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Note

A Note on Constructive Lower Bounds for the Ramsey Numbers $R(3, t)$

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We present a simple explicit construction, in terms of t, of a graph that is triangle-free, has independence number t, and contains more than $\frac{5}{6}((t-1)/2)^{\log 6/\log 4} \in \Omega(t^{1.29})$ vertices. This result is a (feasibly) constructive proof that the Ramsey number $R(3, t) \in \Omega(t^{1.29})$. This improves the best previous constructive lower bound of $R(3, t) > t^{(2\log 2)/3(\log 3 - \log 2)} \in \Omega(t^{1.13})$, due to P. Erdős (1966, *J. Combin. Theory* 17, 149–153). Also, our result yields a simple explicit construction, in terms of k, of a triangle-free k-chromatic graph whose size is $O(k^{\log 6/(\log 6 - \log 4)}) \subset O(k^{4.42})$. © 1993 Academic Press, Inc.

The Ramsey number $R(s, t)$ is the smallest integer for which every graph on $R(s, t)$ vertices contains either a clique of size s or an independent set of size t. Ramsey [12] shows that, for all s and t, $R(s, t)$ is well defined. The determination of $R(s, t)$ has proven to be extremely difficult for all but a few values of s and t.

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Several upper and lower bounds on $R(s, t)$ for various values of s and t are known. Many of the lower bounds are proven by nonconstructive methods. These methods, introduced by Erdős [6], establish the existence of graphs of size $n(s, t)$ with clique size s and independence number t (which implies that $R(s, t) > n(s, t)$), but they yield no explicit constructions of these graphs. If a proof also contains an algorithm that, on input s and t, produces an instance of a graph of size $n(s, t)$ with the required properties, and the algorithm runs in time polynomial in $n(s, t)$ then the proof is *(feasibly) constructive*. Many constructive lower bounds are known, but they are considerably weaker than their nonconstructive counterparts.

The problem of finding good constructive lower bounds for the "diagonal" Ramsey numbers $R(t, t)$ has received considerable attention [1, 4, 9–11]. The most recent result, due to Frankl and Wilson [10], yields the strongest constructive lower bound to date,

$$R(t, t) \geqslant \exp\left(\frac{(1-\varepsilon)(\log t)^2}{4 \log \log t}\right),$$

for all $\varepsilon > 0$ and sufficiently large t. There is still a significant gap between this and the best *non*constructive lower bound, which is $R(t, t) \in 2^{\Omega(t)}$.

For the Ramsey numbers of the form $R(3, t)$, Erdős [8] constructively proves that

$$R(3, t) > t^{(2 \log 2)/3(\log 3 - \log 2)} \in \Omega(t^{1.13}),$$

whereas nonconstructive lower bounds of the form $R(3, t) \in \Omega((t/\ln t)^2)$ are known (originally shown by Erdős [7]; improved by a constant factor by Spencer [14]). Also, upper bounds are of the form $R(3, t) \in O(t^2/\log t)$ are known (originally shown by Ajtai, Komlós, and Szemerédi [2, 3]; improved by a constant factor by Shearer [13]).

In this note, we present a simple constructive proof that

$$R(3, t) > \frac{5}{6}\left(\frac{t-1}{2}\right)^{\log 6/\log 4} \in \Omega(t^{1.29}).$$

This is an improvement of the earlier version of these results in [5].

In addition to the connection with Ramsey theory, our result contributes to the problem of explicitly constructing small triangle-free graphs with large chromatic numbers. Our result yields immediately an explicit construction, in terms of k, of a triangle-free k-chromatic graph of size $O(k^{\log 6/(\log 6 - \log 4)}) \subset O(k^{4.42})$.

Our method is based on a construction that transforms a graph G to a graph consisting of six disjoint copies of G connected by additional edges in a particular way. This construction preserves the triangle-freeness property and increases the independence number of the graph by a factor of four, while increasing the size of the graph by a factor of six. By repeatedly applying the transformation, we obtain an explicit construction in terms of t of triangle-free graphs with independence number t and size $\Omega(t^{\log 6/\log 4})$, constructively proving that $R(3, t) \in \Omega(t^{\log 6/\log 4})$.

DEFINITION 1. For any graph $G = (V(G), E(G))$, the *fibration* of G is the graph $H = (V(H), E(H))$ defined below. Roughly speaking, H consists of six disjoint copies of G with extra edges that connect the vertices of different copies of G together. Formally, $V(H) = V(G) \times \{0, 1, 2, 3, 4, 5\}$, and $E(H)$ consists of precisely the following edges.

1. For all $i \in \{0, 1, 2, 3, 4, 5\}$, and for all $(u, v) \in E(G)$, $((u, i), (v, i)) \in E(H)$.
2. For all $i, j \in \{0, 1, 2, 3, 4, 5\}$ with $j \equiv i + 1 \pmod 6$, and for all $(u, v) \in E(G)$, $((u, i), (v, j)) \in E(H)$ and $((u, j), (v, i)) \in E(H)$.
3. For all $i, j \in \{0, 1, 2, 3, 4, 5\}$ with $j \equiv i + 3 \pmod 6$, and for all $u \in V(G)$, $((u, i), (u, j)) \in E(H)$.

For each $i \in \{0, 1, 2, 3, 4, 5\}$, define $V_i = \{(u, i) \in V(H)\}$. For any set $U \subset V(H)$, let $H[U]$ be the subgraph of H induced by U. We define the projection π: $V(H) \to V(G)$ as $\pi(u, i) = u$, for all $(u, i) \in V(H)$.

LEMMA 1. *If G is triangle-free and H is the fibration of G then H is triangle-free.*

Proof. Let (u, i), (v, j), and (w, k) be any three distinct vertices in H. If $i = j = k$ then, since G is triangle-free, the three vertices do not form a triangle in H. Also, if $\{i, j, k\}$ contains two values whose difference modulo 6 is more than 1 then, by the structure of H, the three vertices cannot form a triangle in H. The remaining cases to consider are those where $\{i, j, k\}$ contains exactly two distinct values whose difference modulo 6 is 1. By the symmetry of H, no generality is lost if we assume that $i = j = 0$ and $k = 1$. If $(u, 0)$ and $(v, 0)$ are both joined to $(w, 1)$ in H then, by Definition 1, $(u, 0)$ and $(v, 0)$ are both also joined to $(w, 0)$. Thus, if $(u, 0)$, $(v, 0)$, and $(w, 1)$ form a triangle in H then $(u, 0)$, $(v, 0)$, and $(w, 0)$ form a triangle in H, which contradicts the fact that G is triangle-free. ∎

DEFINITION 2. For a graph G, the *independence number* of G, denoted by $\alpha(G)$, is the size of the largest independent set in G.

LEMMA 2. *If H is the fibration of G then $|H| = 6 \cdot |G|$ and $\alpha(H) \leqslant 4 \cdot \alpha(G)$.*

Proof. Obviously, $|H| = 6 \cdot |G|$. The interesting part is to show that $\alpha(H) \leqslant 4 \cdot \alpha(G)$.

Now, let S be an arbitrary independent set in H. We need to show that $|S| \leqslant 4 \cdot \alpha(G)$. For each $i \in \{0, 1, 2, 3, 4, 5\}$, let $S_i = \{(u, i) \in S\}$.

We shall first show that

$$|S_0| + |S_1| \leqslant \alpha(G) + |\pi(S_0) \cap \pi(S_1)|$$

$$|S_2| + |S_3| \leqslant \alpha(G) + |\pi(S_2) \cap \pi(S_3)|$$

$$|S_4| + |S_5| \leqslant \alpha(G) + |\pi(S_4) \cap \pi(S_5)|.$$

Fix $i \in \{0, 2, 4\}$. The equality

$$|\pi(S_i)| + |\pi(S_{(i+1) \bmod 6})| = |\pi(S_i) \cup \pi(S_{(i+1) \bmod 6})| + |\pi(S_i) \cap \pi(S_{(i+1) \bmod 6})|$$

follows trivially because set cardinality is a modular function. It is therefore sufficient to prove $\pi(S_i) \cup \pi(S_{(i+1) \bmod 6})$ is an independent set in G, since this implies that $|\pi(S_i) \cup \pi(S_{(i+1) \bmod 6})| \leqslant \alpha(G)$. Clearly, both S_i and $S_{(i+1) \bmod 6}$ are independent sets in $H[V_i]$ and $H[V_{(i+1) \bmod 6}]$ and, therefore, $\pi(S_i)$ and $\pi(S_{(i+1) \bmod 6})$ are independent sets in G. Now assume that some $u \in \pi(S_i)$ is adjacent to some $v \in \pi(S_{(i+1) \bmod 6})$ in G. Then, by Definition 1, there is an edge in $E(H)$ joining $(u, i) \in S_i$ and $(v, (i+1) \bmod 6) \in S_{(i+1) \bmod 6}$. This is a contradiction, since S is assumed to be independent. Therefore, $\pi(S_i) \cup \pi(S_{(i+1) \bmod 6})$ is independent in G and thus the three inequalities hold.

Summing the above inequalities, we obtain

$$|S_0| + |S_1| + |S_2| + |S_3| + |S_4| + |S_5| \leqslant 3 \cdot \alpha(G) + |\pi(S_0) \cap \pi(S_1)| + |\pi(S_2) \cap \pi(S_3)| + |\pi(S_4) \cap \pi(S_5)|.$$

Now, to complete the proof, it is sufficient to show that

$$|\pi(S_0) \cap \pi(S_1)| + |\pi(S_2) \cap \pi(S_3)| + |\pi(S_4) \cap \pi(S_5)| \leqslant \alpha(G).$$

First, note that $\pi(S_0) \cap \pi(S_1)$, $\pi(S_2) \cap \pi(S_3)$, and $\pi(S_4) \cap \pi(S_5)$ are mutually exclusive. To see why, suppose, without loss of generality, that $v \in \pi(S_0) \cap \pi(S_1)$ and $w \in \pi(S_2) \cap \pi(S_3)$. Then, in particular, $v \in \pi(S_0)$ and $w \in \pi(S_3)$. Thus, if $v = w$ then, by the structure of H, $((v, 0), (w, 3)) \in H$, contradicting the fact that S is independent in H.

Furthermore, $(\pi(S_0) \cap \pi(S_1)) \cup (\pi(S_2) \cap \pi(S_3)) \cup (\pi(S_4) \cap \pi(S_5))$ is an independent set. This follows from the observation that if, without loss of generality, $v \in \pi(S_0) \cap \pi(S_1)$ and $w \in \pi(S_2) \cap \pi(S_3)$ then, in particular,

$v \in \pi(S_1)$ and $w \in \pi(S_2)$. Thus, if $(v, w) \in G$ then $((v, 0), (w, 1)) \in H$, contradicting the fact that S is independent in H.

Therefore, since $\pi(S_0) \cap \pi(S_1)$, $\pi(S_2) \cap \pi(S_3)$, and $\pi(S_4) \cap \pi(S_5)$ are mutually exclusive and their union is independent in G,

$$|\pi(S_0) \cap \pi(S_1)| + |\pi(S_2) \cap \pi(S_3)| + |\pi(S_4) \cap \pi(S_5)| \leqslant \alpha(G),$$

which completes the proof. ▮

THEOREM 3. *There exists a feasible method for constructing a triangle-free graph with independence number less than t, whose size is greater than* $\frac{5}{6}((t-1)/2)^{\log 6/\log 4}$. *This constructively proves that* $R(3, t) > \frac{5}{6}((t-1)/2)^{\log 6/\log 4} \in \Omega(t^{1.29})$.

Proof. Construct a sequence of graphs $G_0, G_1, G_2, \ldots$ as follows. Let G_0 be a 5-cycle, and let G_{i+1} be the fibration of G_i. G_0 is triangle-free so, by Lemma 1, for all i, G_i is triangle-free. Clearly, $|G_0| = 5$ and $\alpha(G_0) = 2$. By Lemma 2, for all i, $|G_{i+1}| = 6 \cdot |G_i|$ and $\alpha(G_{i+1}) \leqslant 4 \cdot \alpha(G_i)$. Therefore, for all i, $|G_i| = 5 \cdot 6^i$ and $\alpha(G_i) \leqslant 2 \cdot 4^i$. If the sequence is constructed until $i = \lfloor \log((t-1)/2)/\log 4 \rfloor$ then $\alpha(G_i) \leqslant t-1$ and $|G_i| > \frac{5}{6}((t-1)/2)^{\log 6/\log 4}$. ▮

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