On the proper rainbow saturation numbers of cliques, paths, and odd cycles

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Abstract

Given a graph H, we say a graph G is *properly rainbow H-saturated* if there is a proper edge-coloring of G which contains no rainbow copy of H, but adding any edge to G makes such an edge-coloring impossible. The *proper rainbow saturation number*, denoted $\text{sat}^*(n, H)$, is the minimum number of edges in an n-vertex rainbow H-saturated graph. We determine the proper rainbow saturation number for paths up to an additive constant and asymptotically determine $\text{sat}^*(n, K_4)$. In addition, we bound $\text{sat}^*(n, H)$ when H is a larger clique, tree of diameter at least 4, or odd cycle.

1 Introduction

For a fixed graph H, how many edges can a graph G on n vertices have if it does not contain H as a subgraph? We say that G is *H-saturated* if G contains no copy of H, but for any $x, y \in V(G)$ with $xy \notin E(G)$, the graph $G+xy$ on vertex set $V(G)$ with edge set $E(G) \cup \{xy\}$ contains a copy of H.

A classical question in extremal combinatorics asks for the maximum number of edges in an n-vertex H-saturated graph. This is called the *Turán number* $\text{ex}(n, H)$, and has been extensively studied following work of Mantel [4] and Turán [11] determining $\text{ex}(n, K_t)$ for $t \geq 3$. Also of interest is the other extremal case, in which we seek the minimum number of edges in an n-vertex H-saturated graph, called the *saturation number* $\text{sat}(n, H)$. The study of the saturation number was initiated by work of Zykov [12] and independently Erdős, Hajnal, and Moon [3]. Many different generalizations of these two problems have been studied over the years, including analogous questions in the setting of edge-colored graphs.

An edge-coloring $c : E(G) \to C$ is *proper* if $c(e) \neq c(f)$ for all incident edges e, f and *rainbow* if $c(e) \neq c(f)$ for all edges $e, f \in E(G)$. Given graphs G and H and a proper edge-coloring c of G, we say that G is *rainbow H-free under c* if G contains no copy of H

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which is rainbow under the coloring c. A graph G is *(properly) rainbow H-saturated* if the following two conditions hold:

1. There exists a proper edge-coloring c of G such that G is rainbow H-free under c, and
2. For any edge $e \notin E(G)$, any proper edge-coloring of $G + e$ contains a rainbow copy of H.

Keevash, Mubayi, Sudakov, and Verstraëte [8] introduced the *rainbow extremal number* $\mathrm{ex}^*(n, H)$ which is the maximum number of edges in an n-vertex rainbow H-saturated graph. More recently, Bushaw, Johnston, and Rombach [2] initiated the study of the *proper rainbow saturation number* $\mathrm{sat}^*(n, H)$ which is the minimum number of edges in an n-vertex rainbow H-saturated graph.

It is worth noting that other versions of rainbow saturation problems have been considered as well. For example, Behague, Johnston, Letzter, Morrison, and Ogden [1] explored the problem without the restriction that colorings be proper. Another variant, introduced by Hanson and Toft [6], requires that colorings, while not necessarily proper, are restricted to a set of only t colors.

Throughout the remainder of the paper, we will exclusively study proper rainbow saturation problems; thus we drop the qualifier "proper" as this is assumed.

There are few graphs for which the rainbow saturation number is known asymptotically, namely $\mathrm{sat}^*(n, P_4)$ determined by Bushaw, Johnston, and Rombach [2] and $\mathrm{sat}^*(n, C_4)$ determined by Halfpap, Lidický, and Masařík [5]. Adding to this list, we determine $\mathrm{sat}^*(n, K_4)$ asymptotically.

Theorem 1.1. *Let $\frac{1}{2} > \alpha > 0$. For any n such that $\alpha^2 n \geq 7$ and $\alpha n > 220$, we have*

$$\frac{7}{2}n - 8\alpha n \leq \mathrm{sat}^*(n, K_4) \leq \frac{7}{2}n + O(1).$$

We also give bounds on the rainbow saturation number for larger cliques.

Our next main result is to determine the rainbow saturation numbers for paths up to an additive constant. Throughout the paper, we use P_k to denote the path on k vertices (with $k - 1$ edges).

Theorem 1.2. *For $k \geq 5$ and $n \geq (k-1)2^{k-4}$, we have*

$$n - 1 \leq \mathrm{sat}^*(n, P_k) \leq n + O(2^k).$$

In particular, taking k fixed relative to n, we have that $\mathrm{sat}^*(n, P_k)$ is asymptotically equal to n. We also remark that the lower bound of Theorem 1.2 can be extended to a larger class of graphs, showing that $\mathrm{sat}^*(n, T) \geq n - 1$ for any tree T of diameter at least 4.

Finally, we consider the rainbow saturation number of cycles. Recently, Halfpap, Lidický, and Masařík [5] proved that $\mathrm{sat}^*(n, C_4) = \frac{11}{6}n + o(n)$. They also gave upper bounds for other short cycles, showing that $\mathrm{sat}^*(n, C_5) \leq \left\lfloor \frac{5n}{2} \right\rfloor - 4$ for $n \geq 9$ and that $\mathrm{sat}^*(n, C_6) \leq \frac{7}{3}n + O(1)$. In the following theorem, we give an upper bound for the rainbow saturation number for longer odd cycles. For context, no disconnected graph can be rainbow C_k-saturated for any k, so we trivially have $n - 1 \leq \mathrm{sat}^*(n, C_k)$ for all k.

Theorem 1.3. *For odd $k \geq 7$ and for $n \geq 3k-2$, we have*

$$\operatorname{sat}^*(n, C_k) \leq \left(\frac{k-1}{2}\right) n - \binom{\frac{k+1}{2}}{2}.$$

The remainder of the paper is organized as follows. In Section 2, we prove Theorem 1.1 and bounds for larger cliques. In Section 3, we prove Theorem 1.2. Finally, in Section 4, we prove Theorem 1.3.

Remark. While completing this paper, we learned that Lane and Morrison simultaneously and independently derived partially-overlapping results for the proper rainbow saturation number of various graphs. Among other results in [9], they gave the upper bound (but not the lower bound) on $\operatorname{sat}^*(n, K_4)$ in Theorem 1.1 as well as more general upper bounds for larger cliques. They also obtained the upper bound on $\operatorname{sat}^*(n, C_k)$ for odd k in Theorem 1.3 and gave a similar upper bound for even k. In [10], they used a similar approach to ours to show that $\operatorname{sat}^*(n, P_k) = n + O(k)$ and studied other families of trees such as brooms and caterpillars which we did not consider.

1.1 Notation and Preliminary Definitions

Throughout the paper, we will use the following notation. We denote the *degree* of a vertex $v \in V(G)$ by $d(v)$ and the minimum vertex degree of a graph G by $\delta(G)$. For a vertex $v \in V(G)$ we use $N(v)$ to denote the *neighborhood* of v by $N(v) := \{u \in V(G) : uv \in E(G)\}$ so that $d(v) = |N(v)|$. We also use $N[v]$ to denote the *closed neighborhood* of v, that is, $N[v] := N(v) \cup \{v\}$. Given graphs G and H, let $G \vee H$ denote the *join* of G and H which has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. Let E_n denote the empty graph on n vertices. Given two sets A, B we denote by $A \Delta B$ the *symmetric difference*, that is $A \Delta B = \big(A \setminus B\big) \cup \big(B \setminus A\big)$. In a graph G, the *distance* between two vertices u and v, denoted $d(u, v)$, is the length of the shortest uv-path in G. The *diameter* of G is the length of a longest shortest path in G, i.e., $\max\limits_{u,v \in V(G)} d(u, v)$.

Often, we will wish to modify one edge-coloring of some graph G to obtain another edge-coloring. Formally, if c is an edge-coloring of G and $E' \subseteq E(G)$, we *recolor* E' by selecting edge-colors $c_i \neq c(e_i)$ for each $e_i \in E'$, and defining a new edge-coloring

$$c'(e) = \begin{cases} c(e) & \text{if } e \notin E', \\ c_i & \text{if } e_i \in E'. \end{cases}$$

In a slight abuse of notation, when we modify edge-colorings in this way, we will refer to both the original and the new edge-colorings as c. Given a recoloring of E' and $e_i \in E'$, we say that $c(e_i)$ is a *new color for* c if $c(e_i) \neq c(e)$ for all $e \notin E'$.

Given a forbidden graph F, we will in particular wish to modify one rainbow F-free edge-coloring c of G to obtain another. We say $E' \subseteq E(G)$ is *unrestricted* relative to a rainbow F-free edge-coloring c if G remains rainbow F-free under any recoloring of E'. Typically, c and F will be clear from context, and we will simply refer to such an edge set as unrestricted. In the case that $E' = \{e\}$, we simply say that the edge e is unrestricted.

2 Cliques

2.1 Bounds for general cliques

To begin this section, we correct a proof from [2], which states that for any $r \geq 4$, we have $\mathrm{sat}^*(n, K_r) \geq (r-1)n + O(1)$.

Lemma 2.1. *If G is a rainbow K_r-saturated graph, then for any $u, v \in V(G)$ with $uv \notin E(G)$, one of the following holds:*

(1) $|N(u) \cap N(v)| \geq r - 1$,

(2) $d(u) + d(v) \geq \binom{r}{2} - 1$.

Proof. Let G be a rainbow K_r-saturated graph, c be a proper edge-coloring of G containing no rainbow copy of K_r, and $u, v \in V(G)$ with $uv \notin E(G)$. Suppose $|N(u) \cap N(v)| < r - 1$. Note that the common neighborhood of any two nonadjacent vertices in a rainbow K_r-saturated graph has size at least $r-2$, so we may assume $|N(u) \cap N(v)| = r - 2$. Let $G' = G[\{u, v\} \cup (N(u) \cap N(v))]$. Then, G' must be a copy of K_r^- (that is, K_r with the edge uv removed) which is rainbow under the coloring c, as otherwise, we may add the edge uv to G and extend c without introducing a rainbow copy of K_r. Furthermore, if $|N(u) \triangle N(v)| < \binom{r-2}{2}$, then there exists an edge $xy \in G[N(u) \cap N(v)]$ such that $c(xy)$ does not appear on any edges incident to u or v. In this case, we may add the edge uv to G and assign it the color $c(xy)$, contradicting that G is rainbow K_r-saturated. Therefore, it must be the case that $|N(u) \triangle N(v)| \geq \binom{r-2}{2}$. Thus, we have

$$\begin{aligned} d(u) + d(v) &= |N(u) \triangle N(v)| + 2|N(u) \cap N(v)| \\ &\geq \binom{r-2}{2} + 2(r-2) \\ &= \binom{r}{2} - 1. \end{aligned}$$

□

Lemma 2.2. *Let $r \geq 4$. If G is a rainbow K_r-saturated graph, then G has at most one vertex of degree $r-2$.*

Proof. Let G be a rainbow K_r-saturated graph and c be a proper edge-coloring of G with no rainbow K_r. Suppose for contradiction that G has two vertices u, v of degree $r-2$.

Note that if $uv \notin E(G)$, then since G is rainbow K_r-saturated, it must be the case that $N(u) = N(v)$ and $G[N(u) \cup \{u, v\}]$ is a rainbow copy of K_r^- under c. But then, we may add the edge uv to G and color this edge with any color appearing in $G[N(u)]$ without creating a rainbow copy of K_r, a contradiction.

Therefore, we must have $uv \in E(G)$. However, in this case, adding any new edge to u does not create a copy of K_r, a contradiction. □

Proposition 2.3. *Let $r \geq 4$ and $t \geq r-1$. If G is a rainbow K_r-saturated graph with $\delta(G) = t$, then*

$$e(G) \geq \left(\frac{r+t-2}{2}\right) n + O(1).$$

Proof. Let G be a rainbow K_r-saturated graph with minimum degree $t \geq r-1$. Let $u \in V(G)$ with $d(u) = t$. For any $v \in V(G) \setminus N[u]$, because G is rainbow K_r-saturated, then u and v have at least $r-2$ common neighbors and there exists a K_{r-2} in $G[N(u) \cap N(v)]$. Furthermore, v has $d(v) - |N(u) \cap N(v)|$ neighbors outside of $N(u)$. Since this is true for any vertex not incident with u, we have:

$$\begin{aligned}
e(G) &= e(N[u], V(G) \setminus N[u]) + e(G[V(G) \setminus N[u]]) + e(G[N[u]]) \\
&\geq (r-2)(n-t) + \frac{1}{2}(t-(r-2))(n-t-1) + \binom{r-2}{2} \\
&= \left(r - 2 + \frac{t-(r-2)}{2}\right) n + O(1) \\
&= \left(\frac{t+r-2}{2}\right) n + O(1).
\end{aligned}$$

□

Together with Theorem 1.1, Proposition 2.4 allows us to recover the rainbow saturation lower bound for cliques as stated in [2].

Proposition 2.4. *For $r \geq 5$, $sat^*(n, K_r) \geq (r-1)n + O(1)$.*

Proof. Let G be a rainbow K_r-saturated graph. If $\delta(G) \geq r$, we get the desired bound by Proposition 2.3. So, we may assume that $\delta := \delta(G) \in \{r-2, r-1\}$. Let u be a vertex of minimum degree and let $G' = G[V(G) \setminus N[u]]$. By Lemma 2.1, for $v \notin N[u]$, either $|N(u) \cap N(v)| \geq r-1$ or $d(u) + d(v) \geq \binom{r}{2} - 1$.

Let $T_1 = \{v \in V(G) \setminus N[u] : |N(u) \cap N(v)| \geq r-1\}$ and let $T_2 = V(G) \setminus (N[u] \cup T_1)$. Then, for all $v \in T_2$, $d(v) \geq \binom{r}{2} - 1 - \delta$. Furthermore, since for all $v \in T_2$, $|N(u) \cap N(v)| = r-2$, we obtain $d_{G'}(v) \geq \binom{r}{2} - 1 - \delta - (r-2)$. Therefore, we have:

$$\begin{aligned}
e(G) &\geq (r-1)|T_1| + (r-2)|T_2| + \frac{1}{2}|T_2| \left(\binom{r}{2} - 1 - \delta - (r-2)\right) + O(1) \\
&= (r-2)(|T_1| + |T_2|) + |T_1| + \frac{|T_2|}{2} \left(\binom{r}{2} - 1 - \delta - (r-2)\right) + O(1) \\
&\geq (r-2)(|T_1| + |T_2|) + |T_1| + \frac{|T_2|}{2} \left(\binom{r-2}{2} - 1\right) + O(1) \\
&\geq (r-2)(|T_1| + |T_2|) + |T_1| + \frac{|T_2|}{2} (2) + O(1) \\
&= (r-1)n + O(1).
\end{aligned}$$

□

We give an upper bound construction. The key ingredient to this construction is Lemma 2.5. Given a graph G, a subgraph H of G, an edge-coloring c_H of H, and an edge-coloring c_G of G. We say that c_H *extends* to c_G if for any edge $e \in E(H)$ we have $c_H(e) = c_G(e)$, in which case we call c_G the *extension of c_H to G*.

Lemma 2.5. *Let $G = K_r$ and let $c : E(G) \to \mathbb{N}$ be a rainbow edge-coloring of G. Let $H = E_{r\binom{r-1}{2}+1}$ and set $G' := G \vee H$. Then, for any extension of c to G' that is a proper edge-coloring, there exists some vertex $v \in V(H)$ such that $G'[V(G) \cup \{v\}]$ is a rainbow K_{r+1} under c.*

Proof. Let G' be as described and let $c : E(G') \to \mathbb{N}$ be a proper edge-coloring of G' such that $G'[V(G)]$ is a rainbow K_r. We will call a vertex $v \in V(H)$ bad if $G'[V(G) \cup \{v\}]$ is not a rainbow K_{r+1}. For each $u \in V(G)$, there are at most $\binom{r-1}{2}$ vertices in $V(H)$ that may repeat a color appearing on an edge of $G - u$. Since $|V(G)| = r$, then there are at most $r\binom{r-1}{2}$ bad vertices of H. Since $|V(H)| \geq r\binom{r-1}{2} + 1$, we are guaranteed to have at least one vertex which is not bad. □

The following construction allows us to obtain an explicit upper bound on the rainbow saturation for all cliques on at least four vertices. In a way, it refines the general upper bound proof in [2] when restricted to cliques.

Construction 1. *Let $r \geq 3$ and let $n \geq r\binom{r-1}{2} + 2 + \sum_{i=3}^{r} \left(i\binom{i-1}{2} + 1\right)$ and set $n' := n - 1 + \sum_{i=3}^{r} \left(i\binom{i-1}{2} + 1\right)$. Now, let $G(r, n)$ be defined as follows:*

$$G(r,n) = E_1 \vee \left(\bigvee_{i=3}^{r} E_{r\binom{r-1}{2}+1} \right) \vee E_{n'}.$$

We will call the E_i's the parts of $G(r, n)$ and we will call the part of size n' the leftover part of $G(r, n)$. Observe that $G(r, n)$ is a complete r-partite graph (and therefore, K_{r+1}-free).

Proposition 2.6. *Let $r \geq 3$ and $n \geq r\binom{r-1}{2} + 2 + \sum_{i=3}^{r} \left(i\binom{i-1}{2} + 1\right)$. Then, $G(r, n)$ as defined in Construction 1 is properly rainbow K_{r+1}-saturated.*

Proof. We prove this by induction on r. To begin, let $r = 3$ and n large enough. As mentioned before, since $G(3, n)$ is 3-partite, then it is clearly rainbow K_4-free for any proper edge-coloring. Now, suppose we add an edge e to $G(3, n)$ and let c be any proper edge-coloring of $G(3, n) + e$. Because $G(3, n)$ is a complete 3-partite graph, then e must be contained within one of the parts of size greater than 1. In this case, either e is contained in the part of size 4 or the leftover part, which has size at least 4. In either case, e is contained in a triangle with the part of size 1. Finally, since any proper-edge coloring of a triangle leads to a rainbow triangle, then this triangle together with the part of size greater than 1 not containing e must contain a rainbow K_4 by Lemma 2.5.

Next, suppose $r \geq 4$ and let n be large enough (as defined in Construction 1). Suppose we have show that $G(r - 1, m)$ is properly rainbow K_r-saturated for all m large enough. We claim that $G(r, n)$ is properly rainbow K_{r+1}-saturated. As mentioned above, $G(r, n)$ is r-partite and therefore, clearly rainbow K_r-free for any proper edge-coloring. Denote the part of size $r\binom{r-1}{2} + 1$ by S_r and denote the leftover part by S_ℓ. Let $n' = |S_\ell|$ and let $n_1 = n - |S_r|$, $n_2 = n - n'$. Observe that $G_1 := G(r, n)[V(G(r, n)) \setminus S_r] = G(r - 1, n_1)$ and $G_2 := G(r, n)[V(G(r, n)) \setminus S_\ell] = G(r - 1, n_2)$. Now, suppose we add an edge uv to $G(r, n)$ and let c be any proper edge coloring of $G(r, n) + uv$. Observe that the vertices u and v

must be contained in at least one of $V(G_1)$ or $V(G_2)$. Without loss of generality, suppose u and v are contained in $V(G_1)$. Then, since $G_1 = G(r-1, n_1)$, by induction, G_1 contains a rainbow copy of K_r, K. Now, by Lemma 2.5, $K \vee S_r$, a subgraph of $G(r, n)$ must contain a rainbow copy of K_{r+1}.

As $n \to \infty$, the number of edges of $G(r, n)$ not incident to a vertex in the leftover part is some constant. Therefore, we have:

$$\begin{aligned}\lim_{n\to\infty} e(G(r,n)) &= \left(\sum_{i=2}^{r} r\binom{r}{2} + 1\right) n + O(1)\\ &= \left(\frac{r^4 - 2r^3 - r^2 + 10r - 8}{8}\right) n + O(1)\\ &\leq \left(\frac{r^4}{8}\right) n + O(1).\end{aligned}$$

□

Corollary 2.7. *Let $r \geq 3$. Then, for n large enough, we have:*

$$\text{sat}^*(n, K_{r+1}) \leq \left(\frac{r^4}{8}\right) n + O(1).$$

2.2 Proof of Theorem 1.1

In the case of K_4, we can obtain asymptotically tight bounds. Before we provide an upper bound construction for $\text{sat}^*(n, K_4)$, we make some observations which will be repeatedly useful.

Observation 2.8. *Suppose G is a properly rainbow K_4-saturated graph, and $xy \notin E(G)$. Then $N(x) \cap N(y)$ contains an edge.*

Proof. If G is properly rainbow K_4-saturated, then G has a rainbow K_4-free proper edge coloring c, but there is no rainbow K_4-free proper edge coloring of $G + xy$. In particular, we can properly edge-color $G + xy$ by coloring $E(G)$ according to c and adding edge xy in a color not appearing in c, so this proper edge-coloring of $G + xy$ contains a rainbow K_4-copy. Since G is rainbow K_4-free under c, any rainbow K_4-copy in the described coloring must include xy. In particular, xy is contained in some K_4-copy, say on $\{x, y, u, v\}$, and uv is an edge in $N(x) \cap N(y)$. □

Observation 2.9. *Suppose vertices x, y, z form a K_3-copy, and let $V_j = \{v_1, v_2, \ldots, v_j\}$ be a set of $j \geq 1$ vertices disjoint from $\{x, y, z\}$. Under any rainbow K_4-free proper edge-coloring of $\{x, y, z\} \vee V_j$, each $v_i \in V_j$ is adjacent to one of x, y, z via an edge with color in $\{c(xy), c(xz), c(yz)\}$.*

Proof. Under any proper edge-coloring of $\{x, y, z\} \vee V_j$, the K_3 on $\{x, y, z\}$ must be rainbow. Moreover, for any $v_i \in V_j$, the edges xv_i, yv_i, zv_i must receive distinct colors in any proper edge-coloring. Thus, if the K_4-copy on $\{x, y, z, v_i\}$ is not rainbow, then

$$\{c(xy), c(xz), c(yz)\} \cap \{c(xv_i), c(yv_i), c(zv_i)\} \neq \emptyset.$$

□

Observation 2.9 has two particular implications. Note that in a proper edge-coloring, $c(xy) \notin \{c(xv_i), c(yv_i)\}$, so if $c(xy) \in \{c(xy), c(xz), c(yz)\} \cap \{c(xv_i), c(yv_i), c(zv_i)\}$, then $c(zv_i) = c(xy)$. Since z is incident to at most one edge of color $c(xy)$, this implies that for at most one $v_i \in V_j$ do we have $c(xy) \in \{c(xy), c(xz), c(yz)\} \cap \{c(xv_i), c(yv_i), c(zv_i)\}$. Thus, as a consequence of Observation 2.9, any properly rainbow K_4-saturated graph is $K_3 \vee E_4$ free. Moreover, if G is a properly edge-colored, rainbow K_4-free graph containing a copy of $K_3 \vee E_3$, then the edge-colors used in this K_3-copy must be repeated on a matching between $V(K_3)$ and $V(E_3)$. Using Observation 2.9, we can now quickly show that Construction 2 is properly rainbow K_4-saturated. This construction will provide the upper bound in Theorem 1.1.

Construction 2. *For $n \geq 6$, consider the n-vertex graph $G(n)$ obtained as follows. Let $G'(n)$ be the $n-2$ vertex graph on $k := \lceil \frac{n-2}{4} \rceil$ components $C_1, C_2, \ldots, C_k$, with $C_i = K_4$ if $i \leq \lfloor \frac{n-2}{4} \rfloor$. We label the vertices of C_i as $\{v_{i,j} : 1 \leq j \leq 4\}$. If $n-2 \equiv m \mod 4$ with $m \neq 0$, then let $C_k = K_m$, and label the vertices of C_k as $\{v_{k,j} : 1 \leq j \leq m\}$. We take $G(n) = K_2 \vee G'(n)$, labeling the vertices of the K_2-copy joined to $G'(n)$ as x, y.*

We edge-color $G(n)$ as follows. Color $G(n)[C_1 \cup \{x, y\}]$ with five colors using the perfect matching decomposition of a K_6. Suppose $c(xy) = 0$ in this coloring and the other colors used are $\{1, 2, 3, 4\}$. For $1 < i \leq \lfloor \frac{k-2}{4} \rfloor$, color the K_6 induced by $C_i \cup \{x, y\}$ the exact same (most importantly, ensuring $c(xy) = 0$), switching colors $\{1, 2, 3, 4\}$ with colors $\{4i+1, 4i+2, 4i+3, 4i+4\}$. Finally, if $|V(C_k)| < 4$, we color the edges incident to $V(C_k)$ in any legal fashion (using at most 4 new colors) which avoids a rainbow K_4-copy on $V(C_k) \cup \{x, y\}$.

We illustrate the described coloring of an edge xy and the components C_1 and C_2 in Figure 1. The proof of Proposition 2.10 yields the upper bound of Theorem 1.1.

Proposition 2.10. *Construction 2 is properly rainbow K_4-saturated for all $n \geq 6$.*

Proof. Let c be the edge-coloring of $G(n)$ described in Construction 2. Since there are no edges between any C_i, then any possible rainbow K_4 under c must be contained in the graph induced by $C_i \cup \{x, y\}$ for some $1 < i \leq \lfloor \frac{k-2}{4} \rfloor$. However, for any i, $G(n)[C_i \cup \{x, y\}]$ contains at most 5 colors, ensuring that no rainbow K_4 can exist.
Now, suppose we add an edge $v_{i,j}v_{i',j'}$ to $G(n)$. Without loss of generality, suppose $i \leq \lfloor \frac{n-2}{4} \rfloor$. In particular, $|C_i| = 4$. Then, observe that $v_{i,j}, x, y$ form a triangle in $G(n)$ and the vertices $v_{i',j'}, v_{i,j+1}, v_{i,j+2}, v_{i,j+3}$ (where the second indices are taken mod 4) are joined to every vertex of the triangle by an edge in $G(n) + v_{i,j}v_{i',j'}$. That is, $G(n) + v_{i,j}v_{i',j'}$ contains a copy of $K_3 \vee E_4$. Therefore, by Observation 2.9, $G(n) + v_{i,j}v_{i',j'}$ contains a rainbow K_4 under any coloring. □

We now turn to proving the lower bound in Theorem 1.1. We begin with a lemma which greatly restricts the number and behavior of very low-degree vertices in a properly rainbow K_4-saturated graph.

Lemma 2.11. *If G is rainbow K_4-saturated, then the vertices of G with degree at most 3 form a clique.*

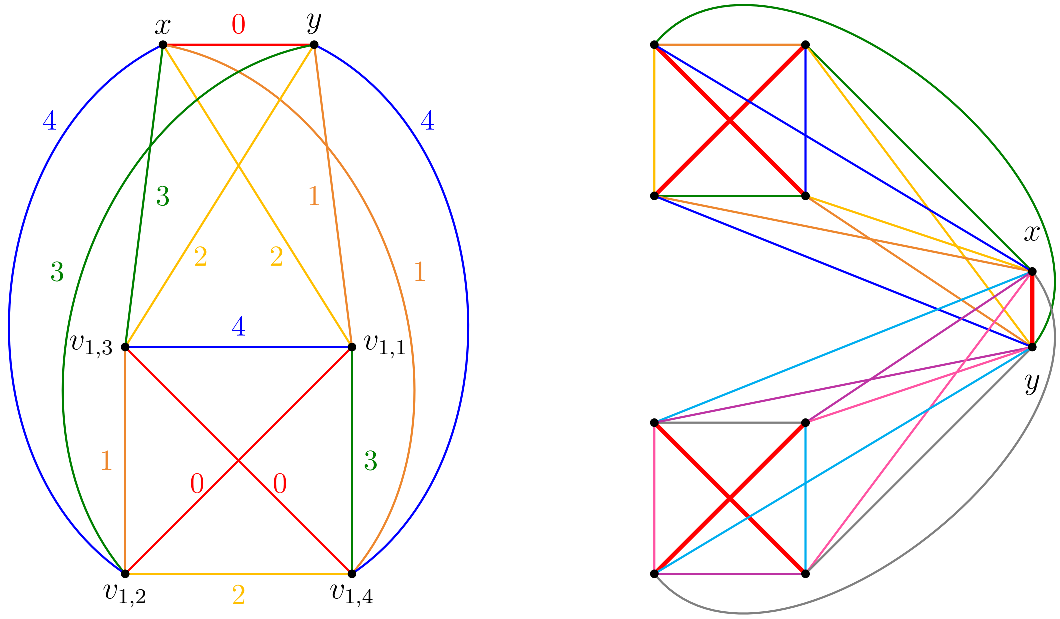

Figure 1: Construction 2 on 6 and 10 vertices; note the repetition of color 0 when x, y are adjacent to multiple K_4-copies.

Proof. Let u, v be vertices with degree at most 3, and suppose $uv \notin E(G)$. By Observation 2.8, there is an edge xy in $N(u) \cap N(v)$. Without loss of generality, $2 \leq d(u) \leq d(v) \leq 3$. Fix a rainbow K_4-free coloring of G, say c. There are four cases to consider, ordered by complexity.

Case 1: $d(u) = d(v) = 2$.

Observe that the edges ux and vy are not contained in any K_4 copy in G, so $\{ux, vy\}$ is unrestricted. We may thus re-color so that $c(ux) = c(vy)$ is a new color for c. Now, adding the edge uv in another new color creates a proper edge-coloring of $G + uv$. Note that uv is contained in exactly one K_4 copy, on $\{u, v, x, y\}$, which is not rainbow. Thus, $G + uv$ admits a rainbow K_4-free proper edge-coloring, a contradiction.

Case 2: $d(u) = 2, d(v) = 3$.

Let $z \in N(v) \setminus N(u)$. Since $uz \notin E(G)$, there must exist an edge in $N(u) \cap N(z)$, which must be xy since $N(u) = \{x, y\}$. Hence $\{x, y, z\}$ form a clique. The edges ux, uy are in no K_4, so $\{ux, uy\}$ is unrestricted. Both vx and vy are in exactly one K_4, on vertex set $\{v, x, y, z\}$. It follows that either $\{ux, uy, vx\}$ or $\{ux, uy, vy\}$ is unrestricted: if $c(xy) = c(vz)$, then both are unrestricted; if $c(xz) = c(vy)$, then

$\{ux, uy, vx\}$ is unrestricted; if $c(yz) = c(vx)$, then $\{ux, uy, vy\}$ is unrestricted. Without loss of generality, $\{ux, uy, vy\}$ is unrestricted. By recoloring $\{ux, vy\}$ as in Case 1, we can extend c to a rainbow K_4-free edge-coloring of $G + uv$, a contradiction.

Case 3: $d(u) = d(v) = 3$, $N(u) \neq N(v)$.

Let $z \in N(u) \setminus N(v)$ and $w \in N(v) \setminus N(u)$. First, suppose $c(uz) \neq c(xy)$ and $c(vw) \neq c(xy)$. We extend c to $G + uv$ by setting $c(uv) = c(xy)$. This extension of c remains proper, and uv is contained in one K_4 copy, on $\{u, v, x, y\}$, which is not rainbow, a contradiction. Hence either $c(uz) = c(xy)$ or $c(vw) = c(xy)$. Relabeling if necessary, we may assume $c(uz) = c(xy) = 1$. Then $\{ux, uy\}$ is unrestricted. If $\{x, y, w\}$ does not induce a clique, then in fact $\{ux, uy, vx, vy\}$ is unrestricted. If so, we may proceed as in Case 1 to reach a contradiction. Hence we may assume $\{x, y, w\}$ induces a clique. As in Case 2, at least one of $\{ux, uy, vx\}$ and $\{ux, uy, vy\}$ is free. Without loss of generality, $\{ux, uy, vy\}$ is free, and we recolor ux and vy with the same new color, as in previous cases. Adding the edge uv in any color introduces no rainbow K_4, a contradiction.

Case 4: $d(u) = d(v) = 3$, $N(u) = N(v)$.

Say $N(u) = \{x, y, z\}$. If $\{x, y, z\}$ does *not* induce a clique, then the set of all edges incident to u and v is unrestricted. Then we may recolor so that $c(ux) = c(vy) = k_1$, $c(uy) = c(vz) = k_2$, $c(uz) = c(vx) = k_3$, where all k_i are new colors. We then extend c to $G + uv$ letting $c(uv)$ be another new color. The only copies of K_4 that can contain uv in $G + uv$ are on vertex sets $\{u, x, y, v\}$, $\{u, x, z, v\}$, and $\{u, y, z, v\}$, none of which are rainbow under c. Hence $G + uv$ is rainbow K_4-free under c, a contradiction.
Hence, $\{x, y, z\}$ induces a clique. Then $\{u, x, y, z\}$ is a K_4 copy, and so two of its edges must receive the same color under c. Without loss of generality, $c(xy) = c(uz) = 1$. Similarly, two edges induced by $\{v, x, y, z\}$ must repeat a color under c. This can not be the same color repeated in $\{u, x, y, z\}$. Without loss of generality, $c(yz) = c(vx) = 2$. Note $c(xz)$ must be distinct from both of these, say $c(xz) = 3$. Then $\{ux, uy, vy, vz\}$ is unrestricted. We recolor so that $c(ux) = c(vy) = k_1$, $c(uy) = c(vz) = k_2$, where k_i are new colors. We add the edge uv, and extend c to $G + uv$ by setting $c(uv) = 3$. Note, c remains proper since u and v are incident to no edges colored 3 under c in G. Adding the edge uv creates three new K_4 copies: the copy on $\{u, x, y, v\}$ repeats color k_1, the copy on $\{u, x, z, v\}$ repeats color 3, and the copy on $\{u, y, z, v\}$ repeats color k_2. Hence $G + uv$ admits a rainbow K_4-free proper edge-coloring, a contradiction.

In any case, we reach a contradiction. Hence, $uv \in E(G)$. This holds for any pair of vertices of degree at most 3. Hence all vertices of degree at most 3 form a clique. □

We now derive the main tool (Lemma 2.12) needed to prove Theorem 1.1. The broad strategy of our argument is to first argue that any rainbow K_4-saturated graph G has a dominating set D with certain useful properties, and then count edges in G by examining the structure of $G \setminus D$ and the density of edges between D and $V(G \setminus D)$. Towards this end, we begin by arguing that any sparse rainbow K_4-saturated graph must contain a small dominating set satisfying one of two properties. We say that $D \subset V(G)$ is *k-dominating* if for any $v \notin D$, we have $|N(v) \cap D| \geq k$.

Lemma 2.12. *Fix $\frac{1}{2} > \alpha > 0$, and suppose G is an n-vertex, rainbow K_4-saturated graph with n large enough that $\alpha^2 n \geq 7$. Then either $e(G) \geq \frac{7}{2}n - 4\alpha n$, or G contains a dominating set D with $|D| \leq 2\alpha n$ such that D satisfies one of the following conditions:*

1. *D is a 3-dominating set;*
2. *D is a 2-dominating set containing adjacent vertices x, y such that $v \in N(x, y)$ for every $v \in V(G) \setminus D$.*

Proof. Let v be a vertex of minimum degree in G. If $d(v) \geq \alpha n - 1$, then by choice of n, we have $\delta(G) \geq \frac{\alpha}{2}n$, and thus

$$e(G) \geq \frac{1}{2} \cdot \frac{\alpha}{2}n^2 = \frac{\alpha}{4}n^2 > \frac{\alpha^2}{2}n^2,$$

with the last inequality following from $\frac{1}{2} > \alpha$. Now, since $\alpha^2 n \geq 7$, we have

$$e(G) \geq \frac{\alpha^2}{2}n^2 \geq \frac{7}{2}n,$$

and so the desired edge bound holds.

Thus, we may assume that $|N[v]| \leq \alpha n$. If $N[v]$ is a 3-dominating set, then we set $D = N[v]$ and are done. Suppose not. We let $V' := V(G) \setminus N[v]$, and $G' := G[V']$. By Observation 2.8, $|N(u) \cap N(v)| \geq 2$ for any $u \in V'$, so $N[v]$ is a 2-dominating set. We now consider the degrees of vertices in V', and in particular those vertices which have either two neighbors in $N[v]$ or few neighbors in V'.

We define

$$L_1 := \{u \in V' : |N(u) \cap N(v)| = 2\}$$

and

$$L_2 := \{u \in V' : |N(u) \setminus N(v)| \leq 2\}.$$

We also define a set of vertices with many neighbors in $N[v]$:

$$H := \{u \in V' : |N(u) \cap N(v)| \geq 4\}.$$

Note that H and L_1 are disjoint.

We first show that we may assume $L_1 \cap L_2$ is not empty. Indeed, if $L_1 \cap L_2 = \emptyset$, then we can bound

$$e(V', N[v]) \geq 4|H| + 3(|V'| - |H|) - |L_1| = 3|V' \setminus (L_1 \cup L_2)| + 2|L_1| + 3|L_2| + |H| \quad (1)$$

and estimate $e(G')$ as follows. Recall that by Lemma 2.11, the vertices of degree at most 3 in G form a clique. In particular, Lemma 2.11 implies that $L_2 \setminus H$ contains at most one vertex u with $d_{G'}(u) = 0$. Indeed, if $u_1, u_2 \in L_2 \setminus H$ have degree 0 in G', then $d(u_1), d(u_2)$

are both at most 3, which implies u_1u_2 is an edge in G'. Thus, at most one vertex of $L_2 \setminus H$ has degree 0 in G', so we have

$$e(G') = \frac{\sum_{w \in V' \setminus L_2} d(w) + \sum_{u \in L_2} d_{G'}(u)}{2} \geq \frac{3|V' \setminus L_2| + |L_2 \setminus H| - 1}{2}$$
$$= \frac{3}{2}|V' \setminus (L_1 \cup L_2)| + \frac{3}{2}|L_1| + \frac{1}{2}|L_2 \setminus H| - \frac{1}{2}. \tag{2}$$

We combine (1) and (2), noting that since $(L_2 \setminus H) \cup H \supseteq L_2$, we have $3|L_2| + |H| + \frac{1}{2}|L_2 \setminus H| \geq \frac{7}{2}|L_2|$. Thus,

$$e(G) \geq \frac{9}{2}|V \setminus (L_1 \cup L_2)| + \frac{7}{2}|L_1| + \frac{7}{2}|L_2| - \frac{1}{2} \geq \frac{7}{2}|V'| - 1 \geq \frac{7}{2}n - \frac{7\alpha}{2}n - 1 > \frac{7}{2}n - 4\alpha n,$$

yielding the desired edge bound.

So we may assume that $L_1 \cap L_2$ is not empty; let $z \in L_1 \cap L_2$, with $N(v) \cap N(z) = \{x, y\}$. Since $N(v) \cap N(z)$ is not an independent set, we must have $xy \in E(G)$. Our goal now is to show that almost all vertices in V' are in $N(x, y)$. If $d(z) = 2$, then by Observation 2.8, any $u \in V'$ is in $N(x, y)$. So, assume $d(z) \geq 3$. Suppose there exists $w \in V' \setminus [N(x, y) \cup N(z)]$. (So, w is not dominated by xy, and w is not adjacent to z.)

Claim 1. *If $|N(w)| \leq \alpha n$, then $N[v] \cup N[w] \cup N(z)$ is a 3-dominating set of size at most $2\alpha n$.*

Proof of Claim 1. We set $D := N[v] \cup N[w] \cup N(z)$, and begin by bounding $|D|$. Since $z \in L_1 \cap L_2$, we have $|N(z) \cap V'| \leq 2$. Since $zw \notin E(G)$, we must have $|N(z) \cap N(w)| \geq 2$; in particular, since $N(w)$ contains at most one of x, y, we know that $|N(z) \cap V'|$ is non-empty and intersects $N(w)$. Thus, $|N[w] \cup (N(z) \cap V')| \leq |N[w]| + 1 \leq \alpha n + 2$. Moreover, since $vw \notin E(G)$, we must have $|N[w] \cap N[v]| \geq 2$, implying that

$$|D| = |N[v]| + |(N[w] \cup N(z)) \cap V'| \leq |N[v]| + |N[w] \cup (N(z) \cap V')| - 2 \leq 2\alpha n,$$

as desired.

Next, we show that every vertex in $V \setminus D$ has at least 3 neighbors in D. By construction, z has at least 3 neighbors in D since $|N(z)| \geq 3$. Take $u \neq z$ in $V \setminus D$. Thus, u is not adjacent to v, w, or z. In particular, u must have at least two neighbors in each of $N(v), N(w)$, and $N(z)$. We have two cases.

1. $x, y \in N(u)$.

   In this case, since $|\{x, y\} \cap N(w)| \leq 1$, we know that u must have at least one more neighbor in $N(w)$. Thus, $|N(u) \cap D| \geq 3$.

2. $N(u)$ contains at most one of x, y.

   In this case, we know that $|N(u) \cap N(v)| \geq 2$ and that $N(u)$ contains a neighbor of z which is not in $\{x, y\}$. Since $N(v) \cap N(z) = \{x, y\}$, this implies that $N(u)$ intersects $D \setminus N(v)$; thus, $|N(u) \cap D| \geq 3$.

Hence, D is a 3-dominating set, completing the claim. □

Now, by Claim 1, we are done if there exists a vertex $w \in V' \setminus [N(x,y) \cup N(z)]$ with $d(w) \le \alpha n$. On the other hand, if V contains at least $\alpha n - 8$ vertices of degree greater than αn, then

$$e(G) \ge \frac{1}{2}(\alpha n)(\alpha n - 8) = \frac{\alpha^2}{2}n^2 - 4\alpha n \ge \frac{7}{2}n - 4\alpha n,$$

satisfying the desired bound. So $|V' \setminus [N(x,y) \cup N(z)]| < \alpha n - 8$, and thus

$$D := N[v] \cup [V' \setminus N(x,y)]$$

has $|D| \le 2\alpha n$, and D is a 2-dominating set with the property that $u \in N(x,y)$ for any $u \notin D$. □

We are now ready to prove Theorem 1.1, which we restate here for convenience.

Theorem 1.1. *Let $\frac{1}{2} > \alpha > 0$. For any n such that $\alpha^2 n \ge 7$ and $\alpha n > 220$, we have*

$$\frac{7}{2}n - 8\alpha n \le \text{sat}^*(n, K_4) \le \frac{7}{2}n + O(1).$$

Proof. By Proposition 2.10, we have $\text{sat}^*(n, K_4) \le \frac{7}{2}n + O(1)$. Thus, we must only demonstrate the lower bound. Suppose that G is an n-vertex, properly rainbow K_4-saturated graph with $\alpha^2 n \ge 7$. If $e(G) \ge \frac{7}{2}n - 4\alpha n$, we are done, so suppose not. By Lemma 2.12, G contains a dominating set D with $|D| \le 2\alpha n$ satisfying one of the outcomes of Lemma 2.12. We now have two cases, depending upon which outcome occurs. For notational convenience, we set $V' := V(G) \setminus D$ and let G' be the subgraph of G induced on V'.

Suppose first that D is a 3-dominating set. Recall that by Lemma 2.11, the set of vertices in G of degree at most 3 form a clique. Note that if $v \in V'$ has degree at most 3, then in fact $d(v) = 3$ and $N(v) \subseteq D$. Thus, the set of vertices of degree at most 3 intersects V' at most once. We define

$$V_1 := \{v \in V' : d_{G'}(v) = 0\}$$

and

$$V_2 := \{v \in V' : d_{G'}(v) \ge 1\}.$$

By the above observations, V_1 contains at most one vertex of degree 3 in G. We have

$$\begin{aligned} e(G) \ge e(V', D) + e(G') &= \sum_{v \in V_1} d(v) + \sum_{u \in V_2} |N(u) \cap D| + \frac{1}{2}\sum_{u \in V_2} d_{G'}(u) \\ &\ge 4|V_1| - 1 + 3|V_2| + \frac{|V_2|}{2} \ge \frac{7}{2}|V'| - 1. \end{aligned}$$

Since $|V'| = n - |D| \ge (1 - 2\alpha)n$, we have

$$e(G) \ge \frac{7}{2}(1 - 2\alpha)n - 1 = \frac{7}{2}n - 7\alpha n - 1 > \frac{7}{2}n - 8\alpha n,$$

as desired.

Next, suppose D is a 2-dominating set containing adjacent vertices x, y such that $v \in N(x, y)$ for every $v \in V(G) \setminus D$. In this case, note that $V(G) \setminus N(x, y) \subseteq D$, and both are 2-dominating sets of the desired type. For later convenience, we will redefine D if necessary, setting

$$D := V(G) \setminus N(x, y).$$

Note that now, $V' = V(G) \setminus D$ is simply equal to $N(x, y)$. As before, we set $G' = G[V']$.

Now, we begin by making several observations on the structure and coloring of G'. In particular, we will find a matching within G' of edges which receive the same color as xy; to indicate the special role played by this edge-color, we shall say $c(xy) = 0$. We begin by showing that certain vertices in G' must be incident to edges of color 0.

Suppose $z \in V'$ has $d_{G'}(z) \geq 3$. By Observation 2.9, we must have $d_{G'}(z) = 3$, and each of the three vertices in $N(z) \cap V'$ is adjacent to one of x, y, z via an edge with color in $\{0, c(xz), c(yz)\}$. To maintain a proper edge-coloring, only x can be adjacent to $N(z) \cap V'$ via an edge of color $c(yz)$, only y can be adjacent to $N(z) \cap V'$ via an edge of color $c(xz)$, and only z can be adjacent to $N(z) \cap V'$ via an edge of color 0. Thus, the maximum degree in G' is 3, and every vertex of degree 3 in G' is incident to an edge of color 0.

We now investigate vertices with degree 2 in G'. In particular, we define the set of vertices with degree 2 in G' and exactly 2 neighbors in D,

$$S_2^2 := \{v \in V' : d_{G'}(v) = 2 \text{ and } |N(v) \cap D| = 2\}.$$

Note that every $v \in S_2^2$ has $N(v) \cap D = \{x, y\}$. We start by noting that S_2^2 contains very few vertices which are not incident to an edge of color 0.

Indeed, suppose there exist $u, v \in S_2^2$ such that $d(u, v) > 2$ and neither u nor v is incident to an edge of color 0. We can thus add uv to G with $c(uv) = 0$ while maintaining a proper coloring. Moreover, since $d(u, v) > 2$, the addition of uv does not form a triangle in G'. Thus, any copy of K_4 using uv in $G + uv$ must contain two vertices from D. Since $u, v \in S_2^2$, their only neighbors in D are x, y. Hence, $\{u, v, x, y\}$ span the unique copy of K_4 using uv in $G + uv$; since $c(uv) = c(xy) = 0$, this copy is not rainbow. So $G + uv$ can be properly edge-colored while avoiding a rainbow K_4-copy, a contradiction to the assumption that G is rainbow K_4-saturated.

Thus, the subset of vertices in S_2^2 which are not incident to edges of color 0 pairwise are at distance at most 2 in G'. Since $\Delta(G') \leq 3$, for *any* vertex $u \in G'$, we have

$$|\{v \in G' : d_{G'}(u, v) \leq 2\}| \leq 13.$$

While a tighter bound in fact holds for vertices in S_2^2, we shall only require that the subset of vertices in S_2^2 not incident to edges of color 0 is of constant size.

Now, let M_0 be the matching in G' consisting of all edges in $E(G')$ colored 0. We form G'_0 by deleting M_0 from G'. Now, by the above observations, $\Delta(G'_0) \leq 2$. Thus, every component of G'_0 is a path (possibly trivial) or a cycle. We shall use this structure to count edges in G.

Call $v \in V'$ *light* if $N(v) \cap D = \{x, y\}$, and *heavy* if not. (Thus, all S_2^2 vertices are light, but V' may contain light vertices which are not in S_2^2.) Call a component C of G'_0 *light* if at most one vertex of C is heavy, and let L be the set of light components of G'_0.

Similarly, call a component C *heavy* if at least two vertices of C are heavy, with the set of heavy components denoted by H. For a component C of G_0', let L_C denote the set of light vertices in $V(C)$ and H_C the set of heavy vertices in $V(C)$.

Now, $e(G) \geq e(G', D) + e(G')$, which we can bound by summing component-by-component:

$$e(G) \geq \sum_{v \in V'} \left(|N(v) \cap D| + \frac{1}{2} d_G'(v) \right) = \sum_{v \in C, C \in L} \left(|N(v) \cap D| + \frac{1}{2} d_{G'}(v) \right) + \sum_{v \in C, C \in H} \left(|N(v) \cap D| + \frac{1}{2} d_{G'}(v) \right).$$

In particular, if the number of edges incident to component C satisfies

$$i(C) =: \sum_{v \in C} \left(|N(v) \cap D| + \frac{1}{2} d_{G'}(v) \right) \geq \frac{7}{2}|C| \tag{3}$$

for all components C, then the desired bound on $e(G)$ would hold. We will not be able to guarantee that (3) holds for all C, but show that only constantly many components C can fail to satisfy (3).

We shall obtain different bounds on $i(C)$ for different types of components. Firstly, recall that by Lemma 2.11, at most four vertices of G have degree ≤ 3, and we have argued above that at most 13 vertices in S_2^2 are not incident to an edge of color 0 in G'. Call a component C *exceptional* if C contains either a vertex with degree ≤ 3 in G or a vertex in S_2^2 which is not incident to an edge of color 0 in G'. Note that there are at most 17 exceptional components of G_0' in total. We now estimate $i(C)$ for different types of components C.

We first address components of size 1. If C is a component such that some vertex v in C has $d_C(v) = 0$, then $V(C) = \{v\}$. In this case, $i(C) = d(v)$. Note that if C is non-exceptional, then $d(v) \geq 4$, so $i(C) > \frac{7}{2}|C|$.

Next, we consider components of size greater than 1. We treat these in cases, depending upon their classification as light or heavy.

Case 1: $|V(C)| \geq 2$ and $C \in H$.

For a non-exceptional component $C \in H$, we estimate $i(C)$ as follows. If v is a light vertex of C, then (since C is non-exceptional) v either is in S_2^2 and is incident to a color 0 edge in G' or $d_{G'}(v) = 3$. In either case, v is incident to an edge in M_0, so $d_{G'}(v) = d_C(v) + 1$. If v is a heavy vertex of C, then $|N(v) \cap D| \geq 3$, and (since $|V(C)| \geq 2$) $d_C(v) \geq 1$. Now,

$$\begin{aligned} i(C) &= \sum_{u \in L_C} \left(|N(u) \cap D| + \frac{d_{G'}(u)}{2} \right) + \sum_{v \in H_C} \left(|N(v) \cap D| + \frac{d_{G'}(v)}{2} \right) \\ &\geq \sum_{u \in L_C} \left(2 + \frac{d_C(u) + 1}{2} \right) + \sum_{c \in H_C} \left(3 + \frac{d_C(v)}{2} \right). \end{aligned}$$

Since C is either a path or a cycle, at most two vertices of C have degree 1 in C, so

$$\sum_{u\in L_C}\left(2+\frac{d_C(u)+1}{2}\right)+\sum_{c\in H_C}\left(3+\frac{d_C(v)}{2}\right)$$

$$\geq \frac{7}{2}|L_C|+4|H_C|-1=\frac{7}{2}|C|+\frac{1}{2}|H_C|-1\geq \frac{7}{2}|C|$$

since, because C is heavy, $|H_C|\geq 2$.

If C is an exceptional heavy component, then at most 17 vertices in C are either of degree at most 3 in G or are in S_2^2 but not incident to an edge of M_0. Since every vertex in C has at least one neighbor in G' and at least two neighbors in D, every vertex in C has degree at least 3 in G and if C contains a vertex of degree 3, then this is a light vertex. All vertices in S_2^2 are light, so to estimate $i(C)$ when C is an exceptional heavy component, only the edge incidence count for L_C will change. It remains true that at most two vertices of G have degree 1 in C, since C is a cycle or a path, and now at most 17 vertices in L_C are not incident to an edge of M_0. So

$$\sum_{u\in L_C}\left(|N(u)\cap D|+\frac{d_{G'}(u)}{2}\right)\geq \sum_{u\in L_C}\left(2+\frac{d_C(u)+1}{2}\right)-\frac{17}{2}\geq\frac{7}{2}|L_C|-\frac{19}{2}.$$

Thus,

$$i(C)\geq\frac{7}{2}|L_C|+4|H_C|-\frac{19}{2}\geq\frac{7}{2}|C|-\frac{17}{2}.$$

Case 2: $|V(C)|\geq 2$ and $C\in L$.

As in Case 1, if C is a non-exceptional component, then we have

$$i(C)\geq\sum_{u\in L_C}\left(2+\frac{d_C(u)+1}{2}\right)+\sum_{c\in H_C}\left(3+\frac{d_C(v)}{2}\right).$$

Now, either $|H_C|=1$ or H_C is empty. Note that since $2+\frac{d_C(v)+1}{2}<3+\frac{d_C(v)}{2}$, we have

$$i(C)\geq\sum_{u\in V(C)}2+\frac{d_C(u)+1}{2}$$

regardless of the size of H_C. Now, if C is a cycle, we have

$$i(C)\geq\sum_{u\in V(C)}2+\frac{2+1}{2}=\frac{7}{2}|C|.$$

If C is a path, then we have $i(C)\geq\frac{7}{2}|C|-1$.

If C is an exceptional component then, as in Case 1, at most 17 vertices of C either have degree 3 or are in S_2^2 but not incident to an edge from M_0. We can then bound

$$i(C)\geq\sum_{u\in V(C)}\left(2+\frac{d_C(u)+1}{2}\right)-\frac{17}{2}\geq\frac{7}{2}|C|-\frac{19}{2}.$$

Again, the goal is to show that very few components C have $i(C) < \frac{7}{2}|C|$. Among non-exceptional components, only light paths fail to meet the desired value of $i(C)$. Our final step is thus to demonstrate that there are few non-exceptional light paths in G'_0. We first require a more careful description of heavy vertices in light paths.

Recall that if C is a light path, then either C contains no heavy vertices, or precisely one heavy vertex, say v. Note that if v is a heavy vertex of C which is incident to an edge of M_0, then we in fact have

$$i(C) = \sum_{u \in L_C} \left(2 + \frac{d_C(u)+1}{2}\right) + 3 + \frac{d_C(v)+1}{2} = \frac{5}{2}|C| + 1 + \sum_{u \in V(C)} \frac{d_C(u)}{2} = \frac{7}{2}|C|.$$

Thus, any light path C with $i(C) < \frac{7}{2}|C|$ either contains no heavy vertex v, or its heavy vertex v is not incident to an edge from M_0. Note that this implies $d_C(v) = d_{G'}(v)$, and in particular, v is not adjacent to any heavy vertex in V'. Similarly, if C contains a heavy vertex with more than 3 neighbors in D, then $i(C) \geq \frac{7}{2}|C|$, so any light path C with $i(C) < \frac{7}{2}|C|$ either contains no heavy vertex v, or its heavy vertex v is adjacent to exactly 3 vertices in D.

Now, recall that D is precisely $V(G) \setminus N(x, y)$, so there is no triangle in D containing xy. In particular, suppose v is a heavy vertex in G' which has no heavy neighbor in G', such that $N(v) \cap D = \{x, y, z\}$. Then at most one neighbor of v is adjacent to z, so vz is an unrestricted edge. Among light paths, we will thus be able in large part to argue without distinguishing their heavy vertices.

Suppose G'_0 contains two non-exceptional light paths $P_1 = v_1, v_2, \ldots, v_k$ and $P_2 = w_1, w_2, \ldots, w_\ell$ such that M_0 contains no edges between $\{v_1, v_2, v_{k-1}, v_k\}$ and $\{w_1, w_2, w_{\ell-1}, w_\ell\}$. Note that we can find two such non-exceptional light paths if G'_0 contains more than 5 non-exceptional light paths in total. We will contradict that G is properly rainbow K_4-saturated by showing that an edge between an endpoint of P_1 and an endpoint of P_2 may be added without creating any rainbow K_4-copy.

To identify the correct place in which to add this edge, we will fix an orientation of both paths, as follows. Observe, since $c(v_1v_2) \neq 0$ and the K_4-copy on $\{x, y, v_1, v_2\}$ is not rainbow, either $c(v_1x) = c(v_2y)$ or $c(v_1y) = c(v_2x)$. We will say edge v_iv_{i+1} of P_1 is *left-oriented* if $c(v_ix) = c(v_{i+1}y)$, and *right oriented* if not. Observe, if v_1v_2 is left-oriented, then $c(v_2y)$ cannot equal $c(v_3x)$, since x is already adjacent to v_1 via an edge colored with $c(v_2y)$. Thus, if v_1v_2 is left-oriented, then v_2v_3 must also be left-oriented. Inductively, if v_1v_2 is left-oriented, then in fact v_iv_{i+1} must be left-oriented for all $i \leq k-1$, and similarly, if v_1v_2 is right-oriented, then v_iv_{i+1} is right-oriented for every $i \leq k-1$.

Say P_1 is left-oriented if all of its edges are left-oriented, and right-oriented if all of its edges are right-oriented. Observe that orientation is purely a function of a path's vertex labelling: if P_1 is right-oriented, we may relabel its vertices, changing v_i to v_{n-i+1}, to view P_1 as left-oriented. So, we relabel P_1 and P_2 if necessary to ensure that both are left-oriented. (Note that under this relabelling, it remains the case that no M_0 edge connects $\{v_1, v_2, v_{k-1}, v_k\}$ and $\{w_1, w_2, w_{\ell-1}, w_\ell\}$.) Now, we will add either v_kw_1 or $w_\ell v_1$ to G, choosing whichever edge contains fewer heavy vertices. Since P_1, P_2 are light, one of v_kw_1, $w_\ell v_1$ contains at most one heavy vertex. Without loss of generality, say we add v_kw_1.

Because at most one of v_k, w_1 is heavy, the addition of v_kw_1 does not create any K_4-copy

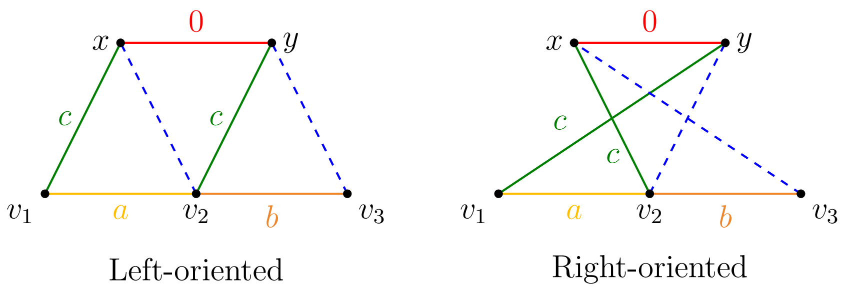

Figure 2: Left and right orientations of v_1v_2. Note that in each case, to avoid a rainbow K_4-copy, the orientation of v_2v_3 must match that of v_1v_2.

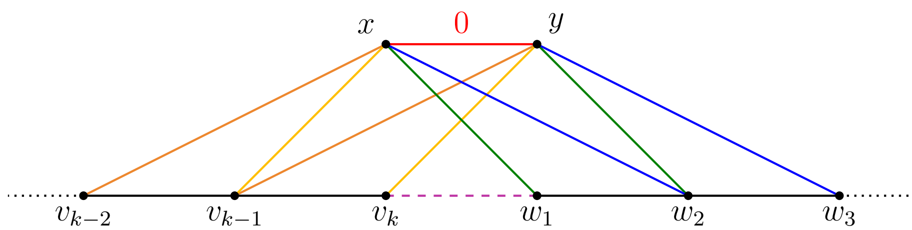

Figure 3: P_1 and P_2, both left-oriented. We add the dashed edge v_kw_1 to G.

which intersects $D \setminus \{x, y\}$. In particular, if P_1, P_2 contain heavy vertices v_i, w_j, say adjacent to $z_i, z_j \in D$ (possibly with $z_i = z_j$), then $c(v_iz_i)$ and $c(w_jz_j)$ are unrestricted, and even in $G + v_kw_1$, the edges v_iz_i and w_jz_j are not contained in any K_4-copy. We choose distinct new colors for each of v_iz_i, w_jz_j to ensure that these edges do not share a color with any edge of G; we will also never reuse these new colors in subsequent recoloring steps. Thus, any recoloring of edges incident to v_i, w_j will not conflict with $c(v_iz_i)$ or $c(w_jz_j)$. For the remainder of the proof, we may omit consideration of v_iz_i and w_jz_j, if they exist.

The addition of v_kw_1 creates one K_4-copy, on $\{x, y, v_k, w_1\}$. Since no edge of M_0 connects $\{v_1, v_2, v_{k-1}, v_k\}$ with $\{w_1, w_2, w_{\ell-1}, w_\ell\}$, the addition of v_kw_1 does not create any triangle in G', so in fact, $\{x, y, v_k, w_1\}$ span the unique K_4-copy containing v_kw_1. If xv_k and yw_1 are both unrestricted edges, we can recolor so that $c(xv_k) = c(yw_1)$, using an edge-color not yet appearing in G. Coloring v_kw_1 with any legal color then creates a proper edge-coloring of $G + v_kw_1$ which is rainbow K_4-free, a contradiction.

Thus, suppose xv_k is restricted. Since P_1 is left-oriented, the K_4-copy on $\{x, y, v_{k-1}, v_k\}$ has $c(xv_{k-1}) = c(xv_k)$, so for xv_k to be restricted, xv_k must be contained in another K_4-copy. This is only possible if $v_{k-2}v_k \in M_0$, a situation we depict in Figure 4. Note that the coloring depicted in Figure 4 is without loss of generality; in particular, since P_1 is left-oriented, we have $c(xv_{k-2}) = c(xv_{k-1})$ and $c(xv_{k-1}) = c(yv_k)$, none of which can be in $\{c(v_{k-2}v_{k-1}), c(v_{k-1}v_k)\}$ since the edge-coloring is proper.

Observe that if xv_k is restricted, then to avoid a rainbow K_4-copy, we must have $c(yv_{k-2}) = c(v_{k-1}v_k)$ (color 1 in Figure 4) and $c(xv_k) = c(v_{k-1}v_{k-2})$ (color 2 in Figure 4). Now, if $v_{k-2}v_{k-1}$ is only in the K_4-copies on $\{v_{k-2}, v_{k-1}, x, y\}$, $\{v_{k-2}, v_{k-1}, v_k, x\}$, and $\{v_{k-2}, v_{k-1}, v_k, y\}$, then we can recolor $v_{k-2}v_{k-1}$ and xv_k simultaneously, so long as we maintain $c(v_{k-2}v_{k-1}) = c(xv_k)$.

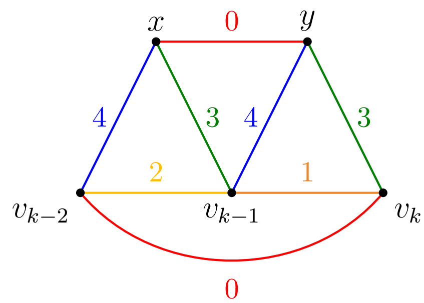

Figure 4: The structure of G near v_k if xv_k is restricted

If $v_{k-2}v_{k-1}$ is in another K_4 copy in G, then this copy must include a triangle in G', implying $v_{k-3}v_{k-1} \in M_0$. We depict this in Figure 5.

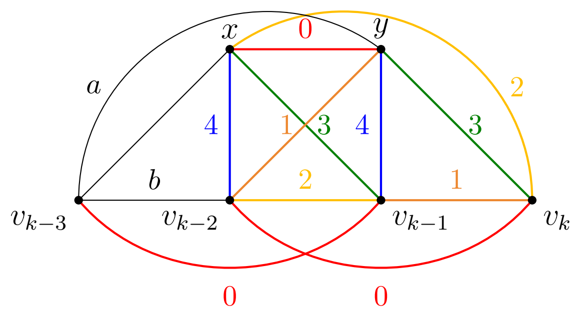

Figure 5: The structure of G near v_k if both $v_{k-2}v_k$ and $v_{k-3}v_{k-1}$ are in M_0

Now, consider $a = c(yv_{k-3})$ and $b = c(v_{k-3}v_{k-2})$. To maintain a proper coloring, $a \notin \{0, 1, 3, 4\}$ and $b \notin \{0, 1, 2, 4\}$. So, since $y, v_{k-1}, v_{k-2}, v_{k-3}$ cannot span a rainbow K_4-copy in G, we must have $a = 2$. Because P_1 is left-oriented, we know that $a = c(yv_{k-3})$ is equal to $c(xv_{k-4})$, if v_{k-4} exists. However, such a color assignment would be impossible if $a = 2$, since x is already incident to v_k via an edge of color 2.

Thus, if $v_{k-3}v_{k-1}$ is in M_0, we conclude that P_1 is a four-vertex light path with $v_{k-3}v_{k-1}$ and $v_{k-1}v_k$ in M_0. In this case, P_1 is not only a component of G'_0 but of G' and, because P_1 is light, every K_4-copy in $G + v_{k-3}v_k$ which intersects P_1 must use only vertices from $V(P_1) \cup \{x, y\}$. We can thus add the edge $v_{k-3}v_k$, since $P_2 \vee K_4$ can be properly edge-colored without creating a rainbow K_4-copy, a contradiction.

Thus, either xv_k is unrestricted, or $c(xv_k) = c(v_{k-2}v_{k-1})$ and we can freely change the value of $c(xv_k)$ so long as we also change $c(v_{k-2}v_{k-1})$ to maintain $c(xv_k) = c(v_{k-2}v_{k-1})$. Analogously, either yw_1 is unrestricted, or $c(yw_1) = c(w_2w_3)$ and we can freely change the value of $c(yw_1)$ so long as we also change the value of $c(w_2w_3)$. Also note that because P_1, P_2 are separate components, the edges $xv_k, v_{k-1}v_{k-2}, yw_1$, and w_2w_3 are pairwise vertex disjoint, so it is possible to maintain a legal edge-coloring if some or all of their colors are equal. We recolor so that $c(xv_k) = c(yw_1)$ via a color not yet used in G, also recoloring $v_{k-2}v_{k-1}$ and w_2w_3 if necessary. Under this new edge-coloring, G remains rainbow K_4-free. Moreover, we have constructed this new coloring to allow the addition of $v_k w_1$ in any legal color without creating a rainbow K_4-copy.

Thus, if we can find P_1, P_2 in G_0', then G is not properly rainbow K_4-saturated, a contradiction. If we cannot find P_1, P_2 as above, then G_0' contains at most 5 non-exceptional light paths. We know that G_0' contains at most 17 exceptional components in total, so contains at most 22 components C with $i(C) < \frac{7}{2}|C|$. We have seen that every component has $i(C) \geq \frac{7}{2}|C| - \frac{19}{2} > \frac{7}{2}|C| - 10$, so in total

$$e(G) \geq \sum_C i(C) \geq \sum_C \frac{7}{2}|C| - 22 \cdot 10 = \frac{7}{2}(n - |D|) - 220 \geq \frac{7}{2}(n - 2\alpha n) - 220 > \frac{7}{2}n - 8\alpha n$$

since we have $\alpha n > 220$. □

3 Paths

In this section, we prove Theorem 1.2. We begin by proving the desired lower bound in the following proposition. We remark that this argument can be adapted to show that $\text{sat}^*(n, T) \geq n - 1$ for all trees T with diameter at least 4.

Proposition 3.1. *If a graph G is rainbow P_k-saturated for $k \geq 5$, then G has at most one acyclic component. Consequently,* $\text{sat}^*(n, P_k) \geq n - 1$.

Proof. We first show that G does not contain two acyclic components of diameter at most 2. For the sake of a contradiction, assume two such components exist. Note that these two components are stars, and adding an edge between their centers will create no new P_k-copies. Thus, G is not rainbow P_k-saturated, a contradiction. It follows that G has at most one acyclic component of diameter at most 2. If this component has at least 3 vertices, then adding an edge between two leaves will not create any new P_k-copies. So again, G is not rainbow P_k-saturated, a contradiction. Thus, such a component has at most 2 vertices, meaning it is isomorphic to K_1 or K_2.

We now show that, in fact, G cannot contain two acyclic components of any diameter. Suppose towards a contradiction that G has two acyclic components, T_1 and T_2. By the above observations, one of T_1 or T_2 must have diameter at least 3. Without loss of generality, say this is T_1. Fix $u \in V(T_1)$ and $x \in V(T_2)$ to be endpoints of longest shortest paths (i.e., paths which realize the diameters of T_1 and T_2). Then u and x are necessarily leaves. By the above, we may assume T_1 is a tree of diameter at least 3, while T_2 may be either K_1, K_2, or a tree of diameter at least 3.

Let v be the unique neighbor of u. Since T_1 has diameter at least 3, v must have exactly one non-leaf neighbor, say w. Let c be a rainbow P_k-free proper edge-coloring of G. Without loss of generality, $c(uv) = 1$ and $c(vw) = 2$. Now, we consider the edge-coloring of T_2. If T_2 is not a K_1-copy, let y be the neighbor of x, and if T_2 is not a K_2-copy, let z be the unique non-leaf neighbor of y. By relabeling edge-colors within T_2 only, we can set $c(xy) = 1$ (if y exists) and $c(yz) = 2$ (if z exists).

Now we claim we can add the edge ux with $c(ux) = 2$ without creating any rainbow P_k-copy. Indeed, observe that since $k \geq 5$, any copy of P_k which contains ux either contains

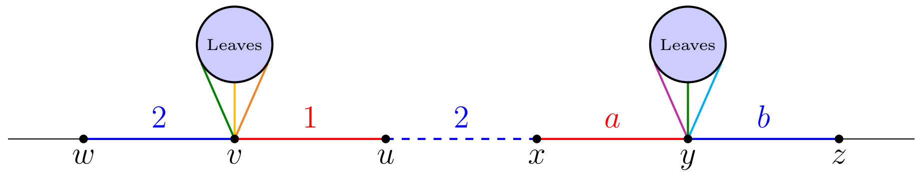

Figure 6: Leaves in T_1 and T_2 which can be connected by an edge of color 2. If necessary, we can relabel colors within T_2 only so that $a = 1$ and $b = 2$

one of vw or yz, or else both of uv and xy. Thus, $G + ux$ can be properly edge-colored without rainbow P_k-copies, contradicting the assumption that G is properly rainbow P_k-saturated. ☐

We now work to show the upper bound in Theorem 1.2. The substance of this argument will be showing that a particular construction is properly rainbow P_k-saturated. In order to do this, we must define a family of graphs upon which our construction is based, and develop an understanding of the proper edge-colorings of this family.

Definition 3.2. *For $w \geq 3$, let $V = \{0,1\}^w/\equiv$, where $\equiv$ is the equivalence relation $u \equiv v$ iff $u_i = 1 - v_i$ for each $i \in [w]$. Let $uv \in E$ if and only if $u, v \in V$ have respective representatives that differ in exactly one bit. Define $H_w/2 := (V, E)$.*

We note that $H_w/2$ is precisely the $(w-1)$-dimensional hypercube H_{w-1} with edges joining antipodal vertices, sometimes called the *folded hypercube*. Moreover, $H_w/2$ is w-regular, and $\text{diam}(H_w/2) = \lfloor w/2 \rfloor$. The folded hypercube has previously been examined in the study of rainbow Turán numbers for paths; in the cases where the rainbow Turán number of P_k is known, folded hypercubes provide an extremal construction.

The following lemma describes a construction of Johnston and Rombach [7] for a coloring of $H_w/2$ which avoids rainbow P_{w+1}-copies. In this lemma and the rest of this section, we will find it convenient to describe the edges of $H_w/2$ by the component in which their endpoints differ. We refer to the elements of the binary strings defining the vertices as *bits* and say that edge $e \in E(H_w/2)$ corresponds to an i^{th} *bit-flip* if (some representations of) its endpoints differ in precisely bit i.

Lemma 3.3. *For $w \geq 3$, $H_w/2$ has a proper edge-coloring that is rainbow P_{w+1}-free.*

Proof. Define a coloring $c : E \to \mathbb{N}$ by setting $c(e) = i$ if e corresponds to an i^{th} bit-flip. Then for any $u \in V$, a rainbow walk starting from u with w edges flips every bit exactly once, and therefore forms a cycle. Hence, $H_w/2$ has no rainbow P_{w+1} under this coloring. ☐

While we have exhibited a proper edge-coloring of $H_w/2$ which is rainbow P_{w+1}-free, this is not necessarily the unique such coloring. In order to use the family of folded hypercubes to construct the desired properly rainbow P_k-saturated family of graphs, we must understand the set of all rainbow P_{w+1}-free proper edge-colorings of $H_w/2$. Towards this end, we show that every rainbow P_{w+1}-free proper edge-coloring of $H_w/2$ contains a rainbow cycle of particular structure.

Definition 3.4. *A total bit-flip cycle (TBFC) is a cycle of length w in $H_w/2$ in which every edge corresponds to a different bit-flip. A total bit-flip rainbow cycle (TBFRC) is a TBFC with a rainbow edge-coloring.*

Lemma 3.5. *For $w \geq 3$, every proper edge-coloring of $H_w/2$ that is rainbow P_{w+1}-free contains a TBFRC.*

Proof. We construct a rainbow path $P_k = (x_0, \ldots, x_k)$ for all $0 \leq k \leq w-1$ iteratively. Set $P_0 = (x_0)$, for some fixed vertex x_0. Given P_k, the manner in which we extend to P_{k+1} will depend upon the value of k. We will say that a color c is *new* for P_k if $c \neq c(x_{i-1}x_i)$ for any $i \in \{1, \ldots, k\}$. Let $d := \operatorname{diam}(H_w/2) = \lfloor w/2 \rfloor$. While $k \leq d$, we say a color c is *bad* for P_k if c satisfies the following conditions:

1. $c = c(x_{i-1}x_i)$ for some $i \in \{1, \ldots, k\}$;

2. For some vertex $y \notin \{x_0, \ldots, x_{k-1}\}$, $x_k y$ is an edge with $c(x_k y) = c$.

For $0 \leq k \leq d$, extend P_k to P_{k+1} by adding an edge $x_k x_{k+1}$ such that $c(x_k x_{k+1})$ is new (for convenience we may assume this is color $k+1$) and $x_k x_{k+1}$ corresponds to a yet unused bit-flip. Such an edge exists, since the number of bad colors at this step is at most $k-1$, and the edges of P_k correspond to k bit-flips, leaving $w-k$ new bits available to flip. Indeed, if $k \leq d$, then $2k-1 \leq 2\lfloor w/2 \rfloor - 1$, and thus, we have $k-1 < w-k$.

In this manner, we produce a rainbow path P_{d+1} of length $d+1$, where each edge uses a different bit-flip. Now, for $k \geq d+1$, we continue an extension process, obtaining P_{k+1} by adding an edge $x_k x_{k+1}$ to P_k so that $c(x_k x_{k+1})$ is new and $x_k x_{k+1}$ corresponds to a specified type of bit-flip. However, to ensure that such an edge $x_k x_{k+1}$ exists, we must somewhat relax the condition on which bit-flips may correspond to $x_k x_{k+1}$. To describe this new condition, we introduce the following notation.

Given $d+1 \leq k \leq w-1$, suppose P_k has been defined. We denote by S_d the subpath $(x_0, \ldots, x_d)$ of P_k, and for $d+1 \leq j \leq k$, we denote by S_j the subpath $(x_{2j-w}, \ldots, x_j)$ of P_k. We say that bit-flip i is *used by* S_j if some edge of S_j corresponds to bit-flip i. Rather than extending P_k by adding an edge $x_k x_{k+1}$ which does not correspond to the same bit-flip as any previous edge in P_k, we will only demand that $x_k x_{k+1}$ corresponds to a bit-flip not used by S_k. For $d \leq j < k$, we will denote by S_j^+ the concatenation of S_j and $x_j x_{j+1}$. Thus, to determine an acceptable choice for x_{j+1} as we are building our path, we ensure that $x_j x_{j+1}$ does not repeat a bit-flip used by S_j; once we add x_{j+1} to our path, the subpath S_j^+ contains no repeated bit-flips.

Claim 2. *For $d+1 \leq k \leq w-1$, there exists a rainbow path $P_k = (x_0, \ldots x_k)$ in $H_w/2$ such that for every j with $d \leq j \leq k-1$, S_j^+ contains no two edges corresponding to the same bit-flip. Moreover, $x_i x_j \notin E(H_w/2)$ if $i+1 \not\equiv j \mod w$.*

Proof of Claim 2. We proceed by induction. For $k = d+1$, we take P_{d+1} as described above. By construction, P_{d+1} is rainbow, and each edge corresponds to a different bit-flip. It is immediate that S_d^+ contains no two edges corresponding to the same bit-flip, and that two vertices x_i, x_j either are adjacent in P_{d+1} or are connected by a path corresponding to $\ell \geq 2$ distinct bit-flips. Note that if $2 \leq \ell < w-1$, then x_i, x_j are not adjacent in $H_w/2$. If

$\ell = w-1$, then $x_i = x_0$ and $x_j = x_{d+1}$, and we must have $d+2 \equiv 0 \mod w$. Thus, the statement holds for $k = d+1$.

Let $d+1 < k \leq w-1$ and suppose that there exists a rainbow path $P_{k-1} = (x_0, \ldots, x_{k-1})$ satisfying all conditions in the statement of Claim 2. We shall obtain a rainbow path P_k by extending P_{k-1}. Let N be the set of neighbors of x_{k-1} which are not on P_{k-1}. Since $k \leq w-1$, by the inductive hypothesis, x_{k-1} is adjacent to no vertex on P_{k-1} except for x_{k-2}, so $|N| = w-1$. Note that at most $k-2$ elements of N are adjacent to x_{k-1} via an edge whose color is not new for P_{k-1} (since the color of $x_{k-2}x_{k-1}$ is not new for P_{k-1}, but is not used for any edge between x_{k-1} and N). Moreover, exactly $w-k+1$ bit-flips are used by S_{k-1}, so at most $w-k$ elements of N are adjacent to x_{k-1} via an edge corresponding to a bit-flip used by S_{k-1} (since the bit-flip corresponding to $x_{k-2}x_{k-1}$ does not correspond to any edge from x_{k-1} to N). In total, there are at most $(k-2)+(w-k) = w-2$ elements x of N such that either $c(x_{k-1}x)$ is not new for P_{k-1} or $x_{k-1}x$ corresponds to a bit-flip used by S_{k-1}. Thus, there exists an element x_k of N such that $c(x_{k-1}x_k)$ is new for P_{k-1} and $x_{k-1}x_k$ does not correspond to a bit-flip used by S_{k-1}. We set $P_k = (x_0, \ldots, x_{k-1}, x_k)$.

Now, it remains to show that P_k satisfies all conditions in Claim 2. By construction, P_k is rainbow. For $j < k-1$, the inductive hypothesis on P_{k-1} implies that S_j^+ contains no two edges corresponding to the same bit-flip. By construction, S_{k-1}^+ contains no two edges corresponding to the same bit-flip, since $x_k x_{k+1}$ does not correspond to a bit-flip used by any edge of S_{k-1}, and since S_{k-1} is contained in S_{k-2}^+, implying that no two edges of S_{k-1} correspond to the same bit-flip. Thus, we only need verify that $x_i x_j \notin E(H_w/2)$ if $i+1 \not\equiv j \mod w$. Again, the inductive hypothesis on P_{k-1} guarantees this if both i, j are smaller than k. Thus, we consider pairs x_i, x_k. There are several cases to consider.

Case 1: $i = 0$ and $k \leq w-2$

Since S_d^+ has no repeated bit-flips, we have $d(x_0, x_{d+1}) = w-(d+1)$. Then since $d(x_0, x_{d+1}) \leq d(x_0, x_k) + d(x_k, x_{d+1})$, we have

$$w-(d+1) \leq d(x_0, x_k) + k - (d+1).$$

This implies $2 \leq w-k \leq d(x_0, x_k)$. Therefore, $x_0 x_k \notin E(H_w/2)$.

Case 2: $i = 1$

By assumption, S_{d+1}^+ has no repeated bit-flips and has length $w-d$, so we have $d(x_{2d-w+2}, x_{d+2}) = d$. Thus, $d(x_{2d-w+2}, x_{d+2}) \leq d(x_{2d-w+2}, x_1) + d(x_1, x_k) + d(x_k, x_{d+2})$, which implies

$$d \leq (2d-w+1) + d(x_1, x_k) + k - (d+2).$$

As a result, we have $w-k+1 \leq d(x_1, x_k)$, and hence $2 \leq d(x_1, x_k)$. Therefore, $x_1 x_k \notin E(H_w/2)$.

Case 3: $2 \leq i \leq 2(k-1)-w$

Let $j = \left\lfloor \frac{i+w}{2} \right\rfloor$. Note that $j \geq \left\lfloor \frac{2+w}{2} \right\rfloor = d+1$, and $j \leq \left\lfloor \frac{2(k-1)-w+w}{2} \right\rfloor = k-1$. Hence by assumption, S_j^+ has no repeated bit-flips, and is a path from x_{2j-w} to x_{j+1} with length $w-j+1$. We also note that $i-1 \leq 2j-w \leq i$ and $i < j+1$. If $i=2$ and w is odd, then $j = d+1$ and S_j^+ contains $w-(d+1)+1 = w-d = d+1$ edges. In this case, $2j-w = 1$ and $i = 2$, so $d(x_i, x_{j+1}) = d$, and we note that $d = j+1-i$. If w is even or $i \geq 3$, then S_j^+ contains $w-j+1 \leq d$ edges, and it follows that $d(x_i, x_{j+1}) = j+1-i$. We combine these observations with the inequality $d(x_i, x_{j+1}) \leq d(x_i, x_k) + d(x_k, x_{j+1})$, which implies

$$j+1-i \leq d(x_i, x_k) + (k-(j+1)).$$

Thus, we have $2+2j-i \leq d(x_i, x_k) + (w-1)$, which gives $2 \leq d(x_i, x_k)$. Therefore, $x_i x_k \notin E(H_w/2)$.

Case 4: $2(k-1)-w < i \leq k-2$ By assumption, S_{k-1}^+ has no repeated bit-flips and is a path from $x_{2(k-1)-w}$ to x_k with length $w-k+2$. Then x_i lies on this path, with $d(x_i, x_k) = k-i \geq 2$. Therefore, $x_i x_k \notin E(H_w/2)$.

□

With Claim 2 established, we construct a TBFRC in $H_w/2$ as follows. Let $P_{w-1} = (x_0, \ldots x_{w-1})$ be a rainbow path satisfying the conditions of Claim 2. Without loss of generality, $c(x_{i-1}x_i) = i$ for each $i \in \{1, \ldots, w-1\}$. Since the only neighbors of x_{w-1} on P_{w-1} are x_{w-2} and potentially x_0, there are at least $w-2$ ways to extend P_{w-1} to a path with w edges. By hypothesis, $H_w/2$ is rainbow P_{w+1}-free, so each of these $w-2$ edges must use colors from $\{1, \ldots, w-2\}$. This also means that there must exist an edge $x_{w-1}x_0$, and it must use a new color, w. Let $C_w := P_{w-1} + x_{w-1}x_0$. Thus, C_w is a rainbow cycle of length w. It remains to check that C_w is a TBFC.

Fix a representation of x_0. For $1 \leq i \leq w$, if x_{i-1} and x_i differ in bit j, then fix the representation of x_i obtained by flipping the jth bit of x_{i-1}. In this manner, we select representations for $x_0, \ldots, x_w$ so that for each $1 \leq i \leq j$, there is precisely one bit of difference between x_{i-1} and x_i. With these representations fixed, we proceed as follows. Since $x_0 x_{w-1} \in E(H_w/2)$, we have that (the fixed representations of) x_0 and x_{w-1} either differ in exactly one bit or differ in $w-1$ bits. Observe that if x_{w-1} differs from x_0 in $w-1$ bits, then C_w is a TBFC. Thus, it suffices to show that x_0 and x_{w-1} differ in more than one bit. Note that x_0 and x_{w-1} differ in bit i if and only if an odd number of edges in P_{w-1} correspond to the ith bit-flip. So if x_0 and x_{w-1} differ in exactly one bit, then the number of edges of P_{w-1} corresponding to bits in which x_0 and x_{w-1} do not differ is even, and the number of edges corresponding to the unique bit in which they do differ is odd. If w is odd, then P_{w-1} has an even number of edges, and it is thus impossible for x_0 and x_{w-1} to differ in exactly one bit. If w is even, consider the path $P = (x_d, x_{d+1}, \ldots x_0)$ on C_w. Because no bit-flips are repeated within S_d^+, we know that x_d in fact has distance $d = \frac{w}{2}$ from x_0. Since P is a path of length d from x_0 to x_d, P must also contain no two edges which correspond to the same bit-flip. Thus, the bit-flip corresponding to $x_d x_{d+1}$ is repeated nowhere in P_{w-1}. Thus, if x_0 and x_{w-1} differ only in one bit, then this is the same bit in which x_d differs from x_{d+1}; moreover, every bit-flip which corresponds to another edge in P_{w-1} in fact corresponds to at least two edges of P_{w-1}.

Now, we partition P_{w-1} into two sub-paths: $P_1 = (x_0, \ldots, x_d)$ and $P_2 = (x_{d+1}, \ldots x_{w-1})$. Since both P_1 and P_2 contain no repeated bit-flips, it must be the case that every bit-flip corresponding to an edge of P_1 also corresponds to an edge of P_2. However, P_1 has length

d and P_2 has length $d-2$, and thus some bit-flip corresponding to an edge of P_1 does not correspond to an edge of P_2. We conclude that x_0 and x_{w-1} differ in more than one bit; thus, C_w is a TBFRC. □

Now, the existence of a TBFRC in any rainbow P_{w+1}-free edge-coloring of $H_w/2$ will imply that in fact, all rainbow P_{w+1}-free edge-colorings of $H_w/2$ are equivalent up to relabeling of colors. To see this, we also require the following lemma.

Lemma 3.6. *For $w \geq 5$, a TBFRC in $H_w/2$ can be uniquely extended to a rainbow P_{w+1}-free edge-coloring of $H_w/2$.*

Proof. Suppose that we have a TBFRC $C_w := (x_1, x_2, \ldots, x_w, x_1)$, where x_1 is an arbitrary starting point. Reindexing if necessary, we may assume that $x_i x_{i+1}$ corresponds to the ith bit-flip in $H_w/2$. We identify any TBFC starting at x_1 with the permutation on $[w]$ given by the order of bit-flips corresponding to the edges of this cycle; thus, C_w corresponds to the identity permutation. We may also assume without loss of generality that $c(x_i x_{i+1}) = i$ (with subscripts taken modulo $w+1$).

Note that if x is on a TBFRC C, and $y \in N(x)$ is not on C, then $c(xy)$ must be a color appearing on C to avoid a rainbow P_w-copy. Also note that the only vertices on C which are adjacent to x are the two neighbors of x along C, since a vertex at distance at least 2 from x in C differs from x in at least two bit-flips. Thus, x is incident to only edges receiving colors used on C. Since $d(x) = w$, this implies that x is incident to edges of all colors used on C.

We now consider TBFC's in $H_w/2$ which interact with C_w. For a fixed $2 \leq i \leq w$, let y be the vertex such that $x_{i-1}y$ corresponds to the ith bit-flip in $H_w/2$. Note that yx_{i+1} is an edge of $H_w/2$ corresponding to the $(i-1)$st bit-flip. So,

$$C(i, i-1) := (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots x_w, x_1)$$

is a TBFC not containing x_i, which corresponds to the transposition $(i, i-1)$. Since we assume $w \geq 5$, note also that $N(x_{i-1}, x_{i+1})$ is exactly equal to $\{x_i, y\}$. (If $w = 4$, this statement does not hold; common neighborhoods in $H_w/2$ have size 3.) We now consider the edge-coloring of $C(i, i-1)$.

We claim that $c(x_{i-1}y) = i$ and $c(yx_{i+1}) = i-1$. Indeed, if this did not hold, then by the above observations on the edges incident to x_{i-1} and x_{i+1}, there would exist edges $x_{i-1}u$ and $x_{i+1}v$ such that $u \neq v$ and $u, v \notin V(C(i, i-1))$ with $c(x_{i-1}u) = i$ and $c(x_{i+1}v) = i-1$. (We allow for the possibility that one of u, v is equal to y.) This would yield that $(u, x_{i-1}, x_{i-2}, \ldots, x_1, x_w, \ldots, x_{i+1}, v)$ is a rainbow P_{w+1}-copy in $H_w/2$, a contradiction. Thus, $c(x_{i-1}y) = i$ and $c(yx_{i+1}) = i-1$, so $C(i, i-1)$ is a TBFRC. Moreover, the edge-coloring of $C(i, i-1)$ is uniquely determined, with the property that each edge of $C(i, i-1)$ has the same color as the bit-flip to which it corresponds.

Repeatedly applying the above argument, if C' is a TBFC containing x_1 which corresponds to a product of transpositions of the form $(i, i-1)$, then C' has a rainbow edge-coloring determined by the permutation to which it corresponds. That is, if u, v are adjacent on C' and differ in bit i, then $c(u, v) = i$. Note that every TBFC containing x_1 corresponds to some permutation of $[w]$, and it is well-known that every permutation of $[w]$ can be written as the product of permutations of the form $(i, i-1)$. Thus, if uv is an edge of $H_w/2$ such that uv is in some TBFC containing x_1, then $c(uv)$ is equal to the bit in which u, v differ. We observe that given x_1 and uv (possibly with $x_1 = u$), it *is* possible to construct a TBFC beginning at x_1 and containing uv. Indeed, let P_u be a shortest path between x_1 and u, and P_v a shortest path between x_1 and v. Thus, P_u and P_v do not repeat bit-flips. Without loss of generality, we may assume $|e(P_u)| \leq |e(P_v)|$. Suppose that uv corresponds to bit-flip i, and there is also an edge of P_u corresponding to bit-flip i. Let B be the set of bit-flips different from i which correspond to edges of P_u. Note that v must differ from x_1 in precisely the bits of B, so there is a path of length $|B| < |e(P_u)|$ from x_1 to v, contradicting the assumption that $|e(P_u)| \leq |e(P_v)|$. Thus, $P_u + uv$ is a path with no repeated bit-flips containing x_1 and uv. It is clear that $P_u + uv$ can be extended to a TBFC. Thus, uv is colored by the bit in which u, v differ. □

We are now equipped to prove our main result, Theorem 3.7, on the edge-colorings of $H_w/2$.

Theorem 3.7. *For $w \geq 3$, there is a unique (up to relabeling) rainbow P_{w+1}-free edge-coloring of $H_w/2$.*

Proof. If $w \geq 5$, this follows quickly from Lemmas 3.3,3.5, and 3.6. Indeed, suppose c and c' are two rainbow P_{w+1}-free edge-colorings of $H_w/2$. (By Lemma 3.3, at least one rainbow P_{w+1}-free edge-colorings of $H_w/2$ exists.) By Lemmas 3.5 and 3.6, each of c and c' contain a TBFRC, and in fact every TBFC in $H_w/2$ is rainbow under both c and c'. Fix in particular the TBFC in $H_w/2$ whose ith edge corresponds to bit-flip i, which we shall call $C = (x_1, x_2, \ldots, x_w)$. Relabel the colors of c' so that $c(x_ix_{i+1}) = c'(x_ix_{i+1})$ for every $i \in \{1, \ldots, w\}$. By Lemma 3.6, every color which appears in c' appears on some edge of C, so this relabeling of c' indeed extends to a relabeling of every edge-color under c'. Now, under the described relabelling, c and c' agree on a TBFRC, so by Lemma 3.6, c and the relabelling of c' must be equal.

Thus, it remains only to verify the result for $w \in \{3, 4\}$. If $w = 3$, then we have $H_3/2 = K_4$. The only proper edge-coloring of K_4 that avoids a rainbow P_4-copy is the 3-coloring under which each color class forms a perfect matching.

For $w = 4$, observe that $H_4/2 = K_{4,4}$. We show that there exists one rainbow P_5-free edge-coloring of $K_{4,4}$ up to isomorphism. Let $A \cup B$ be the bipartition of $V(K_{4,4})$ where every pair $a_i \in A, b_j \in B$ forms an edge a_ib_j. It is not difficult to check that, to avoid a rainbow P_5-copy, $K_{4,4}$ must be 4 edge-colored. Thus, say we edge-color $K_{4,4}$ using colors $\{1, 2, 3, 4\}$. Without loss of generality, let $c(a_1b_j) = j$ and $c(a_ib_1) = i$.

Consider the edge a_ib_i for $i \geq 2$. If $c(a_ib_i) = j \neq 1$, then there exists a path $a_ib_ia_1b_1a_k$ for $k \neq j$ that is rainbow. Therefore $c(a_ib_i) = 1$.

Consider the edge a_ib_j for $i, j \geq 2, i \neq j$. To maintain a proper edge-coloring, $c(a_ib_j)$ is not i, j, or 1, and thus $c(a_ib_j)$ must be the unique color in $[4] \setminus \{1, i, j\}$. Thus, up to relabelling, there is a unique rainbow P_5-free edge-coloring of $K_{4,4}$. For reference, we depict the described edge-coloring in Figure 7. □

Now that we have established a precise understanding of the rainbow P_{w+1}-free colorings of $H_w/2$, we are ready to define and study a set of constructions arising from the folded hypercubes. Theorem 3.7 implies that, while it is possible to properly edge-color $H_w/2$ while

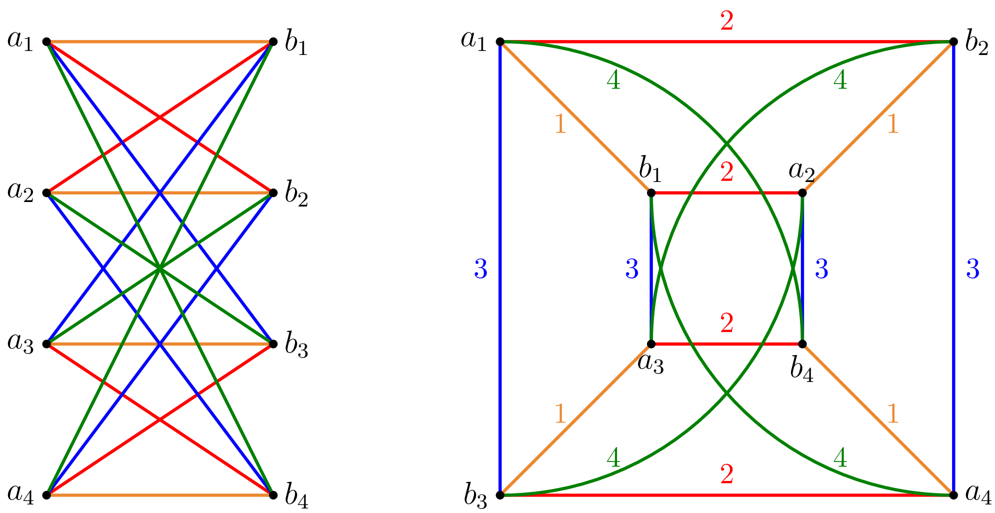

Figure 7: The unique coloring of $H_4/2$, viewed as $K_{4,4}$ and as a folded hypercube

avoiding a rainbow P_{w+1}-copy, any such edge-coloring contains many rainbow P_w-copies. In particular, every sequence of $w-1$ distinct bit-flips corresponds to a rainbow P_w-copy in $H_w/2$. In the following observation, we note that it is possible to construct such paths from any desired starting point, and which avoid any desired vertex or bit-flip.

Observation 3.8. *Let $w \geq 3$, $j \in [w]$, and $u, v \in V(H_w/2)$ be distinct vertices. Then there exist paths P and P' of length $w-1$ starting at u in which each edge corresponds to a distinct bit-flip, such that no edge in P corresponds to bit-flip j and v is not a vertex in P'.*

Using this observation, we will show the desired upper bound on $\text{sat}^*(n, P_k)$. Consider the following construction.

Construction 3. *Fix $k \geq 6$ and $n \geq (k-1)2^{k-4}$. Let $G_k(n)$ be the graph obtained from $H_{k-3}/2$ by attaching $(k-2)$ pendant edges to each vertex of $H_{k-3}/2$ except for 0^{k-3}, and $n-(k-1)2^{k-4}+k-2$ pendant edges to 0^{k-3}.*

We call the subgraph of $G_k(n)$ isomorphic to $H_{k-3}/2$ the *core* of $G_k(n)$, and call a vertex of $G_k(n)$ not in the core a *pendant vertex*. We illustrate $G_6(20)$ and a more general depiction of $G_6(n)$ in Figure 8.

Note that when $n = (k-1)2^{k-4}$, every vertex of the core of $G_k(n)$ is incident to exactly $k-2$ pendant vertices. In general, we have

$$\begin{aligned} |E(G_k(n))| &= E(H_{k-3}/2) + (k-2)(2^{k-4}-1) + n - (k-1)2^{k-4} + k - 2 \\ &= n + (k-3)2^{k-5} + (k-2)2^{k-4} - (k-2) - (k-1)2^{k-4} + k - 2 \\ &= n + (k-3)2^{k-5} - 2^{k-4} \\ &= n + (k-5)2^{k-5}. \end{aligned}$$

We will show that $G_k(n)$ is properly rainbow P_k-saturated in Theorem 3.10 with the help of the following simple lemma.

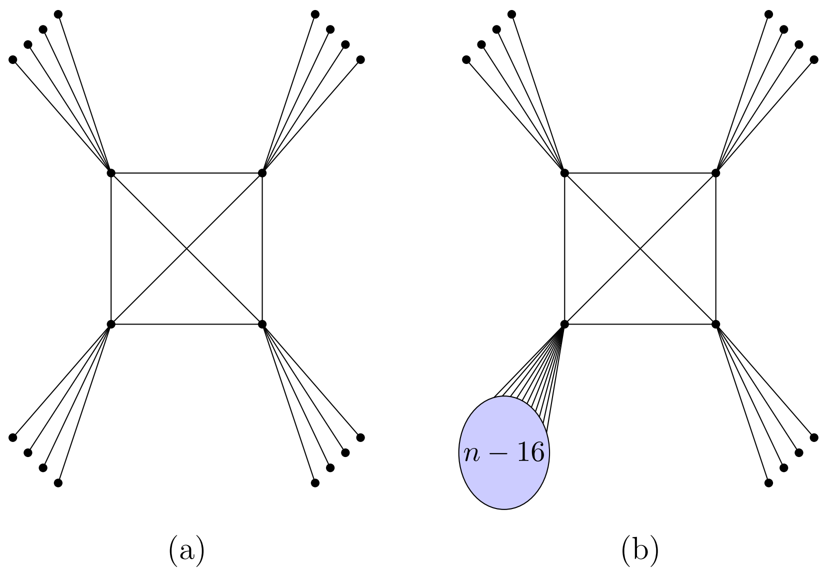

Figure 8: (a): The construction $G_6(n)$ for $n = 20$; (b): for general n

Lemma 3.9. *Let $k \geq 3$ let C be an n-vertex graph with $n \geq 1$. Let G be a graph obtained from C by attaching at least $k-2$ pendant edges to each vertex of C. Then a proper edge-coloring c of G contains a rainbow P_k-copy if and only if the restriction of c to C contains a rainbow P_{k-2}-copy.*

Proof. First, suppose G contains a rainbow copy of P_k, say $P = (x_0, x_1, \ldots, x_k)$. Note that P contains at most two pendant edges of G, and any pendant edge contained in P must contain either x_0 or x_k. Thus, the sub-path $(x_1, \ldots, x_{k-1})$ is a rainbow P_{k-2}-copy contained C.

On the other hand, suppose C contains a rainbow P_{k-2}-copy, say P', with endpoints u and v. Note that u and v are each incident to at least $k-2$ pendant edges in G. Furthermore, one of the $k-3$ edges on P' is incident to u and another is incident to v. So, only $k-4$ of the edge-colors used on P' may appear among the $k-2$ pendant edges incident to u, and a (possibly different) set of $k-4$ of these colors may appear among the $k-2$ pendant edges incident to v. In particular, there are 2 pendant edges incident to u and two pendant edges incident to v whose edge-colors are distinct from those on P'. We may extend P' via two of these pendant edges to a rainbow P_k-copy in G. □

Observe that Lemma 3.9 can be applied to $G_k(n)$, with the core of $G_k(n)$ acting as C; however, we will also apply Lemma 3.9 in a case where C may not equal the core of $G_k(n)$.

Now, we are ready to prove that $G_k(n)$ is properly rainbow P_k-saturated.

Theorem 3.10. *For $k \geq 6$ and $n \geq (k-1)2^{k-4}$, $G_k(n)$ is rainbow P_k-saturated. In particular,* $\mathrm{sat}^*(n, P_k) \leq n + (k-5)2^{k-5}$.

Proof. By Theorem 3.7, there is (up to relabelling of colors) a unique rainbow P_{k-2}-free coloring of the core, and by Lemma 3.9, this extends to a rainbow P_k-free coloring of $G_k(n)$.

Next, we show that for any edge $e \notin E(G_k(n))$, any proper edge-coloring of $G_k(n) + e$ contains a rainbow P_k-copy. Assume, for the sake of contradiction, that some coloring c of $G_k(n) + e$ is rainbow P_k-free for some $e \notin E(G_k(n))$. By Lemma 3.9, the restriction of c to the core must be rainbow P_{k-2}-free. By Theorem 3.7, the core must be colored so that edges corresponding to the same bit-flip receive the same color.

Now, either both endpoints of e are pendant vertices, or else e is incident to the core. Suppose first that both endpoints x, y of e are pendant vertices. Say x, y have neighbors u and v, respectively, in the core. As a consequence of Theorem 3.7, the core is colored with precisely $k - 3$ edge-colors, say $\{1, 2, \dots, k - 3\}$, and in particular, u and v are each incident to edges of every color $i \in \{1, 2, \dots, k - 3\}$ within the core. So $c(ux)$ and $c(vy)$ are not in $\{1, 2, \dots, k - 3\}$. Now, whether or not $c(xy) \in \{1, 2, \dots, k - 3\}$, Observation 3.8 implies that there exists a rainbow P_{k-3}-copy P in the core which has u as an endpoint and does not contain an edge of color $c(xy)$. Let w be the endpoint of this path not equal to u. Of at least $k - 2$ pendant edges incident to w, at least $k - 3$ are incident to neither x nor y. As $k - 3 \geq 3$, at least one such pendant edge, say wz, has $c(wz) \notin \{c(xy), c(ux)\}$. Observe that the concatenation of (y, x, u), P, and (w, z) is a rainbow P_k-copy in $G_k(n) + e$, a contradiction.

Next, suppose e is incident to the core. Let C be the subgraph of $G_k(n)$ induced on the vertices of the core (so either C is the core, or C is the core with an added edge). By Lemma 3.9, to avoid a rainbow P_k-copy, we must color so that C is rainbow P_{k-2}-free. We claim that this is not possible if both endpoints of e are incident to the core.

Indeed, to avoid an immediate contradiction, all core edges which are not equal to e still must be colored as described by Theorem 3.7. If both endpoints u, v of e are contained in the core, then by Observation 3.8, there exists a rainbow P_{k-3}-copy P in the core which has u as an endpoint and does not contain v. Note also that, since u, v are already incident to edges of all colors from $\{1, 2, \dots, k - 3\}$, we must have $c(uv) \notin \{1, 2, \dots, k - 3\}$. Thus, concatenating uv with P yields a rainbow P_{k-2}-copy in the core, which can be extended to a rainbow P_k-copy in $G_k(n) + e$ by Lemma 3.9, a contradiction.

Finally, we consider the case where one endpoint u of e is in the core and the other, say x, is a pendant vertex. Note again that $c(ux) \notin \{1, 2, \dots, k - 3\}$ and since $ux \notin E(G_k(n))$, x is adjacent to a core vertex distinct from u, say v. Now, using Observation 3.8, there is a rainbow P_{k-3}-copy P in the core with v as an endpoint which does not contain u. Concatenating P with (u, x, v) and a pendant edge incident to the other endpoint of P will again yield a rainbow P_k-copy in $G_k(n) + e$, a contradiction. ☐

Finally, for the sake of completeness, we derive Theorem 1.2

Theorem 1.2. *For $k \geq 5$ and $n \geq (k-1)2^{k-4}$, we have*

$$n - 1 \leq \text{sat}^*(n, P_k) \leq n + O(2^k).$$

Proof. We have the lower bound by Proposition 3.1 and the upper bound for $k \geq 6$ by Theorem 3.10. For $k = 5$, consider the n-vertex graph $G_5(n)$ obtained by attaching $n - 4$ pendant edges to one vertex of a K_4-copy; we again call this K_4-copy the *core* of $G_5(n)$ and say that vertices outside the core are *pendant vertices*. Note that we can extend the perfect matching coloring of the core to a rainbow P_5-free proper edge-coloring of $G_5(n)$. We claim that in fact, $G_5(n)$ is properly rainbow P_5-saturated.

Let x be the core vertex to which all pendant vertices of $G_5(n)$ are adjacent. Since $n \geq 4 \cdot 2 = 8$, x has at least 4 pendant vertex neighbors. Since the core is complete, any edge added to $G_5(n)$ either connects two pendant vertices u, v or connects a core vertex $y \neq x$ to a pendant vertex u. Note that if we add edge uv to $G_5(n)$, then $\{u, v, x\}$ must span a (rainbow) triangle, while if we add edge uy to $G_5(n)$, then there exist two other pendant vertices v, w such that $c(ux), c(vx), c(wx)$, and $c(uy)$ are pairwise distinct. We depict these two cases in Figure 9, also labeling the other core vertices of $G_5(n)$.

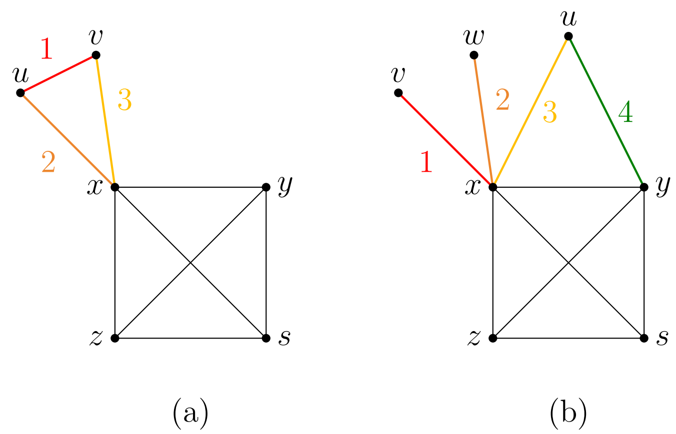

Figure 9: (a): $G_5(n)$ with uv added between pendant vertices; (b): $G_5(n)$ with uy added between a pendant vertex and a core vertex

In $G_5(n) + uv$, note that at most one of $c(xy), c(xz), c(xs)$ is equal to 1; without loss of generality, $c(xy) \neq 1$. Thus, both (u, v, x, y) and (v, u, x, y) are rainbow P_4-copies in $G_5(n) + uv$. Note also that $c(yz), c(ys)$ are not equal to $c(xy)$, and at most one is equal to 1. Without loss of generality, $c(yz) \neq 1$. Then either (u, v, x, y, z) or (v, u, x, y, z) is a rainbow P_5-copy in $G_5(n) + uv$. Thus, any proper edge-coloring of $G_5(n) + uv$ contains a rainbow P_5-copy.

In $G_5(n) + uy$, note that at most one of $c(yz), c(ys)$ is equal to 3. Without loss of generality, $c(yz) \neq 3$. Then either (v, x, u, y, z) or (w, x, u, y, z) is a rainbow P_5-copy in $G_5(n) + uy$. Thus, any proper edge-coloring of $G_5(n) + uy$ contains a rainbow P_5-copy.

We conclude that $G_5(n)$ is properly rainbow P_5-saturated. Note that $|E(G_5(n))| = n + 2$. Thus, the desired upper bound also holds for $k = 5$. □

4 Cycles

Theorem 1.3. *For odd $k \geq 7$ and for $n \geq 3k - 2$, we have*

$$\mathrm{sat}^*(n, C_k) \leq \binom{k-1}{2} n - \binom{\frac{k+1}{2}}{2}.$$

Proof. Fix odd $k \geq 7$, and suppose $n \geq 3k - 2$. Let $G = X \vee Y$, where $X = K_{\frac{k-1}{2}}$ and $Y = E_{n - \frac{k-1}{2}}$. We will show that G is rainbow C_k-saturated.

Set $V(X) = \{x_1, \dots, x_{\frac{k-1}{2}}\}$ and $V(Y) = \{y_1, \dots, y_{n-|V(X)|}\}$. Since $|V(X)| = \frac{k-1}{2}$ and Y is an independent set, there are no odd cycles of length k or greater in G. Let $G' = G + y_1y_2$ and $c : E(G') \to \mathbb{N}$ be a proper edge-coloring of G'. Observe that the double star $S(1,2)$ is a subgraph of $G'[x_1, x_2, x_3, y_1, y_2]$ with centers y_1, y_2. So, there is a rainbow path of length 3 containing y_1y_2. Without loss of generality, suppose this rainbow path is $P = x_1y_1y_2x_2$.

We begin building our rainbow C_k by setting $C := P$ and greedily building the rest. For an edge $e \in G'$, we call e a bad edge if $c(e) \in c(C)$ and otherwise, e is a good edge. Set

$$S_3 := \{y \in Y : y \notin V(C) \text{ and } x_2y, x_3y \text{ are good edges}\}.$$

Observe that there can be at most two bad edges outside of $E(C)$ incident to x_2 and at most three bad edges incident to x_3. Therefore, if $|V(Y)| \geq 2 + 2 + 3 + 1 = 8$, then $S_3 \neq \emptyset$. Since $|V(Y)| \geq \frac{5}{2}(k-1) + 1 \geq 16$, this inequality holds. By relabeling, we may assume $y_3 \in S_3$. Now, add the edges x_2y_3, y_3x_3 to C. Continuing in this way, after relabeling, suppose $C = x_1y_1y_2x_2y_3x_3 \dots y_ix_i$, $i < \frac{k-1}{2}$. Then, set

$$S_{i+1} := \{y \in V(Y) : y \notin V(C) \text{ and } x_iy, x_{i+1}y \text{ are good edges}\}.$$

As before, observe that there can be at most $|E(C)| - 1 = 2i$ bad edges outside of $E(C)$ incident to x_i and at most $|E(C)| = 2i + 1$ bad edges incident to x_{i+1}. Therefore, $S_{i+1} \neq \emptyset$ if the following inequality holds:

$$|V(Y)| \geq |V(C) \cap V(Y)| + (2i) + (2i+1) + 1 = i + (2i) + (2i+1) + 1 = 5i + 2$$

Since $|V(Y)| \geq \frac{5}{2}(k-1)+1$, this inequality holds for all $3 \leq i < \frac{k-1}{2}$ and we may continue building C. Finally, suppose $C = x_1y_1y_2x_2y_3x_3 \dots y_{\frac{k-1}{2}}x_{\frac{k-1}{2}}$ and set

$$S_{\frac{k-1}{2}} := \{y \in V(Y) : y \notin V(C) \text{ and } x_1y, x_{\frac{k-1}{2}}y \text{ are good edges}\}.$$

Now, there can be at most $|E(C)| - 1 = k - 1$ bad edges outside of $E(C)$ incident to x_1 and same for $x_{\frac{k-1}{2}}$. Therefore, $S_{\frac{k-1}{2}} \neq \emptyset$ if the following inequality holds:

$$|V(Y)| \geq |V(C) \cap V(Y)| + (k-1) + (k-1) + 1 = \frac{k-1}{2} + 2(k-1) + 1 = \frac{5}{2}(k-1) + 1.$$

Again, since $|V(Y)| \geq \frac{5}{2}(k-1)+1$, the inequality holds and we may find $y_{\frac{k+1}{2}} \in S_{\frac{k-1}{2}}$, add the edges $x_1y_{\frac{k+1}{2}}$ and $x_{\frac{k-1}{2}}y_{\frac{k+1}{2}}$ to C, completing our rainbow C_k. Therefore, G' is rainbow C_k-saturated.

Finally, we count the edges of G' as follows:

$$\begin{aligned}
|E(G')| &= |E(X)| + |E(X,Y)| + |E(Y)| \\
&= \binom{\frac{k-1}{2}}{2} + |V(X)|(n - |V(X)|) + 0 \\
&= \binom{\frac{k-1}{2}}{2} + \left(\frac{k-1}{2}\right)\left(n - \frac{k-1}{2}\right) \\
&= \left(\frac{k-1}{2}\right)n - \binom{\frac{k+1}{2}}{2}.
\end{aligned}$$

□

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