BILL, RECORD LECTURE!!!!

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Ramsey Fails for $\mathbb{R}$

Exposition by William Gasarch

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The proof uses AC by using WOP.

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$\mathbb{R}$ can be well ordered. Is that strange?

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Odd? Do these two odd facts make your doubt WOP?

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- This is a stronger result then we originally stated: the homog set can't be of any cardinality bigger than countable. If $\neg CH$ is true this matters.

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all homog sets are countable.

- This is a stronger result then we originally stated: the homog set can't be of any cardinality bigger than countable. If $\neg CH$ is true this matters.
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See next page for how to make Gary happy, which will also make Meatloaf happy.

Making Gary Happy

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2) Work in ZF, not ZFC. Perhaps add some other axioms. I think there has been some work on this but its not stated this way.

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CON You call that an axiom? (TELL STORY)

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If A is finite then 2^A is finite.

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So far ZFC has not been able to show that X does not exist. Most set theorists think that ZFC + $\exists X$ is consistent.

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CON Is LC1 so obvious as to be a axiom?

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Thm If X is a RC then X is inaccessible. Hence we cannot prove RC's exist.