Ergodic Approach to Ramsey Theory

October 9, 2024

0.1 Brouwer's Fixed Point Theorem

We are interesting in the following question: For which function f and domains D is the following true:

If $f : D \to D$ is a function then $(\exists x)[f(x) = x]$.

We give some examples of classes of f where this is true, then classes of f where this is false, then finally state and prove Brouwer's fixed point theorem which gives a wide class of (f, D) where it is true.

Theorem 0.1.1

1. *If f is a continous map from $[0, 1]$ to $[0, 1]$ then there exists an $x \in [0, 1]$ such that $f(x) = x$.*
2. *There exists a continous map f from $(0, 1)$ to $(0, 1)$ such that there is no x with $f(x) = x$.*
3. *There exists a continous map f from $[0, 1] \cup [2, 3]$ to $[0, 1] \cup [2, 3]$ such that there is no x with $f(x) = x$.*
4. *There exists a continous map f from $\mathbb{R}$ to $\mathbb{R}$ such that there is no x with $f(x) = x$.*
5. *There exists a continous map f from $\mathbb{S}^1$ to $\mathbb{S}^1$ such that there is no x with $f(x) = x$. (S^1 is the circle.)*

Proof:
1) Let $g(x) = f(x) - x$. Note that:

$g(0) = f(0) - 0 = f(0) \geq 0$.

$g(1) = f(1) - 1 \leq 0$.

Since g is continous, by the intermediate value theorem, there exists an $x_0 \in [0, 1]$ such that $g(x_0) = 0$. Note that $f(x_0) = x_0$.

2)

$$f(x) = \begin{cases} x + 1 & \text{if } x \in [0, 1] \\ x - 2 & \text{if } x \in [2, 3] \end{cases} \tag{1}$$

3) $f(x) = x^2$ from $(0, 1)$ to $(0, 1)$ works.

4) $f(x) = x + 1$ works.

5) We take $\mathbb{S}^1$ to be the unit circle in the plane. We denote a point p on the circle by the angle θ (in radians) between the x-axis and the line from

the origin to p. Take $f(\theta) = \theta + \pi$. (Any multiple of π that is not an even multiple of π works.) ∎

What property does $[0, 1]$ have that $(0, 1)$, $\mathbb{R}$, and S^1 lack?

1. $(0, 1)$ is not closed. If $X \subseteq \mathbb{R}^1$ that is not closed then (I think) one can construct a continous function $f : X \to X$ with no fixed point.

2. $\mathbb{R}$ is unbounded. If $x \subseteq \mathbb{R}$

We state Brouwer's Fixed Point Theorem in stages, increasing generality as we go.

Theorem 0.1.2

1. *If f is a continous function from $[a, b]$ to $[a, b]$ then f has a fixed point.*

2. *Let $n \in \mathbb{N}$. Let B be a closed ball in $\mathbb{R}^n$. If f is a continous function from B to B then f has a fixed point.*

3. *Let $n \in \mathbb{N}$. Let B be a nonempty convex compact subset of $\mathbb{R}^n$. If f is a continous function from B to B then f has a fixed point.*

We omit the proof
BILL- I MIGHT PUT IN A PROOF LATER.

0.2 Continous functions from $\mathbb{S}^1$ to $\mathbb{S}^1$

The simplest case where Brouwer's Fixed Point Theorem fails is $\mathbb{S}^1$. That is, there is a continous function from $\mathbb{S}^1$ to $\mathbb{S}^1$ with no fixed point. We will present a theorem about continous functions from $\mathbb{S}^1$ to $\mathbb{S}^1$ that can be regarded as an approximate Fixed Point Theorem.

Def 0.2.1 Let M be a matric space. Let f be a function from M to M. Let $x \in M$.

1. x is **fixed** if $f(x) = x$.

2. x is **periodic** if there exists n, $f^{(n)}(x) = x$. n is the *degree of the fixed point.* Note that a fixed point is a periodic point of degree 1.

3. x is **recurrent** if there exists a sequence of naturals $n_1, n_2, \ldots,$ such that $\lim_{i\in\mathbb{N}} f^{(n_i)}(x) = x$. Note that a periodic point is a recurrent point though a boring one.

We give examples of continou functions from $\mathbb{S}^1$ to $\mathbb{S}^1$ to see if they have fixed points, periodic points, or recurrent points.

We take $\mathbb{S}^1$ to be the unit circle in the plane. We denote a point p on the circle by the angle θ (in radians) between the x-axis and the line from the origin to p.

1) $f(\theta) = \theta + \pi$. Every point is periodic with $n = 2$.

2) $f(\theta) = \theta + \frac{\pi}{1011}$. Every point is periodic with $n = 2022$.

3) $f(\theta) = \theta + \pi\sqrt{2}$. We will soon prove that every point is recurrent.

BILL- WILL PROVE THIS IS RECURRENT LATER.

Are there interesting examples of continous functions f from $\mathbb{S}^1$ to $\mathbb{S}^1$ that have a fixed point? No. Are there interesting examples of continous functions f from $\mathbb{S}^1$ to $\mathbb{S}^1$ that have a periodic point? No. Are there interesting examples of continous functions f from $\mathbb{S}^1$ to $\mathbb{S}^1$ that have a recurrent point? Yes.

Theorem 0.2.2 *Let f is a conntinous functions from $\mathbb{S}^1$ to $\mathbb{S}^1$.*

1. *If f has a fixed point then f is the identity function.*

2. *If f has a periodic point of degree n then $f^{(n)}$ is the identity.*

Proof:
1) LATER
2) LATER
$\blacksquare$

Lemma 0.2.3 *Let G be an additive subgroup of the group $[0, 1)$ with addition mod 1. Then either:*

- *G is dense.*

- *There exists $r \in [0, 1)$ such that $G = \{nr : n \in \mathbb{N} \mod 1\}$.*

Proof:
Case 1: there exists a sequence of elements of G, $g_1, g_2, \ldots$ such that $g_1, g_2, g_3, \ldots$ converges to 0.

We show that G is dense. Let $0 \le a < b < 1$. We show that some element of G is between a and b. Let $b - a = d$. Let g be an element of the sequence such that $0 < g < \min\{a, d/2\}$. Look at $g, 2g, 3g, \ldots$. Let n be such that ng is the largest element $< a$. Then $(n+1)g > a$. Since $g < d/2$, $a < (n+a)g < b$.

Case 2: There is no such sequence. Hence there exists $g \in G$ such that $0 < g$ and there is no element of G in $[0, g)$.

Look at $g, 2g, 3g, \ldots$.

We show these are all of the elements of G. Assume not. Then there exists $n \in \mathbb{N}$ and $h \in G$ such that $ng < h < n(g+1)$. Subtract ng from this to get $0 < h - ng < g$. This contradicts having no element in $[0, g)$. ▮

FROM THIS CAN PROVE $f(\theta) = \theta + \alpha\pi$ where $\alpha \in \mathbb{R} - \mathbb{Q}$ has all points recurrent LATER.

We now show that every continous function from $\mathbb{S}^1$ to $\mathbb{S}^1$ has a recurrent point. This is a subcase of Birkoff's Recurrence Theorem. The general Birkoff Recurrence has a different proof.

Def 0.2.4 Let f is a functions from $\mathbb{S}^1$ to $\mathbb{S}^1$. Let $E \subseteq \mathbb{S}^1$. E is *invariant for* f if $f(E) = E$. If the function f is understood we just say *invariant*.

We define what it means for a continous bijection to be *minimal* in two ways that are equivlant.

Def 0.2.5 Let f be a continous bijection from $\mathbb{S}^1$ to $\mathbb{S}^1$.

1. f is *minimal* if for every $x \in \mathbb{S}^1$ the set $\{f^{(n)}(x) : n \in \mathbb{Z}\}$ is dense.

2. f is *minimal* if there is no proper closed subset E such that $f(E) = E$.

BILL- PROVE EQUIVALENCE LATER.

(Used
`https://www.math.cuhk.edu.hk/course_builder/1516/math6081a/lecture1.pdf`

We will need the following notion for the statement and proof of Birhoff's theorem.

Def 0.2.6 Let (X, d) be a metric space.

1. Let $x \in X$ and $\epsilon \in \mathbb{R}^+$. Then
$$B(x, \epsilon) = \{y \colon d(x, y) < \epsilon\}.$$
2. A *basic open set* is a set of the form $B(x, \epsilon)$.
3. An *open set* is a union of basic open sets. The union can be finite or infinite, any cardinality.
4. A *closed set* is the compliment of an open set. It is also know that a closed set contains all of its limit points.
5. (This definition looks unmotivated but its use in the Theorem **??** will show why its just right.) (X, d) is *compact* if for every collection of open set $O_1, O_2, \ldots$ such that $X \subseteq \cup_{i=1}^{\infty}$ there is a finite set of the open sets that also covers X. (I wrote this as the set of open sets is countable, but it could be of any cardinality.)

The next theorem gives examples of compact metric spaces. One of them will be useful in the next problem.

Theorem 0.2.7

1. $[0, 1]$ *is compact.*
2. $\mathbb{S}^1$ *is compact.*
3. $\mathbb{R}$ *is not compact.*

Proof: LATER ∎

Theorem 0.2.8 *Let f be a continous bijection from $\mathbb{S}^1$ to $\mathbb{S}^1$. There is an f-invariant $E \subseteq \mathbb{S}^1$ such that f restricted to E is minimal.*

Proof:

Let $\mathcal{F}$ be the set of all $Y \subseteq \mathbb{S}^1$ with the following properties.

- $Y \neq \emptyset$.

- Y is closed.
- $f(Y) = Y$.

Since $S^1 \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$.

We order $\mathcal{F}$ be reverse inclusion. Note that $\mathbb{S}^1$ is minimal.

We show that this ordering satisfies Zorn's Lemma.

All we need is that if $Y_1 \supset Y_2 \subset Y_3 \subset \cdots$ then $Y = \cap_{i=1} Y_i$ satisfies the three properties.

Items 2 and 3 are easy.

Item 1 is the key one- need that the intersection is nonempty. Assume, by way of contraction, that the intersection of the Y_i's is empty. Then the complement of the intersection is $\mathbb{S}^1$. Hence

$$\mathbb{S}^1 \subseteq \bigcup_{i=1}^{\infty} X - Y_i.$$

Since Y_i is closed, $X - Y_i$ is open. Hence this is an open cover. Since $\mathbb{S}^1$ is compact there exists a finite subcover. Hence there exists an n such that

$$\mathbb{S}^1 \subseteq \bigcup_{i=1}^{n} X - Y_i.$$

Take the complement to get

$$\emptyset \supseteq \bigcap_{i=1}^{n} Y_i = Y_n.$$

Hence $Y_n = \emptyset$ which is a contradiction.

We apply Zorn's lemma to obtain a closed set Y that is maximal in the ordering on $\mathcal{F}$. Hence there is no closed nonempty set Z such that $Z \subset Y$ and which has $f(Z) = Z$.

We show that every $z \in Z$ is recurrent.

Look at $Q(z) = \overline{\{f^{(n)}(z) \colon n \in \mathbb{N}\}}$

Recall that this is $\{f^{(n)}(z) \colon n \in \mathbb{N}\}$ together with all of that sets limit points.

If $z \in Q(z)$ then either $z = f^{(n)}(z)$ for n, so then z is trivially recurrent or elements in the orbit of z get arbitrarily close to z, so z is recurrent.

We show that $z \in Q(z)$.

Clearly $Q(z)$ is nonempty, $f(Q(z)) = Q(z)$, and $Q(z)$ is closed.

Since $f(Y_n) = Y_n$, since $z \in Y_n$, $Q(z) \subseteq Y_n$. But Y_n is maximal in the reverse ordering, so $Q(z) = Y_n$. Hence $z \in Q(z)$. ∎