BILL, RECORD LECTURE!!!!

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An Application of Ramsey's Theorem to Proving Programs Terminate: An Exposition

William Gasarch-U of MD

Who is Who

1. Work by
   1.1 **Floyd**,
   1.2 **Byron Cook, Andreas Podelski, Andrey Rybalchenko**,
   1.3 **Lee, Jones, Ben-Amram**
   1.4 Others

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3. **Pre-Brag**: Not my area-some things may be understandable.
4. This talk is for a PL audience so I will skip the **Intro to Ramsey** stuff in it, even though I will be listed as one of the topics.

Overview I

Problem: Given a program we want to prove it terminates no matter what user does (called TERM problem).

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2. **But** can do this on some simple progs. (We will.)

Overview II

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1. Do examples of **traditional method** to prove progs terminate.
2. **DIGRESSION:** A very short lecture on **Ramsey Theory.**
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5. Do example with **Ramsey Theory** and Matrices.

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3. **Note:** we will want to show that **no matter what the user does** the program will halt.
4. The code

   ```
   (x,y) = (f(x,y),g(x,y))
   ```

   means that **simultaneously** x gets f(x,y) and y gets g(x,y).

Example of Traditional Method

```
(x,y,z) = (input(INT), input(INT), input(INT))
While x>0 and y>0 and z>0
        control = input(1,2,3)
        if control == 1 then
                (x,y,z)=(x+1,y-1,z-1)
        else
        if control == 2 then
                (x,y,z)=(x-1,y+1,z-1)
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Whatever the user does x+y+z is decreasing.
Eventually x+y+z=0 so prog terminates there or earlier.

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General method due to **Floyd:** Find a function f(x,y,z) from the values of the variables to N such that

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Note: Method is more general- can map to a well founded order such that in every iteration f(x,y,z) decreases in that order, and if f(x,y,z) is ever a min element then program must have halted.

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Note: $(4, 10^{100}, 10^{10!}) < (5, 0, 0)$.

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In every iteration (x, y, z) **decreases in this ordering.**
If hits bottom then all vars are 0 so **must halt then or earlier.**

Well Ordering is Key!

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Lex order on $N \times N \times N$ is well founded. Discuss.

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Keep these in mind- our later proof will use a **nice** ordering but will need to reason about a **block** of instructions.

Digression Into Ramsey Theory (Parties!)

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3. If you have 2^{2k-1} people at a party then either k of them mutually know each other of k of them mutually do not know each other.
4. If you have an **infinite** number of people at a party then either there exists an **infinite** subset that all know each other or an **infinite** subset that all do not know each other.

Digression Into Ramsey Theory (Math!)

Def Let $c, k, n \in \mathbb{N}$. K_n is the **complete graph on n vertices (all pairs are edges).** $K_{\mathbb{N}}$ is the **infinite complete graph.** A **c-coloring of K_n** is a c-coloring of the edges of K_n. A **homog set** is a subset H of the vertices such that every pair has the same color (e.g., 10 people all of whom know each other).

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3. For all c-colorings of the K_{N} there exists an infinite homog set.

Alt Proof Using Ramsey

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Proof of termination

If program does not halt then there is infinite sequence $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots,$ representing state of vars.

Reasoning about Blocks

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Look at $(x_i, y_i, z_i), \ldots, (x_j, y_j, z_j)$.

1. If control is ever 1 then $x_i > x_j$.
2. If control is never 1 then $y_i > y_j$.

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Look at $(\mathrm{x}_i, \mathrm{y}_i, \mathrm{z}_i), \ldots, (\mathrm{x}_j, \mathrm{y}_j, \mathrm{z}_j)$.

1. If control is ever 1 then $\mathrm{x}_i > \mathrm{x}_j$.
2. If control is never 1 then $\mathrm{y}_i > \mathrm{y}_j$.

Upshot: For all $i < j$ either $\mathrm{x}_i > \mathrm{x}_j$ or $\mathrm{y}_i > \mathrm{y}_j$.

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If program does not halt then there is infinite sequence $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots,$ representing state of vars.
For all $i < j$ either $x_i > x_j$ or $y_i > y_j$.
Define a 2-coloring of the edges of K_{N}:

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If color is Y then $\mathrm{y}_{i_1} > \mathrm{y}_{i_2} > \mathrm{y}_{i_3} > \cdots$
In either case will have eventually have a var ≤ 0 and hence program must terminate. **Contradiction.**

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3. Trad. proof only had to reason about single steps–simple!
4. Ramsey Proof had to reason about blocks of steps–complicated!

What do YOU think?

VOTE:

1. Traditional Proof!
2. Ramsey Proof!

Another Example

```
(x,y) = (input(INT),input(INT))
While x>0 and y>0
       control = input(1,2)
       if control == 1 then
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       else
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Reasoning about Blocks

If program does not halt then there is infinite sequence $(\mathrm{x}_1, \mathrm{y}_1), (\mathrm{x}_2, \mathrm{y}_2), \ldots,$ representing state of vars.
We look at a block $(x_i, y_i), \ldots, (x_j, y_j)$.

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If color is $X + Y$ then $x_{i_1} + y_{i_1} > x_{i_2} + y_{i_2} > \cdots$

Use Ramsey!

Define a 2-coloring of the edges of K_N:

$$COL(i,j) = \begin{cases} X \text{ if } x_i > x_j \\ X + Y \text{ if } x_i + y_i > x_j + y_j \end{cases} \quad (2)$$

By **Ramsey** there exists homog set $i_1 < i_2 < \cdots$.
If color is X then $x_{i_1} > x_{i_2} > \cdots$
If color is $X + Y$ then $x_{i_1} + y_{i_1} > x_{i_2} + y_{i_2} > \cdots$
In either case will have eventually have a var ≤ 0 and hence program must terminate. **Contradiction.**

Comments

1. The condition
   $x_i > x_j$ OR $x_i + y_i > x_j + y_j$.
   in the last proof is called a **Termination Invariant**. It is used to strengthen the induction hypothesis.
2. The proof was **found by the system** of B. Cook et al.
3. Looking for a Termination Invariant is the hard part to automate but they have automated it.
4. Can we use these techniques to solve a fragment of Termination Problem?

Model control=1 via a Matrix

```
if control == 1 then (x,y)=(x-1,x)
```

Model as a matrix A indexed by x,y,x+y.

$$\begin{pmatrix} -1 & 0 & \infty \\ \infty & \infty & \infty \\ \infty & \infty & \infty \end{pmatrix}$$

For a,b $\in$ {x,y,x+y}
Entry (a,b) is difference between NEW b and OLD a.
Entry (a,a) is most interesting- if neg then a decreased.

Model control=2 via a Matrix

```
if control == 2 then (x,y)=(y-2,x+1)
```

Model as a matrix B indexed by x,y,x+y.

$$\begin{pmatrix} \infty & 1 & \infty \\ -2 & \infty & \infty \\ \infty & \infty & -1 \end{pmatrix}$$

Redefine Matrix Mult

A and B matrices, C=AB defined by

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}.$$

Lemma

If matrix A models a statement s_1 and matrix B models a statement s_2 then matrix AB models what happens if you run $s_1; s_2$.

Matrix Proof that Program Terminates

- A is matrix for control=1. B is matrix for control=2.
- Show: any prod of A's and B's some diag is negative.
- Hence in any finite seg one of the vars decreases.
- Hence, by Ramsey proof, the program always terminates

General Program

```
X = (input(INT),...,input(INT))
While x[1]>0 and x[2]>0 and ... x[n]>0
 control = input(1,2,3,...,m)
 if control==1
    X = F1(X,input(INT),...,input(INT)))
  else
  if control==2
    X = F2(X,input(INT),...,input(INT))
  else...
  else
  if control==m
    X = Fm(X,input(INT),...,input(INT))
```

Fragment of TERM decidable?

Def The **TERMINATION PROBLEM:** Given $F_1, \ldots, F_m$ can we determine if the following holds:

For all ω-seq of inputs the program halts

Much Easier Problem Undecidable

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- Hilbert thought there was such an algorithm and that this was a problem in Number Theory.
- Over time (next slide) it was proven that there is NO such algorithm and that this is a problem in Logic.

Computable and C.E. Sets

Def: A set A is **computable** if there is a Java program (Turing Machine, other models) J (on one var) that halts on all inputs such that
If $x \in A$ then $J(x)$=YES
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Known: There are sets that are c.e. but not computable. Here is one: Let J_x be the xth Java program in some reasonable ordering.

$$\{(x, y) \colon J_x(y) \text{ halts } \} = \{(x, y) \colon (\exists t)[J_x(y) \text{ halts in} \le t \text{ steps}] \}$$

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$$A = \{a : (\exists a_1, \ldots, a_n)[p(a, a_1, \ldots, a_n)]\}$$

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4. From this you can conclude that TERM is undecidable.

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3. **OPEN:** Determine which subsets of F_i make this decidable? Σ_1^1-complete? Other?

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1. Our colorings were transitive.
2. **Transitive Ramsey Thm** is weaker than **Ramsey's Thm**.

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2. **Computability:** There exists a computable 2-coloring of K_{N} with no computable homog set (can even have no Σ_2 homog set). For every transitive computable c-coloring of K_{N} there exists a computable homog set (folklore).

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3. **Proof Theory:** Over the axiom system RCA_0, R implies TR, but TR does not imply R.

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2. Some subcases of **TERMINATION PROBLEM** are decidable. Of interest to **PL** and **Logic.**
3. Full strength of Ramsey not needed. Interest to **Logicians** and **Combinatorists.**