BILL, RECORD LECTURE!!!!

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Slight Improvement on $R(k)$

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$$R(a,b) \leq R(a-1,b) + R(a,b-1)$$

$R(a, b) \leq R(a - 1, b) + R(a, b - 1)$

We proved

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Now lets used it

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$1 \leq 6$.

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Actually they are equal. We prove this on the next slide.

Comb Argument

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DONE

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Proof is long, hard, and I don't know it.