 and 2009 we asked several mathematicians about this, including coworkers of Erdős mentioned in the article. Nobody could recall the proof, nor even its existence. The conclusion we are inclined to draw is that there exists no known proof of this result, though possibly it was known to Erdős. It is thus prudent to consider (a) at the moment to be only a conjecture.

3. Building up lower bounds.

First we present a new constructive result in Theorem 3, which while similar to Theorem 1, gives wider applicability. In what follows, we will also show that in several cases it leads to better lower bounds for some classical two-color Ramsey numbers.

THEOREM 3. *Let $k \geq 4$ and $s, t \geq 2$. Given a (k, s)-graph G and a (k, t)-graph H, let M be a graph isomorphic to induced subgraphs of G and H. If the vertex set VM can be partitioned into two nonempty sets, $VM = W_1 \cup W_2$, so that for every $v \in VM - W_i$ and $i \in \{1, 2\}$ there is no K_{k-1} in $M[W_i \cup \{v\}]$, then*

$$R(k, s+t-1) \geq n(G) + n(H) + n(M) + 1.$$

Proof. Let $p = n(G), q = n(H), m = n(M)$, and $m_i = |W_i|$, $m_i > 0$ for $i \in \{1, 2\}$, so we have $m = m_1 + m_2$. First, we will construct a graph F on the vertex set $VG \cup VH \cup W_1 \cup W_2$ and then prove that it contains no K_k and no independent sets of order $s + t - 1$. Since F has the right number of vertices, this will complete the proof of the theorem.

Let $U = VG = \{u_1, \dots, u_p\}$, $V = VH = \{v_1, \dots, v_q\}$, and $W = VM = \{w_1, \dots, w_m\}$. Without loss of generality, we can assume that the graphs G, H, and M are labeled so that their parts induced on $U_1 = \{u_j \mid 1 \leq j \leq m_1\}$, $U_2 = \{u_j \mid m_1{+}1 \leq j \leq m_1 + m_2\}$, $V_1 = \{v_j \mid 1 \leq j \leq m_1\}$, $V_2 = \{v_j \mid m_1 + 1 \leq j \leq m_1 + m_2\}$, $W_1 = \{w_j \mid 1 \leq j \leq m_1\}$, and $W_2 = \{w_j \mid m_1 + 1 \leq j \leq m_1 + m_2\}$ are isomorphic under the mapping $u_j \to v_j \to w_j$.

The set of edges of F contains the edges of the graphs G, H, $M[W_1]$, and $M[W_2]$. The other edges in EF are defined as follows. For $i \in \{1, \dots, m\}$, $j \in \{1, \dots, p\}$, $i \neq j$, if $(u_i, u_j) \in EG$, then $(w_i, u_j) \in EF$, and for $i \in \{1, \dots, m\}$, $j \in \{1, \dots, q\}$, $i \neq j$, if $(v_i, v_j) \in EH$, then $(w_i, v_j) \in EF$. Finally, EF contains the edges of a matching $(u_i, v_i)_{i=1,\dots,m}$ between $U_1 \cup U_2$ and $V_1 \cup V_2$. Note that EF doesn't contain any edges between W_1 and W_2.

Suppose that there is a complete subgraph K_k in F induced by a set $S \subset VF$. Consider the parts of S and its sizes, $S_i \subset S$, $s_i = |S_i|$ for $1 \leq i \leq 4$, as follows:

$$S_1 = S \cap VG, \quad S_2 = S \cap VH,$$
$$S_3 = S \cap W_1, \quad S_4 = S \cap W_2.$$

Case 1 ($s_1, s_2 > 0$). Observe that we must have $s_1 = s_2 = 1$, since the only edges in EF between VG and VH form a partial matching. Hence, we have that $u_j \in S_1$ and $v_j \in S_2$ for some special $1 \leq j \leq m$. Further, since $s_3 s_4 = 0$, we can assume that $s_3 = k - 2$, $S_4 = \emptyset$. This in turn would imply that the induced graph $M[S_3 \cup \{w_j\}]$ contains K_{k-1}, contrary to the assumptions of the theorem.

Case 2 ($s_1 s_2 = 0$). Without loss of generality, assume that $s_2 = s_4 = 0$. From the construction we can easily see that the induced subgraph $G[S_1 \cup \{u_j \in VG \mid w_j \in W_1\}]$ contains K_k, which is a contradiction.

It remains to be shown that F has no independent sets of order $s + t - 1$. Here, we will use a method very similar to the proof of Theorem 2 (Theorem 9 in [10]). Consider any independent set I in F, and its parts I_i, as follows:

$$I_1 = I \cap VG, \quad I_2 = I \cap VH,$$
$$I_3 = I \cap W_1, \quad I_4 = I \cap W_2,$$
$$I_5 = \{u_j \in I_1 \mid w_j \in I_3\} \cup \{v_j \in I_2 \mid w_j \in I_3\},$$
$$I_6 = \{u_j \in I_1 \mid w_j \in I_4\} \cup \{v_j \in I_2 \mid w_j \in I_4\},$$
$$I_7 = I_1 \setminus (I_5 \cup I_6), \quad I_8 = I_2 \setminus (I_5 \cup I_6).$$

Directly from the construction it can be concluded that

$$|I_3 \cup I_6 \cup I_7| \leq s - 1 \text{ and } |I_4 \cup I_5 \cup I_8| \leq t - 1$$

by considering isomorphic embedding of all parts I_j into the vertex set VG and VH, respectively. Since clearly $I = \cup_{i=1}^4 I_i = \cup_{i=3}^8 I_i$, we obtain $|I| \le s+t-2$. This finishes the proof of the theorem. □

For nontrivial choices of the graphs G, H, and M, the construction will produce triangles, so we need $k \ge 4$. Even in the case of $k = 4$, when M is K_3-free, one should use the simpler construction of Theorem 1. In the computational results presented in what follows we have also used a slightly weaker version of Theorem 3. Namely, it is sufficient to find any partition of the vertex set VM into $W_1 \cup W_2$ so that neither of $M[W_1]$ or $M[W_2]$ contains a K_{k-2}. Then the assumptions of Theorem 3 hold, and thus the same construction works. Another observation related to this theorem concerns a possible special case, namely, $H = M = K_{k-1}$, and the corresponding partition of VM into K_2 and K_{k-3}. If we use $t = 2$ and any critical (k, s)-graph as G, then, for $k \ge 5$, we obtain a $(k, s+1)$-graph F (note that for $k = 4$ this special case of the partition doesn't satisfy the assumptions). This implies $R(k, s+1) \ge R(k, s) + 2k - 2$, which improves by one Corollary 1 for $s = 2$, as in the second part of the next corollary.

COROLLARY 3. *If* $k \ge 5$ *and* $2 \le s, t$, *then* $R(k, s+t-1) \ge R(k, s)+R(k, t)+k-2$. *In particular, we have* $R(k, s+1) \ge R(k, s) + 2k - 2$.

We are ready to derive some new lower bounds for two-color Ramsey numbers, as in the following theorem. The new bounds improve on those given in [6].

THEOREM 4. $R(6, 12) \ge 263$, $R(7, 8) \ge 217$, $R(7, 12) \ge 417$, $R(9, 10) \ge 581$, *and* $R(12, 12) \ge 1639$.

Proof. Using Corollary 3 with the best known lower bounds for $R(k, s)$ recorded in [6], we obtain

$$\begin{array}{rcll} R(6,12) & \ge & R(6,11) + 2 \times 6 - 2 & \ge 263, \\ R(7,8) & \ge & R(7,7) + 2 \times 7 - 2 & \ge 217, \\ R(7,12) & \ge & R(7,11) + 2 \times 7 - 2 & \ge 417, \\ R(9,10) & \ge & R(9,9) + 2 \times 9 - 2 & \ge 581, \\ R(11,12) & \ge & R(11,11) + 2 \times 11 - 2 & \ge 1617, \\ R(12,12) & \ge & R(12,11) + 2 \times 12 - 2 & \ge 1639. \end{array}$$ □

The next improvements of the lower bounds required help from some computer algorithms. The results are collected in Theorem 5, which improves the bounds from [8] or those listed in [6].

THEOREM 5. $R(5, 17) \ge 388$, $R(5, 19) \ge 411$, $R(5, 20) \ge 424$, $R(6, 8) \ge 132$, $R(7, 9) \ge 241$, *and* $R(8, 17) \ge 961$.

Proof. Let us define the graph G_{387} by taking as its edges just one color in the tripling construction with $p = 127$, which builds up on the well known Mathon's construction [5], as described in [9]. With some computations, it can be checked that G_{387} is a $(5, 17; 387)$-graph establishing the bound $R(5, 17) \ge 388$, which improves over the previously listed value of 385 [6]. In the following, we will use the same graph in two other constructions.

Each of the rows in Table 1 presents sizes of the parameters used in an application of Theorem 3, for which a proper partition of the set VM was found. The graph G_{387} was used as G in the third and fourth row. The other cases of G and all graphs H were taken from the standard set of known largest Ramsey graphs for the corresponding parameters (cf. [6, 11]). The common subgraph M and the partition $VM = W_1 \cup W_2$ were found in each case with the help of computer heuristics (larger common parts possibly could be found). Four lower bounds in Theorem 5 are one larger than the entries in the last column of Table 1. Finally, we obtain $R(8, 17) \ge 961$ from $R(7, 9) \ge 241$ by applying Theorem 2. □

TABLE 1
Parameters of Theorem 3 used in Theorem 5.

k	s	t	$\|VG\|$	$\|VH\|$	$\|VM\|$	$\|VF\|$
6	6	3	101	17	13	131
7	7	3	204	22	14	240
5	17	3	387	13	10	410
5	17	4	387	24	12	423

4. New lower bound for $R(8,8,8)$. Theorem 6 below is a special case of a more general result in [10]. The *Paley graph* Q_p is defined for primes p of the form $4t+1$. The vertex set of Q_p is $\mathcal{Z}_p$, and the vertices x and y are joined by an edge if and only if $x-y$ is a square modulo p.

THEOREM 6 (see [10]). *For a prime p of the form $4t+1$, let α_p be the order of the largest clique in the Paley graph Q_p. Then*

$$R(3,\alpha_p+2,\alpha_p+2)\geq 6p+3.$$

Following the construction used in the proof of Theorem 2 in [10], one can easily conclude the following lemma by using Corollary 2.

LEMMA 1. *Suppose $k\geq 2$, $s,t\geq 4$. If G is a $(k,s,t;n)$-coloring, in which the subgraph induced by the edges in color 1 is d-regular, then there exists a $(2k-1,s,t+1;4n)$-coloring, in which the subgraph induced by the edges in color 1 is $(n+3d+1)$-regular.*

The new lower bound on $R(8,8,8)$ will be obtained using Theorem 6 and Lemma 1.

THEOREM 7. $R(8,8,8)\geq 6079$.

Proof. Using Theorem 6 with $p=101$ we can obtain a $(3,7,7;608)$-coloring; furthermore, by following the construction of the proof in [10], we can see that it is regular of degree 202 in color 1. Next, by Lemma 1 we can build a $(5,7,8;2432)$-coloring, which is regular of degree $1215=608+3\times 202+1$. Using Lemma 1 once more, we make a $(9,8,8;9728)$-coloring, which is regular of degree 6078 in color 1. Now the neighborhood in color 1 of any vertex in the last coloring induces a $(8,8,8;6078)$-coloring, which completes the proof. ∎

5. Connectivity of Ramsey graphs. Beveridge and Pikhurko in [1], using Theorem 1, proved that, for any $k,l\geq 3$, the connectivity of any Ramsey-critical (k,l)-graph is no less than $k-1$. In Theorem 8, for $k\geq 5$, we increase this bound on connectivity to k. Similarly, the following Corollary 4 improves over a result in [1] on Ramsey-critical graphs which are Hamiltonian. Our results directly depend on Corollary 3, in a way that they could be further strengthened if the lower bound of Corollary 3 is improved.

THEOREM 8. *If $k\geq 5$ and $l\geq 3$, then the connectivity of any Ramsey-critical (k,l)-graph is no less than k.*

Proof. Suppose some graph F is a Ramsey-critical (k,l)-graph, and its connectivity κ is less than k for some $k\geq 5$ and $l\geq 3$. Thus, by the result of Beveridge and Pikhurko, $\kappa=k-1$. Let C be a cut-set of VF, $|C|=k-1$ such that $G[VF\setminus C]$ is disconnected. Let $V_1\cup V_2\cup C$ be the corresponding partition of VF. Consider the graphs $G_1=F[V_1]$, $n_1=|V_1|$ and $G_2=F[V_2]$, $n_2=|V_2|$, and let $s-1$ and $t-1$ be their independence numbers, respectively. Clearly, we have that G_1 is a $(k,s;n_1)$-graph, G_2 is a $(k,t;n_2)$-graph, $n_1+n_2+(k-1)=|VF|=R(k,l)-1$, and $(s-1)+(t-1)<l$.

This implies

$$R(k,s) + R(k,t) - 2 \geq n_1 + n_2 = R(k,l) - k.$$

Thus by Corollary 3 and monotonicity of Ramsey numbers we have $s + t - 1 = l$. Furthermore, now it also easily follows that $n_1 = R(k,s) - 1$ and $n_2 = R(k,t) - 1$; i.e., both G_1 and G_2 are Ramsey-critical for (k,s) and (k,t), respectively.

Observe that for any vertex $v \in C$, the induced subgraph $F[V_1 \cup \{v\}]$ must have an independent set of s vertices containing v, since G_1 is Ramsey-critical for (k,s). Similarly, $F[V_2 \cup \{v\}]$ has an independent set of t vertices containing the same v. Thus, since C is a cut-set, we have found an independent set of $s + t - 1 = l$ vertices in the graph F, which contradicts the assumption that F is a (k,l)-graph. Therefore, $\kappa \geq k$. □

COROLLARY 4. *If $k \geq l - 1 \geq 1$ and $k \geq 3$, except $(k,l) = (3,2)$, then any Ramsey-critical (k,l)-graph is Hamiltonian.*

Proof. For $l = 2$ the only Ramsey-critical (k,l)-graphs are complete K_{k-1}, which are Hamiltonian for $k \geq 4$. For $k \geq 5$ and $l \geq 3$, by Theorem 8, any Ramsey-critical (k,l)-graph F has connectivity $\kappa \geq k$. Theorem 1 in [3], by Chvátal and Erdős, states that any k-connected graph on at least 3 vertices, without independent sets of order $k+1$, is Hamiltonian. Thus, for $k \geq l - 1$, a direct application of the latter implies that F has a Hamiltonian circuit.

The remaining cases of (k,l) are $(3,4), (3,3), (4,3), (4,4)$, and $(4,5)$ for the Ramsey-critical graphs of orders 8, 5, 8, 17, and 24, respectively. All (3,4;8)-, (3,3;5)-, (4,3;8)-, and (4,4;17)-graphs are known (cf. [6]). There are 3, 1, 3, and 1 of them, respectively, and they can be easily checked to be Hamiltonian. Let F be any $(4,5;24)$-graph. We will prove that it has connectivity $\kappa \geq 5$ (a bound of 4 would suffice), so again Theorem 1 in [3] will imply that F is Hamiltonian. Suppose that the graph F has a cut-set C of 4 vertices disconnecting F, and let $V_1 \cup V_2 \cup C$ be the corresponding partition of the vertex set VF (using the same notation as in the proof of Theorem 8). Since $R(4,2) = 4$, $R(4,3) = 9$, and $R(4,4) = 18$, we can easily see that the only possible parameters of this partition are $n_1 = 3$, $s = 2$, $n_2 = 17$, $t = 4$, and $G_1 = K_3$. $R(4,4) = 18$ implies that F has minimum degree at least 6, and thus each vertex in V_1 must be connected to all 4 vertices in C. Hence each vertex in C forms a K_4 together with V_1, which is a contradiction. □

In particular, for $k \geq 3$, all diagonal Ramsey-critical (k,k)-graphs are Hamiltonian. It is an interesting open question for which $k < l-1$ Ramsey-critical (k,l)-graphs remain Hamiltonian. We expect it to be true at least when k is sufficiently close to l.

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