The Very Large Ramsey Theorem
Exposition by William Gasarch

1 The Large Ramsey Theorem

In most theorems in Ramsey Theory the labels on the vertices did *not* matter. Here they do.

Def 1.1 A finite set $F \subseteq \mathsf{N}$ is called *large* if the size of F is BIGGER than the smallest element of F.

Example 1.2

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 > 1$.

2. The set $\{5, 10, 12, 17, 20\}$ is NOT large: It has 5 elements, the smallest element is 5, and 5 is NOT strictly greater than 5.

3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is NOT large: It has 9 elements, the smallest element is 20, and $9 < 20$.

4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 > 5$.

5. The set $\{101, \ldots, 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic K_m's where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Let COL be a 2-coloring of $\binom{[n]}{2}$. Consider the set $\{1, 2\}$. It is clearly both homogeneous and large (using our definition of large). Hence the statement

"for every $n \geq 2$, every 2-coloring of K_n has a large homogeneous set"

is true but trivial.

What if we used $V = \{k, k + 1, \ldots, n\}$ as our vertex set? Then a large homogeneous set would have to have size at least k.

Notation 1.3 LR(k) is the least n, if it exists, such that every 2-coloring of $\binom{\{k,\ldots,n\}}{2}$ has a large homogeneous set.

Theorem 1.4 *For every k there exists n such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a large homog set.*

Proof: This proof is similar to the standard proof of the finite Ramsey Theorem *from* the infinite Ramsey Theorem.

Suppose, by way of contradiction, that there is some $k \geq 2$ such that no such n exists. For every $n \geq k$, there is some way to color $\binom{\{k,\ldots,n\}}{2}$ so that there is no large homog sets. Hence there exist the following:

1. COL_1, a 2-coloring of $\binom{\{k,k+1\}}{2}$ that has no large homog set.
2. COL_2, a 2-coloring of $\binom{\{k,k+1,k+2\}}{2}$ that has no large homog set.
3. COL_3, a 2-coloring of $\binom{\{k,\ldots,k+3\}}{2}$ that has no large homog set.

   $\vdots$

j. COL_L, a 2-coloring of $\binom{\{k,\ldots,k+L\}}{2}$ that has no large homog set.

   $\vdots$

We will use these 2-colorings to form a 2-coloring COL of $\binom{\{k,k+1,\ldots\}}{2}$. This coloring will have an infinite homog set by the infinite Ramsey Theorem. This will give us a contradiction to the definition of one of the COL_i.

Let $e_1, e_2, e_3, \ldots$ be a list of every element of $\binom{\{k,k+1,\ldots\}}{2}$. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathsf{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise.} \end{cases} \tag{1}$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned} COL(e_1) &= COL_j(e_1) \\ COL(e_2) &= COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) &= COL_j(e_{i-1}) \end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE otherwise.} \end{cases} \tag{2}$$

$$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: Let A be a finite subset of $\{k, k+1, \ldots,\}$. Then there exists an infinite number of i such that COL on $\binom{A}{2}$ agrees with COL_i on $\binom{A}{2}$.

Proof of Claim

Left to the reader.

End of Proof of Claim

By the infinite Ramsey Theorem there is an infinite homog set for COL:

$$H = \{x_1 < x_2 < x_3 < \cdots\}.$$

Look at

$$H' = \{x_1 < x_2 < \cdots < x_{x_1+1}\}$$

This is a homog set with respect to COL. By the claim there is an i (in fact, infinitely many) such that COL and COL_i agree on $\binom{H'}{2}$. Clearly H' is a large homog set for COL_i. This contradicts the definition of COL_i. ∎

Theorem 1.5 *For every k, a, c there exists n such that for all c-colorings of $\binom{\{k,\ldots,n\}}{a}$ there exists a large homog set. We denote this n by* $\mathrm{LR(k, a, c)}$.

Note 1.6 The function $\mathrm{LR(k, a, c)}$ grows rather fast. So fast that the existence of $\mathrm{LR(k, a, c)}$ cannot be proven in Peano Arithmetic.

2 The Very Large Ramsey Theorem

We generalize the definition of large.

Def 2.1 Let $X \subseteq \mathsf{N}$ be a finite set. Let

$$X = \{x_0 < x_1 < \cdots < x_k\}.$$

Let α be an ordinal that is $< \omega^\omega$.

We first give some examples of largeness and then generalize to α.

1. Let $a \in \mathsf{N}$. X is *a-large* if $|X| \ge a$.
2. X is *ω-large* if $|X| > \min(X)$ (this is what we call *large*.
3. X is *$(\omega + 1)$-large* if $\{x_1, \ldots, x_k\}$ is ω-large.
4. X is *$(\omega + 2)$-large* if $\{x_2, \ldots, x_k\}$ is ω-large.
5. X is *$(\omega + \omega)$-large* if $X = X_1 \cup X_2$, $X_1 < X_2$, and both X_1, X_2 are ω-large.
6. X is *ω^2-large* if $X = \min(X) \cup X_1 \cup \cdots \cup X_{\min(X)}$ and each X_i is ω-large.
7. X is *$(\alpha + 1)$-large* if $X - \min(X)$ is α-large.
8. X is *$(\alpha + \omega^n)$-large* if $(\alpha + \omega^{n-1} \min(X))$-large.

Notation 2.2

1. $\mathrm{LR}(\alpha)$ is the least n, if it exists, such that every 2-coloring of $\binom{\{k,\ldots,n\}}{2}$ has an α-large homogeneous set.

2. $\mathrm{LR}(\alpha, \mathrm{a}, \mathrm{c})$ is the least n, if it exists, such that every c-coloring of $\binom{\{k,\ldots,n\}}{a}$ has an α-large homogeneous set.

3. $\mathrm{LR}(\mathrm{k}, \mathrm{a}, \mathrm{c})$ is the least n, if it exists, such that every c-coloring of $\binom{\{k,\ldots,n\}}{a}$ has an ω^k-large homogeneous set.

4. $\mathrm{LR}^{\mathrm{ord}}(\alpha)$ is the least ordinal β such that, for every β-large X, for every 2-coloring of $\binom{\beta}{2}$ has an α-large homogeneous set.

Theorems about α-large sets and Ramsey are stated in terms of $\mathrm{LR}^{\mathrm{ord}}$. The following are known:

Theorem 2.3

1. $\mathrm{LR}^{\mathrm{ord}}(\omega) \le \omega^6$. *Ketonen-Solovay, 1981.*

2. $\mathrm{LR}^{\mathrm{ord}}(\omega^k) \le \omega^{\omega^{k \cdot 2}}$ *Bigorajska-Kotlarski 2002.*

3. For all k there exists n such that $\mathrm{LR}^{\mathrm{ord}}(\omega^k) \le \omega^n$. *Patey-Yokoyama. 2018.*

4. For all k $\mathrm{LR}^{\mathrm{ord}}(\omega^k) \le \omega^{300k}$. *Aleksander-Wong-Yokoyama 2020.*

`https://arxiv.org/pdf/2005.06854.pdf`

I have not seen the function LR with ordinals defined in the literature.

I speculate that $\mathrm{LR}(\mathrm{k}, \mathrm{a}, \mathrm{c})$ might be the fastest growing natural computable function in mathematics. Of course, this may depend on your definition of *natural*.