BILL, RECORD LECTURE!!!!

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The Probabilistic Method: Sum-Free Sets

Exposition by William Gasarch

Sum Free Set Problem

Definition: A set of numbers A is **sum-free** if there is NO $x, y, z \in A$ such that $x + y = z$.

Example: Let $y_1, \ldots, y_m \in (1/3, 2/3)$ (so they are all between 1/3 and 2/3). Note that $y_i + y_j > 2/3$, hence $y_i + y_j \notin \{y_1, \ldots, y_m\}$.

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Proof left to the reader.

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Theorem For all $\epsilon > 0$, for all $A \subseteq \mathbb{R}$, $|A| = n$, there is a sum-free subset $X \subseteq A$ such that $|X| \geq (1/3 - \epsilon)n$.

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This opens up a new line of research: Improve $g(n)$.