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LOWER BOUNDS FOR SOME RAMSEY NUMBERS

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Abstract. The symbol $n \to (u)_k$ means that if the edges of a complete graph on n vertices are colored arbitrarily in k colors then there results a complete subgraph on u vertices, all of whose edges have the same color. $n \nrightarrow (u)_k$ is the negation of $n \to (u)_k$. Various results of the form $n \nrightarrow (u)_k$ are proved by constructive arguments. The main one is that $n \nrightarrow (n^{\alpha})_2$ for some $\alpha < \frac{1}{2}$.

If $n \geq 2$ is an integer, the symbol $\langle n \rangle$ will denote the complete graph on n vertices. We sometimes use the symbol $\langle n \rangle$ even when n is not an integer. It is then to be interpreted as $\langle [n] \rangle$. If k is a positive integer and if $u \geq 2$, then

$$(1) \qquad n \to (u)_k ,$$

means that if the edges of an $\langle n \rangle$ are colored arbitrarily in k colors there results a $\langle u \rangle$ all of whose edges have the same color. It follows from Ramsey's Theorem [10] that if u and k are given, (1) holds for all sufficiently large n. $n \nrightarrow (u)_k$ will mean the negation of (1).

It is known [4, 6] that

$$n \to \left(\frac{\log n}{2 \log 2}\right)_2 ,$$

and that

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(2) $$n \not\to \left(\frac{2 \log n}{\log 2}\right)_2 .$$

It is also known that (see, for example [6] or [9])

(3) $$n \to \left(\frac{c \log n}{k \log k}\right)_k ,$$

where c is a positive absolute constant* and in [7] it is remarked that the arguments used in [4], to prove (2) can be used to prove

(4) $$n \not\to \left(\frac{c \log n}{\log k}\right)_k .$$

In [1] it was shown that

(5) $$p \not\to (u)_s \text{ and } q \not\to (u)_t \text{ implies } pq \not\to (u)_{s+t} ,$$

and it was deduced from (2) and (5) that

$$n \not\to \left(\frac{c \log n}{k}\right)_k ,$$

which is superior to (4).

The proof of (2) makes use of probabilistic arguments. In [5], Erdös remarks that it would be very desirable to have a constructive proof of (2) but points out that he is not able to give a constructive proof of the much weaker result

(6) $$n \not\to (\epsilon n^{\frac{1}{2}})_2 ,$$

for every $\epsilon > 0$ $n \geq n_0(\epsilon)$. We remark that $n \not\to (c n^{\frac{1}{2}})_2$, and more generally $n \not\to (c n^{1/k})_k$, have been established (see for example, [2] or [8]).

* We use the letter c to denote positive absolute constants. The numerical value of c will not necessarily be the same at each occurrence. For example, in (3) one can take $c = 1$ and in (4) one can take $c = 2$.

In this note we prove, by constructive arguments, the following result which is stronger than (6).

Theorem. *Let* $\alpha = \log 2/\log 5$. *Then*

$$n \nrightarrow (c\,n^{\alpha})_2 . \tag{7}$$

Before proving the theorem we shall need to prove the following lemma:

Lemma. *If* $p \nrightarrow (s)_k$ *and* $q \nrightarrow (t)_k$ *then* $pq \nrightarrow (st - s - t + 2)_k$.

Proof. Let $\langle p\rangle$ have vertices $V_1, V_2, \ldots, V_p$, and color the edges of $\langle p\rangle$ in k colors $C_1, C_2, \ldots, C_k$ in such a way that there does not result a monochromatic $\langle s\rangle$. Similarly, let $\langle q\rangle$ have vertices $U_1, U_2, \ldots, U_q$ and color the edges of $\langle q\rangle$ in the colors $C_1, C_2, \ldots, C_k$ without getting a monochromatic $\langle t\rangle$. Let $\langle pq\rangle$ have vertices W_{ij}, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. Color the edges of $\langle pq\rangle$ as follows: Let e be the edge joining W_{ij} and W_{lm}. If $i = l$, color e the same as the edge joining U_j and U_m in $\langle q\rangle$. If $i \neq l$, color e the same as the edge joining V_i and V_l in $\langle p\rangle$. Let $u = st - s - t + 2$ and let $\langle u\rangle$ be any complete subgraph of $\langle pq\rangle$. We need to show that $\langle u\rangle$ is not monochromatic. Suppose that it is, and suppose all of its edges are colored C_1, say. We distinguish two cases:

Case 1. There are at least s distinct values of i such that W_{ij} is a vertex of $\langle u\rangle$. Then it is clear that, according to our coloring scheme, $\langle p\rangle$ contains a monochromatic $\langle s\rangle$, and this is a contradiction.

Case 2. There are at most $s - 1$ distinct values of i such that W_{ij} is a vertex of $\langle u\rangle$. Then there must be at least t distinct values of j, say $j_1, j_2, \ldots, j_t$ and a number i such that $W_{ij_1}, W_{ij_2}, \ldots, W_{ij_t}$ are vertices of $\langle u\rangle$. (Otherwise we would have that the number of vertices of $\langle u\rangle$ is at most $(s-1)(t-1) = st - s - t + 1 < u$.) This clearly means, by our coloring scheme, that the points $U_{j_1}, U_{j_2}, \ldots, U_{j_t}$ are the vertices of a monochromatic $\langle t\rangle$ in $\langle q\rangle$. This is a contradiction. Hence our lemma is proved.

Proof of the Theorem. It is well known and easy to verify (see, for example, [9]) that

(8) $$5 \nrightarrow (3)_2 .$$

It follows from (8) and the lemma, by an easy induction argument, that for all positive integers l,

(9) $$5^l \nrightarrow (2^l + 1)_2 .$$

Let $n \geq 5$ be given and let l be the integer defined by

(10) $$5^{l-1} < n \leq 5^l .$$

Then (9) and (10) imply

(11) $$n \nrightarrow (2^l + 1)_2 .$$

It now follows easily from (10) and (11) that (7) holds. Hence our theorem is proved.

We conclude with the following remarks:

Remark 1. The value $\alpha = \log 2/\log 5$ can be replaced by a smaller value simply by taking as a starting point something different from (8). For example, the known result $17 \nrightarrow (4)_2$ gives the value $\alpha = \log 3/\log 17$.

Remark 2. The lemma can be used to obtain results of the form $n \to (c\,n^\alpha)_k$ for values of $k \geq 3$. For example, the known result $16 \nrightarrow (3)_3$ leads to $n \nrightarrow (c\,n^{\frac{1}{2}})_3$. For large values of k, the best result that we have been able to obtain is the following: Let α be any constant satisfying $\alpha > \log 16/\log 89$. Then there is a $k_0 = k_0(\alpha)$ such that if $k \geq k_0$ then $n \nrightarrow (n^{\alpha/k})_k$ for all sufficiently large n. This result is obtained from our lemma and the results in [3].

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