# Ramsey theory

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# What is Ramsey theory?

*«According to a 3500-year-old cuneiform text, an ancient Sumerian scholar once looked to the stars in the heavens and saw a lion, a bull and a scorpion. A modern astronomer would be more likely to describe a constellation as a temporary collection of stars, which we earthlings observe from the edge of an ordinary galaxy. Yet most stargazers would agree that the night sky appears to be filled with constellations in the shape of straight lines, rectangles and pentagons. Could it be that such geometric patterns arise from unknown forces in the cosmos?*

*Mathematics provides a much more plausible explanation.»*

*(1990, Graham R. L., Spencer J.H.)*

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# What is Ramsey theory?

- Frank Plumpton Ramsey (Cambridge, 1903-Londra, 1930) proved that patterns like those we see in the stars are always present in any large enough structure.
- Ramsey theory is an entire branch of mathematics whose purpose is to find ordered substructure in arbitrary structures.
- First contribution to Ramsey theory given by Ramsey in 1928; other contributions given by the Hungarian mathematicians Paul Erdös (Budapest, 1913-Warsaw, 1996) and George Szekeresh (Budapest, 1911-Adelaide, 2005).

# Ramsey Theory and graphs: a historical example


- William Gasarch, a computer scientist, studied the work of a scholar of Pre-Christian England, named Sir Woodsor Kneading, who observed a strange phenomenon.
- 577 b. C. : 5 lords in Wessex, all with armies. No war.
- A sixth lord settled and war broke out.
- 552 b. C. : 5 lords in what is now Sout Wales. No war.
- A sixth lord settled and war broke out.

- Phenomenon observed 42 times by Kneading.

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# Why do six lords always mean war?

One of the following holds:

1. There are 3, 4 or 5 allies, and they attack the other kingdoms.
2. There are 3 or more of them who are pairwise enemies, so a «pairwise war» breaks out.
3. (Rare case) All the 6 kingdoms form an alliance and there is peace.

**Conjecture:** whenever there are 6 lords, not all in alliance, there must either be 3, 4 or 5 lords allied one with each other, or 3, 4, 5 or 6 lords who are pairwise enemies.

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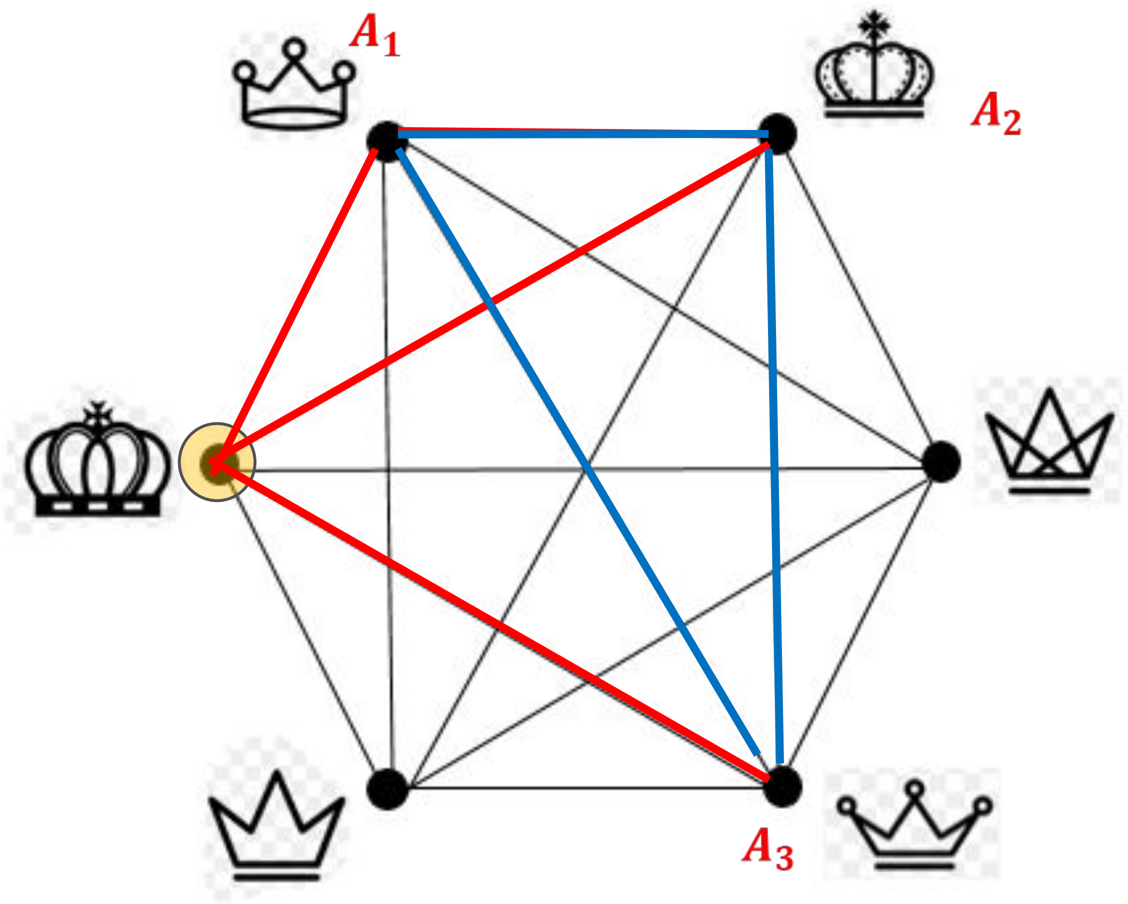
# Formalizing the problem in terms of graphs




- Complete graph ($K_6$), whose edges are all the possible «relationships» between the 6 lords.
- Select one lord (one vertex).
- Suppose there are at least 3 lords allied with the selected one. Colur with red the edges linking these allies. So the selected vertex has at least 3 incident red edges.
- **Case 1: for some $\boldsymbol{i,j \in \{1,2,3\}}$ $\boldsymbol{A_i, A_j}$ are allies.** Then we have (at least) 3 lords who are pairwise allied.
- **Case 2: all $\boldsymbol{A_i, A_j}$ are enemies.** Colour the edges between enemies with blue. Then $A_1A_2A_3$ are pairwise enemies.

# Formalizing the problem in terms of graphs

**Definition 1.** *A k-**edge colouring** of a graph G is a labeling* $f : E(G) \longrightarrow [k]$

The previous conjecture proves in fact the following result.

**Theorem 1.** *For every 2−colouring of* $K_6$*, there is a monochromatic (with respect to edge colourings) subgraph of at least 3 vertices.*

**Central question**

how big a structure has to be, in order to contain *ordered substructures*?

**Edge colourings**

Given an integer , I ask if there exists an integer  such that  contains a monochromatic , for any -edge coluring of .

**Cliques and independent sets**

Given two integers , I ask if there exists an integer  such that <u>every</u> graph  on at least  vertices contains either  or .

# Ramsey's answer

- Ramsey gave positive answer to this question, proving the following theorem, which here is stated in terms of cliques and independent sets.

**Theorem 2.** *Given any integers* $k, l \geq 0$ *there exists a smallest integer, denoted as* $R(k,l)$, *such that every graph on (at least)* $R(k,l)$ *vertices contains either a clique of* $k$ *vertices or an independent set of* $l$ *vertices.*

- The number $R(k,l)$ is the so called *Ramsey number*. If $k = l$, $R(k,k)$ is a *diagonal* Ramsey number.

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# Properties of Ramsey numbers

**Proposition 1.** *The following properties for Ramsey numbers hold:*

1. $R(1,l) = R(k,1) = 1$, *for every* $k, l \geq 0$;
2. $R(2,l) = l, R(k,2) = k$ *for every* $k, l \geq 0$;
3. $R(k,l) = R(l,k)$ *for every* $k, l \geq 0$;
4. *for every* $k, l \geq 2$

   $R(k,l) \leq R(k,l-1) + R(k-1,l)$;

   *strict inequality holds if* $R(k,l-1)$ *and* $R(k-1,l)$ *are both even;*
5. *for every* $k, l \geq 2$

   $R(k,l) \leq \binom{k+l-2}{k-1}$.

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*Proof.* (1), (2), (3) can be easily verified.

(4) Let $G$ be a graph on $R(k, l-1) + R(k-1, l)$ vertices. Let $v \in V(G)$.

The number of vertices adjacent to $v$, plus the number of vertices nonadjacent to $v$ is $R(k, l-1) + R(k-1, l) - 1$.

**Case 1** There exists a set of vertices $S$ of at least $R(k, l-1)$ elements, to which $v$ is nonadjacent;

**Case 2** there exists a set of vertices $T$ of at least $R(k-1, l)$ elements, to which $v$ is adjacent.

If **case 1** holds, $G[S]$ contains either a $k$-clique or an $l-1$-independent set.

Then $G[S \cup v]$ contains either a $k$-clique or an $l$-independent set.

If **case 2** holds, $G[T]$ contains either a $k-1$-clique or an $l$-independent set.

Then $G[T \cup v]$ contains either a $k$-clique or an $l$-independent set.

In any case I can find either a $k$-clique or an $l$-independent set.

Then $R(k, l) \leq R(k, l-1) + R(k-1, l)$.

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Now suppose $R(k, l-1)$ and $R(k-1, l)$ are both even.

Consider a graph $G$ on $R(k, l-1) + R(k-1, l) - 1$ vertices.

The number of vertices is odd, so there exists a vertex $v$ of even degree.

Observe that $v$ cannot be adjacent to $R(k-1, l) - 1$ vertices, or to $R(k, l-1) - 1$ vertices, hence **case 1** or **case 2** must hold.

Repeating the argument used before, I can find a $k$-clique or an $l$-independent set in $G$, thus proving the claim.

**Case 1** There exists a set of vertices $S$ of at least $R(k, l-1)$ elements, to which $v$ is nonadjacent;

**Case 2** there exists a set of vertices $T$ of at least $R(k-1, l)$ elements, to which $v$ is adjacent.

(5) We prove the claim by induction on $k + l$.

Observe it holds if $k + l \leq 5$.

Suppose the claim is true for every $k, l$ such that $5 \leq k + l < m + n$.

Let's show it holds for $m + n$.

Apply (4) to $R(m, n)$, then apply the inductive hypothesis and compute.

$R(m, n) \leq R(m, n-1) + R(m-1, n) \leq \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} = \binom{m+n-2}{m-1}$

□

**Corollary 1.** *For all* $k, l \geq 0$, $R(k, l) \leq 2^{k+l-2}$.

Ramsey numbers are in general difficult to calculate. Only few of them have been found, with the help of the previous properties and of some special graphs, called *Ramsey graphs*.

| $k$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $\ell$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 4 | 5 |
| $r(k,\ell)$ | 6 | 9 | 14 | 18 | 23 | 28 | 36 | 18 | 25 |

Ramsey numbers known to date (Bondy J.A., Murty U.S.R, 2008)

**Definition 2.** *A* $(k,l)$-***Ramsey graph*** *is a graph on* $R(k,l)-1$ *vertices, containing no* $k$*-cliques and no* $l-$*independent sets.*

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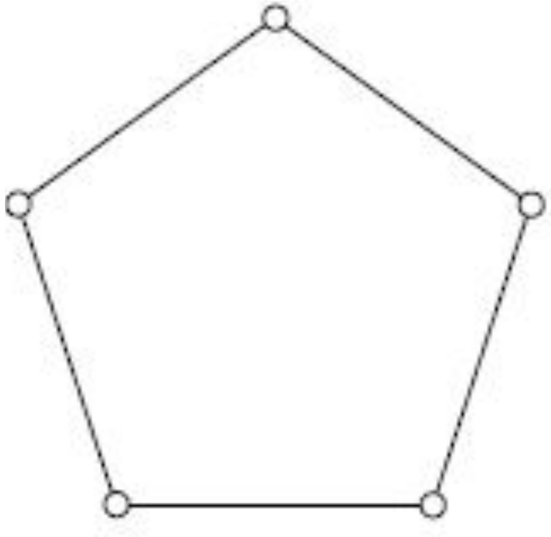
# Example: proof of $R(3,3)$ with the help of the (3,3)-Ramsey graph

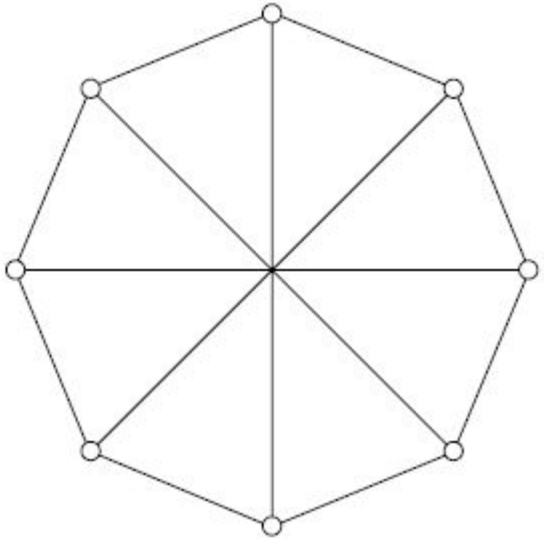
- The graph contains no *3*-cliques and no 3-independent sets.

  So $R(3,3) \geq 6$.

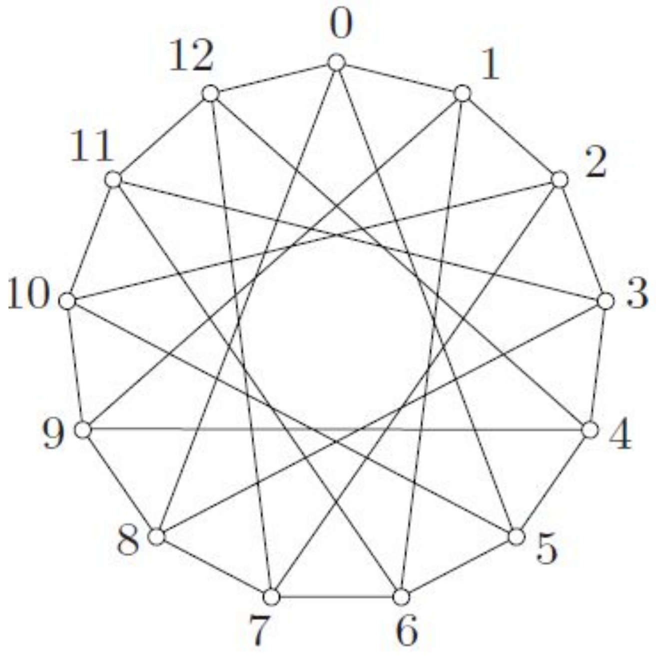
- By *property (4)* we have $R(3,3) \leq R(2,3) + R(3,2) = 3 + 3 = 6$.
- Conclusion: $R(3,3) = 6$.



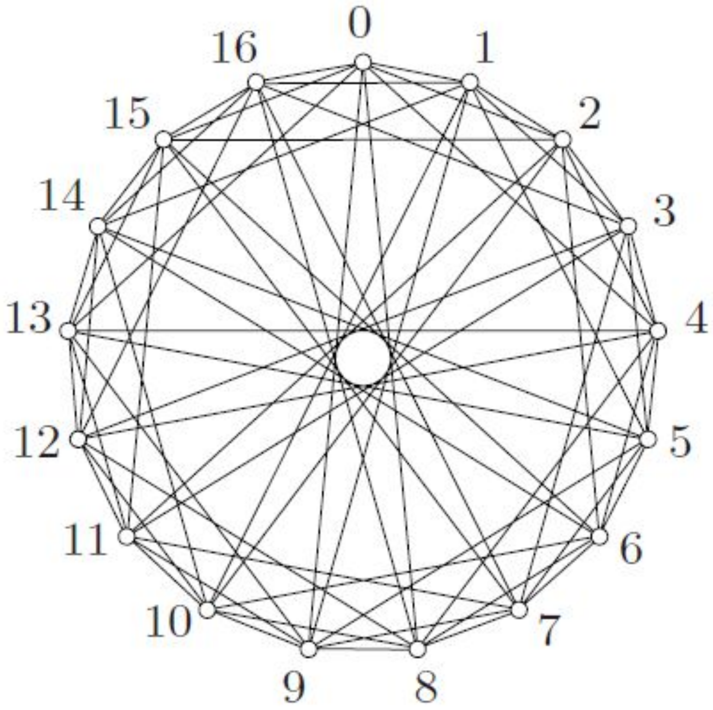
(3,3)-Ramsey graph. (Bondy J.A., Murty U.S.R, 2008)



A (3,4)-Ramsey graph




A (3,5)-Ramsey graph




A (4,4)-Ramsey graph

# Ramsey's result for diagonal Ramsey numbers

**Theorem 3** (Ramsey, 1930). *For every integer $r \geq 0$ there exists an integer $n \geq 0$ such that every graph on at least $n$ vertices cointains either $K_r$ or $\bar{K}_r$.*

*Proof.* Suppose $r \geq 1$, otherwise the assertion is trivial.

Define $n = 2^{2r-3}$ and consider a graph $G$ with $|V(G)| \geq n$.

We construct a sequence of subsets of $V(G)$, say $V_1, V_2, \ldots, V_{2r-2}$, and choose $v_1 \in V_1, \ldots, v_{2r-2} \in V_{2r-2}$ satisfying the following properties:

(i) $|V_i| = 2^{2r-2-i}$, for $1 \leq i \leq 2r-2$;

(ii) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$, for $2 \leq i \leq 2r-2$;

(iii) $v_{i-1}$ is adjacent either to all the vertices of $V_i$, or to no vertex of $V_i$, for $2 \leq i \leq 2r-2$.

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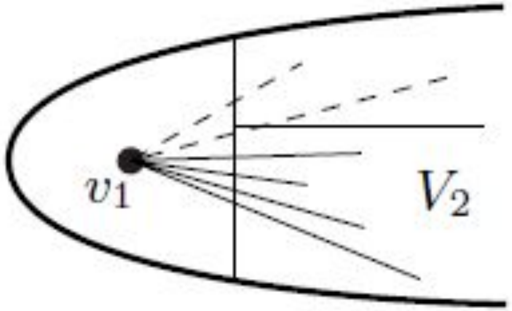
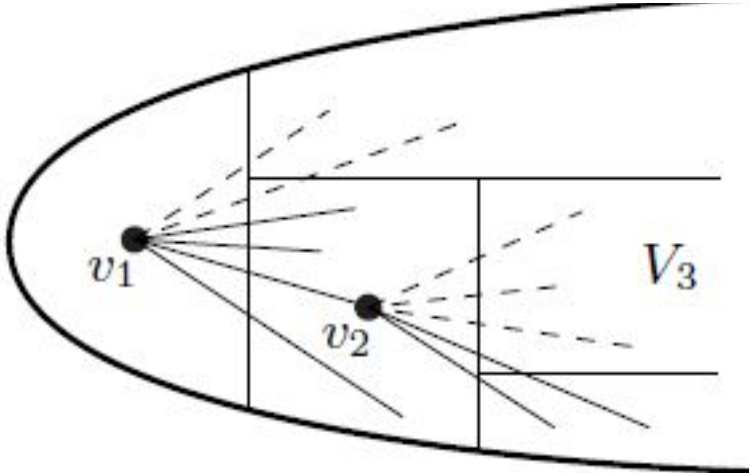
Construction of the sequence with the required properties. (Diestel, 1997)




(i) $|V_i| = 2^{2r-2-i}$, for $1 \leq i \leq 2r-2$;

(ii) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$, for $2 \leq i \leq 2r-2$;

(iii) $v_{i-1}$ is adjacent either to all the vertices of $V_i$, or to no vertex of $V_i$, for $2 \leq i \leq 2r-2$.

Such a sequence can be constructed inductively.

Start by choosing $V_1 \subseteq V(G)$, $|V_1| = n$. Pick $v_1$ arbitrarily.

Suppose to have constructed $V_1, \ldots, V_{i-1}$ for some $i$. Let's show I can define $V_i$.

Choose $v_{i-1} \in V_{i-1}$;
by $(i)$, $|V_{i-1} - \{v_{i-1}\}| = 2^{2r-2-(i-1)} - 1 = 2^{2r-i-1} - 1$.

$|V_{i-1} - \{v_{i-1}\}|$ is an odd number, then:
**either** $v_{i-1}$ is adjacent to more than a half of the vertices in $V_{i-1}$,

**or** $v_{i-1}$ is adjacent to less than a half of the vertices in $V_{i-1}$.

In both cases I can select $V_i \subseteq V_{i-1} - \{v_{i-1}\}$, satisfying $(i), (ii), (iii)$.

We construct a sequence of subsets of $V(G)$, say $V_1, V_2, \ldots, V_{2r-2}$, and choose $v_1 \in V_1, \ldots, v_{2r-2} \in V_{2r-2}$ satisfying the following properties:

(i) $|V_i| = 2^{2r-2-i}$, for $1 \leq i \leq 2r-2$;

(ii) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$, for $2 \leq i \leq 2r-2$;

(iii) $v_{i-1}$ is adjacent either to all the vertices of $V_i$, or to no vertex of $V_i$, for $2 \leq i \leq 2r-2$.

(iii) $v_{i-1}$ is adjacent either to all the vertices of $V_i$, or to no vertex of $V_i$, for $2 \leq i \leq 2r-2$.

Pick any $v_i$. Observe that $2r-2$ is even.

Among $v_1, \ldots, v_{2r-3}$ there is a set of $r-1$ vertices with same behaviour in the sense of $(iii)$.

Together with $v_{2r-2}$ they induce either a $K_r$ or a $\bar{K}_r$.
To understand this, observe that for every $i$, $v_i, v_{i+1}, \ldots, v_{2r-2} \in V_i$.

□

Observe that the theorem gives an upper bound for diagonal Ramsey numbers, i. e.

# A lower bound for Ramsey numbers

**Theorem 4** (Erdös, 1947). *For all* $k \geq 2$

$$R(k,k) \geq 2^{\frac{k}{2}}.$$

*Proof.* The case $k = 2$ is trivial.
If $k = 3$, $6 = R(3,3) \geq 2^{\frac{3}{2}}$.

Suppose $k \geq 4$.

We will show with a probabilistic method that there exists a simple graph on $2^{\frac{k}{2}}$ vertices with no $k$-cliques and no $k-$independent sets.

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**Claim:** $\mathbb{P}$(a simple graph on $n$ vertices contains at least a $K_k$ or a $\bar{K_k}$) $< 1$.

Let $n \leq 2^{\frac{k}{2}}$.

In a simple graph we have that $\mathbb{P}$(a selected set of $k$ elements forms a $K_k$ or a $\bar{K_k}$) $= \frac{2}{2^{\binom{k}{2}}}$.

Hence $\mathbb{P}$(a simple graph on $n$ vertices contains at least a $K_k$ or a $\bar{K_k}$) $\leq \binom{n}{k}\frac{2}{2^{\binom{k}{2}}}$.

Now, $\binom{n}{k}\frac{2}{2^{\binom{k}{2}}} = \binom{n}{k}2^{1-\binom{k}{2}} = 2^{1-\binom{k}{2}}\frac{n!}{k!(n-k)!} < \frac{2n^k}{k!2^{\binom{k}{2}}}$.

Computing $\binom{k}{2}$, and remembering that $n \leq 2^{\frac{k}{2}}$ we obtain

$\frac{2n^k}{k!2^{\binom{k}{2}}} \leq \frac{2^{\frac{k}{2}+1}}{k!} < 1$ (for $k! > 2^k$ for $k \geq 4$). This proves our **claim**.

Then, there must exist a graph on at most $2^{\frac{k}{2}}$ vertices with no $k$-cliques and no $k$-independent sets.

So $R(k,k) > 2^{\frac{k}{2}}$.

$\square$

# Conclusions

*«Yet workers have only begun to explore the implications of the theory. It suggests that much of the essential structure of mathematics consists of extremely large numbers and sets, objects so huge that they are difficult to express, much less understand.*
*As we learn to handle these large numbers, we may find mathematical relations that help engineers to build large communications networks or help scientists to recognize patterns in large-scale physical systems.»*

*(1990, Graham R. L., Spencer J. H.)*

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