BILL, RECORD LECTURE!!!!

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Kruskal's Tree Theorem and Two More Fast Growing Functions

Exposition by William Gasarch

February 27, 2025

Today's Lesson

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Warning Part of this talk will be on the whiteboard.

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Note If $(X, \preceq)$ is a wqo then its both well founded and has no infinite antichains.

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Will use this lemma without pointing it out.

wqo's Closed Under $\times$

Def If $(X, \preceq_1)$ and $(Y, \preceq_2)$ are wqo then we define $\preceq$ on $X \times Y$ as $(x, y) \preceq (x', y')$ if $x \preceq_1 x'$ and $y \preceq_2 y'$.

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If color has DOWN or INC in it then violates wqo.
The color must be UP-UP. This shows that there is an infinite ascending sequence.

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THIS IS SIMILAR TO THE KTT. MIGHT BE A HW.

tree: A Fast Growing Functions

HW Proof that $\exists$ function $\mathrm{tree}(n)$ such that:
$\mathrm{tree}(n)$ is largest number such that $\exists$ a sequence of trees
$T_1, T_2, \ldots, T_{\mathrm{tree}(n)}$
with the following properties.

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Known:

$\mathrm{tree}(1) = 1 \qquad \mathrm{tree}(2) = 5 \qquad \mathrm{tree}(3) \geq 844,424,930,181,960$

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tree grows **much faster** than Ackermann's function.

TREE: A Faster Growing Functions

HW Show that there is a function $\mathrm{TREE}(n)$ such that the following holds:
$\mathrm{TREE}(n)$ is the largest number such that there exists a sequence of n-colored trees
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Known:
$\mathrm{TREE}(1) = 1$ $\mathrm{TREE}(2) = 3$ $\mathrm{TREE}(3)$: See Next Slide.

tree vs TREE

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Suffice to say that $\mathrm{TREE}(n)$ grows much faster than $\mathrm{tree}(n)$.

TREE

TREE might be the faster growing **natural** function in mathematics.