High School Proofs for Better Bounds on the Quadratic van der Waerden Numbers

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Abstract. A corollary of the polynomial van der Waerden theorem is that, for any polynomial $p(x) \in \mathbb{Z}[x]$ with constant term 0, for any $c \in \mathbb{N}$, there exists $W \in \mathbb{N}$ such that, for all c-colorings of $\{1, \ldots, W\}$ there exists a, d such that a and $a + p(d)$ are the same color. The proof of the polynomial van der Waerden theorem, and even of these corollaries, is difficult and gives enormous upper bounds for W. We consider just quadratic polynomials. For $c = 2, 3$ we obtain reasonable bounds, and for $c = 4$ for some quadratics we obtain reasonable bounds, using simple methods.

1. INTRODUCTION We use the following standard definition.

Definition. Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ be the set of non-negative integers, and $\mathbb{N}^+$ be the set of positive integers. Let $[W]$ be the set $\{1, \ldots, W\}$ (where $W \in \mathbb{N}$).

In this paper we will give *High School Proofs (HS Proof)* of theorems. The term *High School Proof* is not a formal term. We use it to mean a proof that can be explained to a bright high school student. We use the term *High School Proof* since (1) the term *elementary* is ambiguous, and (2) the term *Combinatorial* is not quite right since (a) the rather difficult proof of Szemerédi's Theorem is combinatorial, and (b) the rather difficult proof of Gower's bound is mostly combinatorial.

Recall van der Waerden's Theorem [**1, 2**] (see also the books by Graham-Rothchild-Spencer [**3**] and Landman-Robertson [**4**]).

Theorem 1. *For any $k \in \mathbb{N}$, for any $c \in \mathbb{N}$, there exists $W = W(k, c)$, such that for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \neq 0$, such that $a, a + d, \ldots, a + (k - 1)d$ are all the same color.*

The original proof by van der Waerden was HS but yielded bounds on W that were not primitive recursive [**3**]. Shelah [**5**] gave a HS proof that yielded primitive recursive bounds on W. These bounds were still quite large in that they really cannot be written down nicely. Gowers [**6**] gave a non-HS proof that yielded bounds that can be written down:

$$W(k, c) \leq 2^{2^{c^{2^{2^{k+9}}}}}$$

We discuss a known generalization of van der Waerden's theorem. Note that the conclusion of van der Waerden's theorem is that

$$a, a + d, a + 2d, \ldots, a + (k - 1)d \text{ are the same color.}$$

Can we replace $d, 2d, \ldots, (k - 1)d$ by other functions of d? Yes. We can replace them with polynomials in $\mathbb{Z}[x]$ that have no constant term. Here is the Polynomial van der Waerden Theorem:

Theorem 2. *Let $p_1, \ldots, p_k \in \mathbb{Z}[x]$ such that, for $1 \le i \le k$, $p_i(0) = 0$. Let $c \in \mathbb{N}$. Then there exists $W = W(p_1, \ldots, p_k; c)$ such that, for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \ne 0$, such that $a, a + p_1(d), \ldots, a + p_k(d)$ are all the same color.*

For $k = 1$, this theorem was proven independently by Furstenberg [**7**] and Sárközy [**8**]. Bergelson and Leibman [**9**] proved the general result using ergodic methods (not a HS proof). These proofs yielded no upper bounds on $W(p_1, \ldots, p_k; c)$. Walters [**10**] obtained a HS proof of Theorem 2, but the bounds on W were not primitive recursive. Shelah [**11**] gave a (non HS) proof that yielded primitive recursive bounds on W. These bounds were still quite large in that they really cannot be written down nicely. Nobody has obtained a proof that yields bounds one can write down.

Peluse [**12**] and Peluse and Prediville [**13**] proved density results that can be translated into bounds for some polynomial van der Waerden numbers.

1. Peluse and Prediville [**13**] showed that there exists a d such that for large n, $W(x, x^2; (\log\log n)^d) \le n$.
2. Peluse [**12**] showed that if $p_1, \ldots, p_m \in \mathbb{Z}[x]$ are polynomials of different degrees then there exists a constant d (which depends on $p_1, \ldots, p_m$) such that, for large n, $W(p_1(x), \ldots, p_m(x); (\log\log n)^d) \le n$.
3. Peluse and Prediville [**14**] showed that there exists a d such that for large n, $W(x, x^2; (\log n)^d) \le n$.

These proofs are not HS.

We are interested in the case of $W(ax^2 + bx; c)$ where $c = 2, 3, 4$. Furstenberg's proof showed that $W(x^2; c)$ exists; however, his proof gave no upper bounds. Sárközy's proof showed that $W(x^2; c) \le 2^{O(c^3)}$. Pintz, Steiger, and Szemerédi [**15**] (see also [**16**] for exposition) showed that $W(x^2; c) \le 2^{O(c^{0.0001})}$. The 0.0001 can be replaced with any smaller constant; however, in that case the constant associated with the big-O will increase. It is possible that either Sárközy's proof of $W(x^2; c) \le 2^{O(c^3)}$ or Pintz, Steiger, and Szemerédi proof of $W(x^2; c) \le 2^{O(c^{0.0001})}$ could be modified with a fixed value of c such as 4. That may lead to an improvement on our bound on $W(x^2; 4)$; however, such a proof would not be HS.

Harnel, Lyall, and Rice [**17**] showed that there exists a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ such that

$$W(ax^2 + bx; c) \le 2^{f(a,b)c^{0.0001}}$$

(the 0.0001 can be replaced with any smaller constant; however, in that case the function f will be bigger).

Later Rice [**16**] showed that, for all k, there exists a function $f : \mathbb{Z}^k \to \mathbb{N}$ such that

$$W(a_k x^k + \cdots + a_1 x; c) \le 2^{f(a_k, \ldots, a_1)c^{0.0001}}$$

(the 0.0001 can be replaced with any smaller constant; however, in that case the function f will be bigger). Rice (personal communication) later obtained the following more precise result: for all $\epsilon > 0$, for all $a_1, \ldots, a_k \in \mathbb{Z}$, for $J = |a_1| + \cdots + |a_k|$:

$$W(a_k x^k + \cdots + a_1 x; c) \le 2^{2^{2^{100k^2/\epsilon}}} + 2^{2^{(100k^4 \log J)^{100}}} + 2^{c^\epsilon}$$

In summary, the known bounds on $W(ax^2 + bx; c)$ are large.

In this paper we show that, for some $p \in \mathbb{Z}[x]$ and $c = 2, 3, 4$, one can obtain much better bounds on $W(p(x); c)$. Our proofs will be purely combinatorial and much easier than those of Walters, Shelah, and Peluse. We hasten to point out that they proved the full polynomial van der Waerden theorem whereas we only prove it in very special cases.

We will show the following.

- For all $a \in \mathbb{Z}$, $W(ax; c) = |ac| + 1$. (Theorem 5)
- For all $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 2) \leq 12|a| + 6|b|$. (Theorem 6 We actually obtain more precise bounds than that depending on how a, b are related to each other. In Appendix A is a table of some exact values of $W(ax^2 + bx; 2)$.
- For all $a \in \mathbb{N}$, $a \geq 1$, $W(ax^2 + (a-1)x; 2) = 8a - 3$. (Theorem 8)
- $W(x^2; 3) = 29$ and, for all $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$. (Theorem 10)
- For $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 3) = O(ab^6 + a^5b^2)$. (Theorem 15) In Appendix B is a table of some exact values of $W(ax^2 + bx; 3)$.
- $W(x^2; 4) \leq$ 84,149,474,894,213,522. (Theorem 16) In Appendix C is a table of some upper bounds on $W(ax^2 + bx; 4)$.

2. PRELIMINARIES

Definition. Let $c \in \mathbb{N}^+$ and $W \in \mathbb{N}^+$.

1. A *c-coloring of* $[W]$ is a mapping $[W] \to [c]$.
2. Let $p \in \mathbb{Z}[x]$. A $(p; c)$*-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all distinct $x, y \in [W]$, if $y - x = p(d)$ for some $d \in \mathbb{Z}$, then x and y have different colors. When the context is clear, we will often write *proper c-coloring* or simply *proper coloring*.

Note that the polynomial van der Waerden number, $W = W(p(x); c)$, is the least number such that there is no $(p; c)$-proper coloring of $[W]$.

Although we care about proper $(p; c)$-colorings, we need a more general notion:

Definition. Let $F \subseteq \mathbb{Z}$, $c \in \mathbb{N}^+$, and $W \in \mathbb{N}^+$.

- An $(F; c)$*-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$ with $y - x \in F$, x and y have different colors.
- $W = W(F; c)$ is the least number such that there is no $(F; c)$-proper coloring of $[W]$. If no such number exists, we set $W(F; c) = \infty$.
- In the above definitions F is the set of *forbidden distances*. We will use this term for polynomial van der Waerden numbers as well. For example, if looking at $W(3x^2; c)$ the forbidden distances are $3 \times 1^2, 3 \times 2^2, \ldots$.

We leave the following easy lemma to the reader.

Lemma 3. *Let* $c \in \mathbb{N}^+$.

1. *If* $0 \in F$ *then* $W(F; c) = 1$.
2. *Assume* $f \in F$. *Let* $F' = F \cup \{-f\}$. *Then* $W(F; c) = W(F'; c)$.

Lemma 4. *Let* $p \in \mathbb{Z}[x]$, $a \in \mathbb{N}^+$, *and* $c \in \mathbb{N}$. *Then*

$$W(ap; c) = a(W(p; c) - 1) + 1.$$

Proof.
1) $W(ap;c) \le a(W(p;c)-1)+1$:

Assume, by way of contradiction, that $W(ap;c) \ge a(W(p;c)-1)+2$. Hence there exists COL, an $(ap;c)$-proper coloring of $[a(W(p;c)-1)+1]$. Note that, for all x, $ap(x)$ is a forbidden distance for COL.

We use COL to define COL$'$, a proper $(p;c)$-coloring of $[W(p;c)]$; which contradicts the definition of $W(p;c)$.

For $1 \le i \le W(p;c)$ let

$$\mathrm{COL}'(i) = \mathrm{COL}(a(i-1)+1).$$

Suppose $j-i$ is a forbidden distance for COL$'$. Then there exists x such that $j-i=p(x)$. Then

$$a(j-1)+1-(a(i-1)+1) = a(j-i) = ap(x), \text{ a forbidden distance for COL.}$$

Hence $\mathrm{COL}(a(j-1)+1) \ne \mathrm{COL}(a(i-1)+1)$, so $\mathrm{COL}'(j) \ne \mathrm{COL}'(i)$. Therefore COL$'$ is a proper $(p;c)$-coloring of $[W(p;c)]$.

2) $W(ap;c) \ge a(W(p;c)-1)+1$:

To show $W(ap;c) \ge a(W(p;c)-1)+1$ we need to give a proper $(ap;c)$-coloring of $[a(W(p;c)-1)]$.

Let X be a number to be named later. Let COL$'$ be a proper $(p;c)$-coloring of $[X]$. The reader can easily verify that COL, defined below, is a proper $(ap;c)$-coloring of $[aX]$.

- Color $1,\ldots,a$ with $\mathrm{COL}'(1)$.
- Color $a+1,\ldots,2a$ with $\mathrm{COL}'(2)$.
- $\vdots$
- Color $(X-1)a+1,\ldots,Xa$ with $\mathrm{COL}'(X)$.

Take $X = W(p;c)-1$. By definition there exists COL$'$, a proper $(p;c)$-coloring of $[X]$. Hence COL is a proper $(ap;c)$-coloring of $[aX] = [a(W(p;c)-1)]$ which is what we need.

■

3. THE EXACT VALUE OF $W(AX;2)$

For completeness we cover linear polynomials, for which we obtain a complete solution. The proof is very similar to the proof of Lemma 4.

Theorem 5. *Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}^+$. Then*

$$W(ax;c) = |ac|+1\ .$$

Proof. By Lemma 3.1 we have the case of $a=0$. We will assume $a \ge 1$. The case where $a \le -1$ is similar. By Lemma 3.2 we can assume that a is a forbidden distance.
$W(ax;c) \le ac+1$:

By setting $x=1,2,\ldots,c$ we get forbidden distances $a,2a,\ldots,ca$. So $1,a+1,2a+1,\ldots,ca+1$ must all be different colors, but there are only c colors.
$W(ax;c) \ge ac+1$:

We can properly c-color $[ca]$:

- Color $1, \ldots, a$ with 1.
- Color $a+1, \ldots, 2a$ with 2.
- $\vdots$
- Color $(c-1)a+1, \ldots, ca$ with c.

■

4. UPPER BOUNDS ON $W(AX^2 + BX; 2)$

Theorem 6. *Let $a, b \in \mathbb{N}$.*

1. $W(ax^2 + bx, 2) \le 12a + 6b + 1$.
2. *If $b \ge 3a$ then $W(-ax^2 + bx, 2) \le 6b - 12a + 1$.*
3. *If $2a \le b \le 3a$ then $W(-ax^2 + bx, 2) \le 3b - 3a + 1$.*
4. *If $a \le b \le 2a$ then $W(-ax^2 + bx, 2) \le 9a - 3b + 1$.*
5. *If $0 \le b \le a$ then $W(-ax^2 + bx, 2) \le 12a - 6b + 1$.*
6. *One can obtain bounds for $W(ax^2 - bx, 2)$ easily since it equals $W(-ax^2 + bx, 2)$.*
7. *One can obtain bounds for $W(-ax^2 - bx, 2)$ easily since it equals $W(ax^2 + bx, 2)$.*
8. *For all $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 2) \le 12|a| + 6|b|$. (This follows from the other parts.)*

Proof. If $a = 0$ then Theorem 5 yields the results. Hence we assume $a \ge 1$.

We will need the following claim.

Claim: If COL is a 2-coloring of an initial segment of $\mathbb{N}^+$. Let d be a forbidden distance for COL. then $3d$ is a forbidden distance for COL.

Proof of Claim: Let y and $y + 3d$ be in the domain of COL. Hence $y + d, y + 2d$ are also in the domain of COL. We can assume $\mathrm{COL}(y) = \mathbb{R}$. Then

$$\mathrm{COL}(y) = R \implies \mathrm{COL}(y+d) = B \implies \mathrm{COL}(y+2d) = R \implies \mathrm{COL}(y+3d) = B.$$

End of Proof of Claim

1) $W(ax^2 + bx; 2)$. By plugging in $x = 1, 2, 3$ we find forbidden distances:

$$\{a + b, 4a + 2b, 9a + 3b\}.$$

By the Claim the following are forbidden distances:

$$\{3a + 3b, 3(4a + 2b), 9a + 3b\} = \{3a + 3b, 12a + 6b, 9a + 3b\}.$$

Assume there is a proper $W(x^2; 2)$-coloring of $[12a + 6b + 1]$. We will get a contradiction. We can assume that $\mathrm{COL}(1) = R$. Note that

$$\mathrm{COL}(1) = R \implies \mathrm{COL}(1 + (3a + 3b)) = B \implies \mathrm{COL}(1 + (3a + 3b) + (9a + 3b)) = R.$$

We simplify to obtain $\mathrm{COL}(12a + 6b + 1) = R$.

$$\text{COL}(12a + 6b + 1) = R \implies \text{COL}(12a + 6b + 1 - (12a + 6b)) = B.$$

We simplify to obtain $\text{COL}(1) = B$ which is a contradiction.

The key to the last proof was that

- $(3a + 3b) + (9a + 3b) - (12a + 6b) = 0$.
- COL is defined on $(3a + 3b) + (9a + 3b) + 1 = 12a + 6b + 1$.

For all later proofs we just give nonnegative forbidden distances d_1, d_2, d_3 such that $d_1 + d_2 - d_3 = 0$, and conclude that the bound is $d_1 + d_2 + 1$. We abbreviate *Forbidden Distances* by FD.

We now consider $W(-ax^2 + bx; 2)$:
2) $b \geq 3a$. $\{3b - 3a, 6b - 12a, 3b - 9a\}$ are FDs. Hence $(3b - 3a) + (3b - 9a) - (6b - 12a) = 0$ is a FD.

3) $2a \leq b \leq 3a$. $\{3b - 3a, 6b - 12a, 9a - 3b\}$ are FDs. Hence $(6b - 12a) + (9a - 3b) - (3b - 3a) = 0$ is a FD.

4) $a \leq b \leq 2a$. $\{3b - 3a, 12a - 6b, 9a - 3b\}$ are FDs. Hence $(3b - 3a) + (12a - 6b) - (9a - 3b) = 0$ is a FD.

5) $0 \leq b \leq a$. $\{3a - 3b, 12a - 6b, 9a - 3b\}$ are FDs. Hence $(3a - 3b) + (9a - 3b) - (12a - 6b) = 0$ is a FD.

■

Corollary 7. *For all $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 2) \leq 12|a| + 6|b| + 1$.*

The bounds on $W(ax^2 + bx; 2)$ (and the others) from Theorem 6 hold for all a, b; however, for particular a, b better bounds can often be found. We give a class of examples.

Theorem 8. *Let $a \in \mathbb{N}$ with $a \geq 1$. Then $W(ax^2 + (a-1)x; 2) = 8a - 3$.*

Proof.
1) $W(ax^2 + (a-1)x; 2) \leq 8a - 3$.
Let $\text{COL}\colon [8a - 3] \to \{R, B\}$.
By plugging in $x = 1, 2$ we find forbidden distances: $\{2a - 1, 6a - 2\}$. Since $2a - 1$ is a forbidden distance, so is $3(2a - 1) = 6a - 3$. We will use forbidden distances $\{6a - 3, 6a - 2\}$.
Let $y \leq 2a - 1$. Assume $\text{COL}(y) = R$. Then

$$\text{COL}(y) = R \implies \text{COL}(y + (6a - 2)) = B \implies \text{COL}(y + (6a - 2) - (6a - 3)) = R.$$

Since $y + (6a - 2) - (6a - 3) = y + 1$ we have the following which is the keys fact needed for our proof:

$$y \leq 2a - 1 \implies \text{COL}(y) = \text{COL}(y + 1).$$

(We needed $y \le 2a-1$ since we needed $y+(6a-2) \le 8a-3$ so that $y+(6a-2)$ is in the domain of COL.)

Assume $\mathrm{COL}(1)=R$. Then by applying the above we get $\mathrm{COL}(2)=R, \ldots$, $\mathrm{COL}(2a)=R$. However, since $\mathrm{COL}(1)=R$ and $2a-1$ is a forbidden distance, $\mathrm{COL}(2a)=B$. This is a contradiction.

2) $W(ax^2+(a-1)x;2) \ge 8a-3$.

We give a coloring COL of $[8a-4]$ such that, for all $x,y \in [8a-4]$ with $|x-y| \in \{2a-1, 6a-2\}$, $\mathrm{COL}(x) \ne \mathrm{COL}(y)$. All other forbidden distances are larger than $8a-4$ and hence irrelevant.

Here is the coloring:

1. For $1 \le y \le 2a-1$, $\mathrm{COL}(y)=R$.
2. For $2a \le y \le 4a-2$, $\mathrm{COL}(y)=B$.
3. For $4a-1 \le y \le 6a-3$, $\mathrm{COL}(y)=R$.
4. For $6a-2 \le y \le 8a-4$, $\mathrm{COL}(y)=B$.

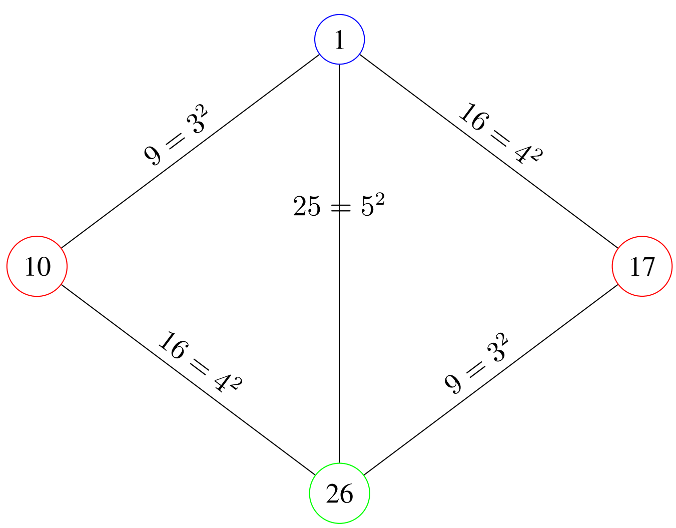
The reader can verify that this coloring suffices. ■

In Appendix A is a table of some exact values of $W(ax^2+bx;2)$.

5. $W(AX^2;3)=28A+1$

In this section we will show that $W(x^2;3)=29$ and then $W(ax^2;3)=28a+1$. We first show a weaker theorem which will be a good warm-up to our work on 4-colorings in Section 7.

Theorem 9. $W(x^2;3) \le 68$.



Proof.

Figure 1. In any proper $(x^2,3)$-coloring, $\mathrm{COL}(10)=\mathrm{COL}(17)$

Assume, by way of contradiction, that COL is an $(x^2;3)$-proper coloring of $[68]$. Figure 1 shows some constraints on COL: COL restricted to $\{10,1,26,17\}$ has to be a proper 3-coloring of the graph (no vertices that have an edge between them are the same color).

We can assume $\mathrm{COL}(10) = R$ and $\mathrm{COL}(1) = B$. By looking at Figure 1 we see that $\mathrm{COL}(26) \notin \{R, B\}$, hence $\mathrm{COL}(26) = G$. Again by looking at Figure 1 we have that $\mathrm{COL}(17) \notin \{B, G\}$, hence $\mathrm{COL}(17) = R$.

Note that we have shown that $\mathrm{COL}(10) = \mathrm{COL}(17)$. More generally we have shown that, for all x, $\mathrm{COL}(x) = \mathrm{COL}(x + 7)$. Not quite. We need that (1) $x - 9$ is in the domain of the coloring, so $x \geq 10$, and $x + 16$ is in the domain of the coloring, so $x \leq 52$. To restate: if $10 \leq x \leq 52$ then $\mathrm{COL}(x) = \mathrm{COL}(x + 7)$.

$$\mathrm{COL}(10) = \mathrm{COL}(17) = \mathrm{COL}(24) = \mathrm{COL}(31) = \mathrm{COL}(38) = \mathrm{COL}(45) = \mathrm{COL}(52) = \mathrm{COL}(59).$$

Since $59 - 10 = 49 = 7^2$, this contradicts COL being an $(x^2; 3)$-proper coloring. ■

The bound in Theorem 9 is not tight. The next theorem gives a tight bound.

The following theorem was proven by Matthew Jordan and William Gasarch.

Theorem 10.

1. $W(x^2; 3) = 29$.
2. *For all* $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$. *This follows from Part 1 and Lemma 4.*

Proof. $W(x^2; 3) \leq 29$: Assume, by way of contradiction, that there exists COL, a proper $(x^2, 3)$-coloring of $\{1, \ldots, 29\}$. Figure 1 shows some constraints on COL: COL restricted to $\{1, 10, 17, 26\}$ has to be a proper 3-coloring of the graph (no vertices that have an edge between them are the same color).

By Figure 1, $\mathrm{COL}(10) = \mathrm{COL}(17)$. By similar reasoning one can show that

$$(\forall x)[10 \leq x \leq 13 \implies \mathrm{COL}(x) = \mathrm{COL}(x + 7)].$$

We refer to this fact as FORCE-SEVEN since the value of $\mathrm{COL}(x)$ forces the value of $\mathrm{COL}(x + 7)$.

We can assume $\mathrm{COL}(10) = R$. Since $11 - 10 = 1^2$ we know that $\mathrm{COL}(10) \neq \mathrm{COL}(11)$, so we can assume $\mathrm{COL}(11) = B$.

17: By FORCE-SEVEN $\mathrm{COL}(17) = \mathrm{COL}(10) = R$

18: By FORCE-SEVEN $\mathrm{COL}(18) = \mathrm{COL}(11) = B$.

10	11	12	13	14	15	16	17	18	19	20
R	B						R	B		

19: Since $\mathrm{COL}(10) = R$, $\mathrm{COL}(19) = \mathrm{COL}(10 + 3^2) \neq R$. Since $\mathrm{COL}(18) = B$, $\mathrm{COL}(19) = \mathrm{COL}(18 + 1^2) \neq B$. Hence $\mathrm{COL}(19) \notin \{R, B\}$, so $\mathrm{COL}(19) = G$.

12: By FORCE-SEVEN $\mathrm{COL}(12) = \mathrm{COL}(19) = G$.

10	11	12	13	14	15	16	17	18	19	20
R	B	G					R	B	G	

20: Since $\mathrm{COL}(11) = B$ and $\mathrm{COL}(19) = G$, $\mathrm{COL}(20) = R$.

13: By FORCE-SEVEN $\mathrm{COL}(13) = \mathrm{COL}(20) = R$.

10	11	12	13	14	15	16	17	18	19	20
R	B	G	R				R	B	G	R

Now we have that $\mathrm{COL}(17) = \mathrm{COL}(13) = R$. But $17 - 13 = 2^2$. This is a contradiction.

$W(x^2, 3) \geq 29$:
We present a proper 3-coloring:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	G	R	G	R	B	G	B	G	R	B	G	B	G

15	16	17	18	19	20	21	22	23	24	25	26	27	28
R	B	R	B	G	R	B	R	B	G	R	G	R	B

We can assume $\mathrm{COL}(10) = R$. Since $11 - 10 = 1^2$ we know that $\mathrm{COL}(10) \neq \mathrm{COL}(11)$, so we can assume $\mathrm{COL}(11) = B$.
17: By FORCE-SEVEN $\mathrm{COL}(17) = \mathrm{COL}(10) = R$
18: By FORCE-SEVEN $\mathrm{COL}(18) = \mathrm{COL}(11) = B$.

10	11	12	13	14	15	16	17	18	19	20
R	B						R	B		

19: Since $\mathrm{COL}(10) = R$, $\mathrm{COL}(19) = \mathrm{COL}(10 + 3^2) \neq R$. Since $\mathrm{COL}(18) = B$, $\mathrm{COL}(19) = \mathrm{COL}(18 + 1^2) \neq B$. Hence $\mathrm{COL}(19) \notin \{R, B\}$, so $\mathrm{COL}(19) = G$.
12: By FORCE-SEVEN $\mathrm{COL}(12) = \mathrm{COL}(19) = G$.

10	11	12	13	14	15	16	17	18	19	20
R	B	G					R	B	G	

20: Since $\mathrm{COL}(11) = B$ and $\mathrm{COL}(19) = G$, $\mathrm{COL}(20) = R$.
13: By FORCE-SEVEN $\mathrm{COL}(13) = \mathrm{COL}(20) = R$.

10	11	12	13	14	15	16	17	18	19	20
R	B	G	R				R	B	G	R

Now we have that $\mathrm{COL}(17) = \mathrm{COL}(13) = R$. But $17 - 13 = 2^2$. This is a contradiction.

$W(x^2, 3) \geq 29$:
We present a proper 3-coloring:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	G	R	G	R	B	G	B	G	R	B	G	B	G

15	16	17	18	19	20	21	22	23	24	25	26	27	28
R	B	R	B	G	R	B	R	B	G	R	G	R	B

■

6. UPPER BOUNDS ON $W(AX^2 + BX; 3)$

We will obtain upper bounds on $W(ax^2 + bx; 3)$ in the case where $a \in \mathbb{N}^+$ and $b \in \mathbb{Z} - \{0\}$. For $b = 0$, Theorem 10.3 yields $W(ax^2; 3) = 28a + 1$. For $a = 0$, Theorem 5 yields $W(bx; 3) = bc + 1$. For $a \le -1$ we can use $W(ax^2 + b; 3) = W(-ax^2 - b; 3)$.

Definition.

(a) A coloring of $[w]$ has *repeat distance* r if x and $x + r$ have the same color, for all $1 \le x \le w - r$.

(b) A coloring of $[w]$ has *repeat distance* r *under one-sided boundary condition* b if x and $x + r$ have the same color, for all $1 \le x \le w - r - b$.

(c) A coloring of $[w]$ has *repeat distance* r *under two-sided boundary condition* b if x and $x + r$ have the same color, for all $b + 1 \le x \le w - r - b$.

Lemma 11. *In any 3-coloring of* $[w]$ *with forbidden distances* $s, t, s + t$*, where* $0 < s < t$*:*

(a) $2s + t$ *is a repeat distance.*

(b) $t - s$ *is a repeat distance under two-sided boundary condition* s*.*

(c) $3s$ *is a repeat distance under one-sided boundary condition* t*.*

Proof. Let $u = s + t$.

(a) Consider a 3-coloring satisfying the conditions of the lemma. Let

$$1 \le x \le w - (2s + t).$$

Without loss of generality, we can assume that x is R. Then $x + s$ is not R, say B, and $x + u = (x + s) + t$ cannot be R or B so it must be G. Then $(x + s) + u = (x + u) + s$ cannot be B or G so it must be R. Since x and $x + u + s$ are both R,

$$(x + u + s) - x = u + s = 2s + t$$

is a repeat distance.

(b) Consider a 3-coloring satisfying the conditions of the lemma. Let

$$s < x \le w - (t - s) - s.$$

Without loss of generality, we can assume that x is R. Then $x - s$ is not R, say B, and $(x - s) + u = x + t$ cannot be R or B so it must be G. Then $(x - s) + t = (x + t) - s$ cannot be B or G, so it must be R. This process requires that $x - s > 0$ and $x + t \le w$. So $(x + t - s) - x = t - s$ is a repeat distance under two-sided boundary condition s.

(c) Take $2s + t$ from part (a) and subtract $t - s$ from part (b). The repeat distance is $(2s + t) - (t - s) = 3s$. There is a one-sided boundary of size $(t - s) + s = t$ from one side of part (b).

■

Lemma 12. *Assume* $[w]$ *has a proper 3-coloring* COL *where* s *is a forbidden distance and* r *is repeat distance under either one-sided or two-sided boundary condition* b*. If* $r|s$ *then*

$$w \leq s + 2b + 1 \ .$$

Proof. We prove the lemma for 2-sided boundary condition. The case of 1-sided boundary condition follows immediately.

Assume, by way of contradiction, that $w \geq s + 2b + 2$. By (a) the definition of repeat distance under two sided boundary condition, (b) $r|s$, and (c) $b + 1 + (\frac{s}{r} - 1)r \leq w - r - b$ (this is equivalent to $w \geq 2b + s + 1$ which follows from $w \geq 2b + s + 2$) we have:

$$\mathrm{COL}(b+1) = \mathrm{COL}(b+1+r) = \mathrm{COL}(b+1+2r) = \cdots$$

$$= \mathrm{COL}\left(b+1+\left(\frac{s}{r}-1\right)r\right) = \mathrm{COL}\left(b+1+\left(\frac{s}{r}\right)r\right) = \mathrm{COL}(b+1+s).$$

But s is a forbidden distance so $b + 1$ and $s + b + 1$ cannot have the same color. Contradiction. ■

We use Lemma 12 to get upper bounds on several quadratic van der Waerden numbers. For one of them we have an exact value.

Theorem 13.

1. *For* $a, b > 0$ *and* $a|b$, $W(ax^2 + bx; 3) \leq \frac{72b^2}{a} + 1$.
2. $W(x^2 + x; 3) = 73$.

Proof.
1) Let $p(x) = ax^2 + bx$. Let

$$x = \frac{5b}{a}, \quad y = \frac{6b}{a}, \quad z = \frac{8b}{a} \ .$$

Then

$$p(x) = \frac{30b^2}{a}, \quad p(y) = \frac{42b^2}{a}, \quad p(z) = \frac{72b^2}{a} \ .$$

Since $p(x) + p(y) = p(z)$, by Lemma 11b, $p(y) - p(x) = \frac{12b^2}{a}$ is a repeat distance under two-sided boundary condition $\frac{30b^2}{a}$. But $p(\frac{3b}{a}) = \frac{12b^2}{a}$ is a forbidden distance. Thus, by Lemma 12, $W(ax^2 + bx; 3) \leq \frac{12b^2}{a} + 2 \cdot \frac{30b^2}{a} + 1 = \frac{72b^2}{a} + 1$.
2) By Part 1 $W(x^2 + x; 3) \leq 73$. We show $W(x^2 + x; 3) \geq 73$ by giving a $(x^2 + x; 3)$-proper coloring of $\{1, \ldots, 72\}$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

■

We need one more lemma before getting an upper bound for $W(p(x);3)$ where $p(x) = ax^2 + bx$.

Lemma 14. *Let*

$$q(a,b) = (4a^3 + 4a^2 + 3a + 1)b^2 \quad \textit{and} \quad r(a,b) = (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2.$$

Then, for all $a \in \mathbb{N}^+$*,* $b \in \mathbb{Z} - \{0\}$*,* $\gcd(2q(a,b) + r(a,b), 3q(a,b))$ *divides* $18b^2$*.*

Proof.
The only values of (a,b) for which $q(a,b) = 0$ either involve $a \notin \mathbb{N}^+$ or $b = 0$. Hence, for the domain we are concerned with, $q(a,b) \neq 0$. The only values of (a,b) for which $r(a,b) = 0$ either involve $a \notin \mathbb{R}$, $a = -1$, or $b = 0$. Hence, for the domain we are concerned with, $r(a,b) \neq 0$. Therefore $\gcd(q(a,b), r(a,b))$ always exists. We use this implicitly.

1) We examine $\gcd(q(a,b), r(a,b))$.
Let $a \in \mathbb{N}^+$ and $b \in \mathbb{Z} - \{0\}$. Let $d_1 = \gcd(q(a,b), r(a,b))$.
The reader can verify the following equation:

$$(-20a^4 - 12a^3 - 4a^2 - 10a - 1)q(a,b) + (20a^2 - 8a + 7)r(a,b) = 6b^2.$$

Since d_1 divides the LHS, d_1 divides the RHS. Hence d_1 divides $6b^2$.

2) We examine $\gcd(2q(a,b) + r(a,b), 3q(a,b))$.
Let $a \in \mathbb{N}^+$ and $b \in \mathbb{Z} - \{0\}$. Let $d = \gcd(2q(a,b) + r(a,b), 3q(a,b))$.
The reader can verify the following equation:

$$3(2q(a,b) + r(a,b)) - 2(3q(a,b)) = 3r(a,b).$$

Since d divides the LHS, d divides the RHS, hence d divides $3r(a,b)$. Since d also divides $3q(a,b)$, d divides

$$\gcd(3r(a,b), 3q(a,b)) = 3\gcd(r(a,b), q(a,b)) = 3d_1.$$

Since d_1 divides $6b^2$, d divides $18b^2$.

■

Theorem 15. *Let $a \in \mathbb{N}^+$ and $b \in \mathbb{Z} - \{0\}$. Let $p(x) = ax^2 + bx$. Then $W(p(x); 3) = O(ab^6 + a^5b^2)$.*

Proof. Let w be such that there is a proper $(p; 3)$-coloring $\mathrm{COL}\colon [w] \to [3]$. We will show that $w = O(ab^6 + a^5b^2)$.

Let

$$x_0 = (2a+1)b, \quad y_0 = (2a^2+2a+1)b, \quad z_0 = (2a^2+2a+2)b\ .$$

Then

$$p(x_0) = (4a^3 + 4a^2 + 3a + 1)b^2$$
$$p(y_0) = (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2$$
$$p(z_0) = (4a^5 + 8a^4 + 12a^3 + 10a^2 + 6a + 2)b^2$$

Thus $p(x_0) + p(y_0) = p(z_0)$. Note that $p(x_0)$, $p(y_0)$, and $p(z_0)$ are (1) positive since $a \in \mathbb{N}^+$ and the only occurence of b is in b^2, and (2) forbidden distances.

1) By Lemma 11a, $2p(x_0) + p(y_0)$ is a repeat distance.
2) By Lemma 11c, $3p(x_0)$ is a repeat distance under one-sided boundary condition $p(y_0)$.
3) By Lemma 14 $\gcd(2p(x_0) + p(y_0), 3p(x_0)) = d \le 18b^2$.

Claim: Let $a \in \mathbb{N}^+$ and $b \in \mathbb{Z} - \{0\}$.

1. $3p(x_0)$ does not divide $2p(x_0) + p(y_0)$.
2. $2p(x_0) + p(y_0)$ does not divide $3p(x_0)$.
3. There is a linear combination over $\mathbb{Z}$ of $2p(x_0) + p(y_0)$ and $3p(x_0)$ that sums to d where one coefficient is < 0 and the other coefficient is > 0.

Proof of Claim

Note that

$$\begin{aligned} 2p(x_0) + p(y_0) = &\quad (4a^5 + 8a^4 + 16a^3 + 14a^2 + 9a + 3)b^2 \\ 3p(x_0) = &\quad (12a^3 + 12a^2 + 9a + 3)b^2 \end{aligned}$$

1) For all $a \in \mathbb{N}^+$, for all $b \in \mathbb{Z} - \{0\}$, $3p(x_0)$ does not divide $2p(x_0) + p(y_0)$.

If we divide $3p(x_0)$ into $2p(x_0) + p(y_0)$ as polynomials in a, b we get the following:

$$(4a^5 + 8a^4 + 16a^3 + 14a^2 + 9a + 3)b^2 =$$

$$\left(\frac{a^2}{3} + \frac{a}{3} + \frac{3}{4}\right)(12a^3 + 12a^2 + 9a + 3)b^2 + \left(a^2 + \frac{5a}{4} + \frac{3}{4}\right)b^2.$$

Since for all $a \in \mathbb{N}^+$, $b \in \mathbb{Z} - \{0\}$, $(a^2 + \frac{5a}{4} + \frac{3}{4})b^2 \neq 0$, $3p(x_0)$ does not divide $2p(x_0) + p(y_0)$.

2) For all $a \in \mathbb{N}^+$, $b \in \mathbb{Z} - \{0\}$, $2p(x_0) + p(y_0)$ does not divide $3p(x_0)$.
For all $a \in \mathbb{N}^+$, $3p(x_0) < 2p(x_0) + p(y_0)$, hence $2p(x_0) + p(y_0)$ does not divide $3p(x_0)$.

3) Since $\gcd(2p(x_0) + p(y_0), 3p(x_0)) = d$, there is a linear combination over $\mathbb{Z}$ of these two quantifies that sums to d. Since both of these quantities are $\geq d$ one coefficient must be ≤ 0 and one must be ≥ 0. By Part 1 and 2, neither coefficient can be 0. Hence one coefficient is < 0 and the other is > 0.
End of Proof of Claim
By Claim (part 3) there exists $j, k \in \mathbb{N}$ such that

$$j(2p(x_0) + p(y_0)) - k(3p(x_0)) = d.$$

By starting at 1 and adding repeat distance $2p(x_0) + p(y_0)$ j times and subtracting repeat distance $3p(x_0)$ k times, we see that d is a repeat distance; however, we need to be careful about the boundary condition. By interspersing the adds and subtracts so that we subtract whenever the sum is greater than $2p(x_0) + p(y_0)$, the one-sided boundary condition is $(2p(x_0) + p(y_0)) + p(y_0) = 2(p(x_0) + p(y_0))$. Hence

4) d is a repeat distance with one-sided boundary condition

$$(2p(x_0) + p(y_0)) + p(y_0) = 2(p(x_0) + p(y_0)).$$

5) $p(db) = ad^2b^2 + b^2d = (ad + 1)db^2 = O(ad^2b^2) = O(ab^6)$ is a forbidden distance. (We use $d \leq 18b^2$.)

By Lemma 12 with $s = p(db) = (ad + 1)db^2$, $r = d$, $b = 2(p(x_0) + p(y_0))$ we get

$$w \leq s + 2b + 1 = (ad + 1)db^2 + 4(p(x_0) + p(y_0)) + 1.$$

Since $p(x_0) = O(a^3b^2)$ and $p(y_0) = O(a^5b^2)$, $4(p(x_0) + p(y_0)) + 1 = O(a^5b^2)$. Hence

$$w \leq s + 2b + 1 = (ad + 1)db^2 + 4(p(x_0) + p(y_0)) + 1 = O(ab^6 + a^5b^2).$$

■

In Appendix B is a table of some exact values of $W(ax^2 + bx; 3)$.

7. UPPER BOUNDS ON $W(X^2; 4)$

Recall that Figure 1 was the key to showing $W(x^2; 3) \leq 68$. We now derive parameters for a new figure that will be the key to an upper bound on $W(x^2; 4)$.
We need to find $a, b, c, d, e, f, x, y, z \in \mathbb{N}^+$ such that the following figure can be drawn:
Hence we need to find solutions in $\mathbb{N}^+$ to the following system of equations:

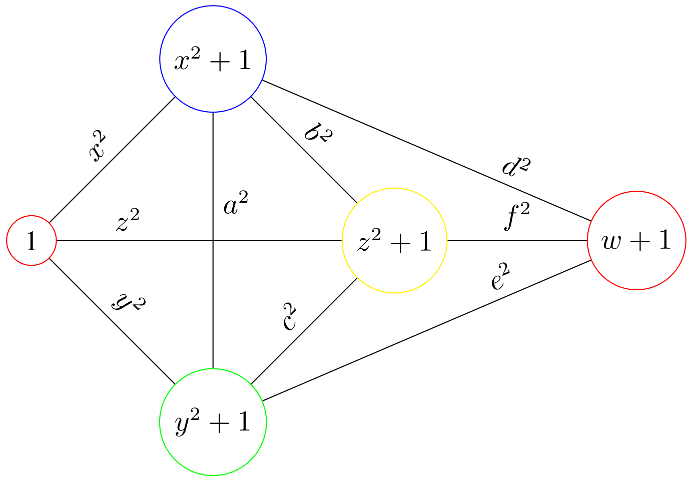


Figure 2. In any $(x^2;4)$-proper coloring, $\text{COL}(1) = \text{COL}(1+w)$

$$\begin{aligned} x^2+a^2 &= y^2 \\ x^2+b^2 &= z^2 \\ y^2+c^2 &= z^2 \\ x^2+d^2 &= w \\ y^2+e^2 &= w \\ z^2+f^2 &= w \end{aligned}$$

The first three equations are overlapping Pythagorean triples—we have three numbers (x, y, z) whose squares have all square pairwise differences. From the first three equations one can derive the following:

$$\begin{aligned} c^2+f^2 &= e^2 \\ b^2+f^2 &= d^2 \\ a^2+c^2 &= b^2 \end{aligned}$$

We give one example by deriving $c^2+f^2=e^2$ algebraically. From $y^2+c^2=z^2$ and $z^2+f^2=w$ we get $y^2+c^2=w-f^2$, and hence

$$c^2+f^2 = w - y^2.$$

From $y^2+e^2=w$ we get that the left hand side is e^2. Hence

$$c^2+f^2=e^2.$$

Since the first three equations are Pythagorean triples, they can be generated by using Euclid's formula: all Pythagorean triples are of the form $(k(m^2-n^2), k(2mn), k(m^2+n^2))$ where $\gcd(m,n)=1$ and $m \not\equiv n \pmod 2$. We can use the Farey sequence as an efficient algorithm to generate coprime pairs m, n. (See Routledge [**18**] or the Wikipedia entry on Farey sequences for the definition of Farey sequences and the algorithm.)

We used a computer program and obtained the following:

Theorem 16. $W(x^2;4) \le 1 + (290{,}085{,}289)^2 = 84{,}149{,}474{,}894{,}213{,}522$

Proof.

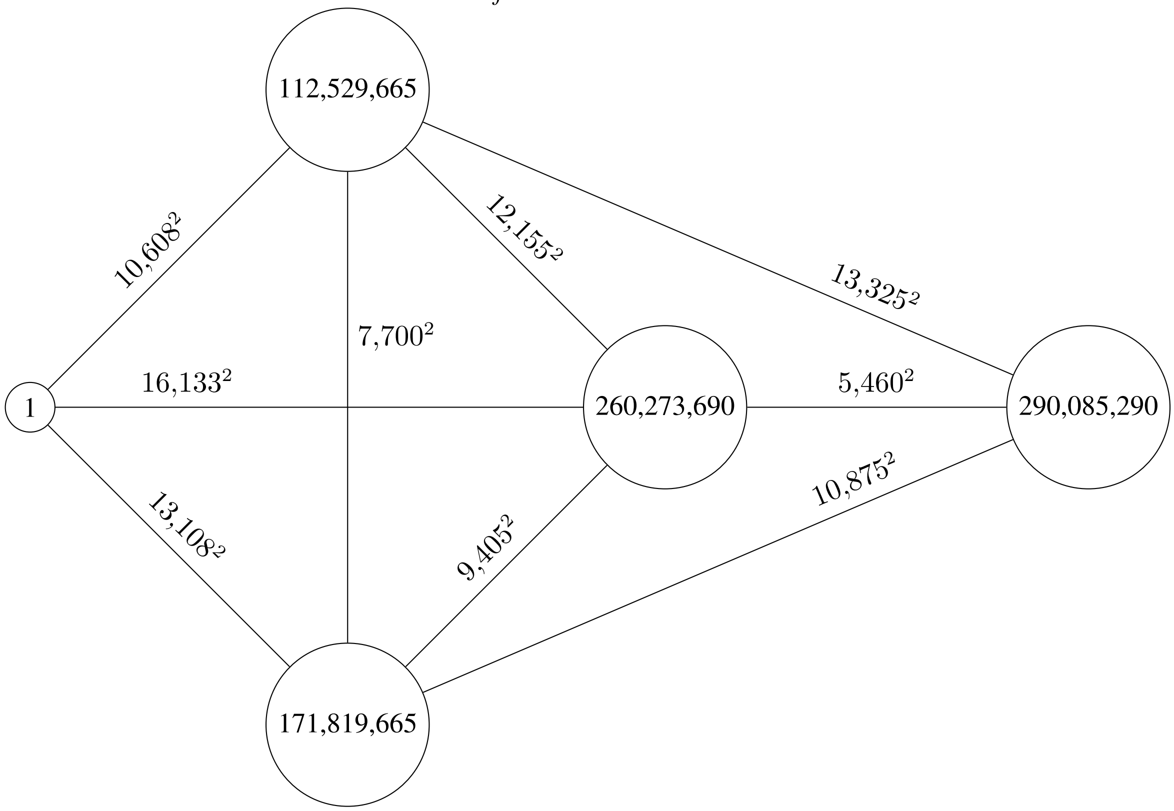


Figure 3. In any $(x^2; 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + 290,085,290)$

Assume, by way of contradiction, that there exists COL, a proper $(x^2; 4)$-coloring of $[1 + (290{,}085{,}289)^2]$. Figure 3 shows some constraints on COL: COL restricted to the numbers on the vertices has to be a proper 4-coloring of the graph (no vertices that have an edge between them are the same color).

By Figure 3 we know that

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 290{,}085{,}289^2).$$

More generally we have shown that, for all x,

$$\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}289^2).$$

Hence

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 290{,}085{,}289)) = \mathrm{COL}(1 + 2 \times 290{,}085{,}289)) = \cdots = \mathrm{COL}(1 + (290{,}085{,}289)^2).$$

This contradicts COL being an $(x^2; 4)$-proper coloring. ∎

Theorem 16 gave an enormous upper bound on $W(x^2; 4)$. The proof was found by a computer program; however, it is a HS proof and human-verifiable. Four colors seems to be at the limit of what computers can find. That is, we have been unable to use a program to find a human-verifiable proof for a bound on $W(x^2;5)$.

Usually a HS proof gives better bounds than a proof that uses advanced mathematics. However, our HS proof of a bound on $W(x^2;4)$ gives such a large bound that its possible a proof using more advanced mathematics would yield a better result. In particular, its possible that if the proofs of the results of Sarkozy [**8**], Pintz-Steiger-Szemerédi [**15**], or Harnel-Lyall-Rice [**17**] were looked at more carefully then one could obtain better bounds for $W(x^2;c)$ for some small values of c. However, these would not be HS proofs.

8. UPPER BOUNDS ON $W(AX^2+BX;4)$

To find upper bounds on $W(Ax^2+Bx;4)$ we have several overlapping equations of the form

$$(Ax^2+Bx)+(Ay^2+By)=(Az^2+Bz).$$

We need a way to generate such triples (x,y,z) much like the generation of Pythagorean triples. First, we use the quadratic formula to express z in terms of x and y.

$$z=\frac{-B+\sqrt{4A^2(x^2+y^2)+4AB(x+y)+B^2}}{2A}$$

We rewrite as

$$4A^2(x^2+y^2)+4AB(x+y)+B^2=(2Az+B)^2.$$

Simple algebra allows us to rewrite this as:

$$(2Ax+B)^2+(2Ay+B)^2=(2Az+B)^2+B^2.$$

If $m=2Ax+B$, $n=2Ay+B$, and $k=(2Az+B)$ then we can rewrite this as $m^2+n^2=k^2+B^2$. A parameterization of $m^2+n^2=k^2+B^2$ would imply one for (x,y,z), and luckily this equation is easier. Using the *Bramagupta-Fibonacci identity* with $bc-ad=B$, we get:

$$(ac-bd)^2+(ad+bc)^2=(ac+bd)^2+B^2$$

So, with parameters a,b,c,d and some tedious algebra we get

$$x=\frac{ac-bd-B}{2A},\ y=\frac{ad+bc-B}{2A},\ z=\frac{ac+bd-B}{2A}$$

with constraints $bc-ad=B, ac-bd>B, 2A|ac-bd-B, 2A|ad+bc-B$.

Rather than searching all (a,b,c,d), we can eliminate parts of the parameter space that do not contain solutions. For fixed a and d, the first constraint implies that bc is some factorization of $ad+B$. We can pre-compute a table of factorizations and use that to cut the search space down to almost $O(n^2)$. You can see the code for this on GitHub at `https://github.com/zaprice/polyvdw`

We can get bounds for $W(x^2 + Bx; 4)$ with this method with rather large values of B, but only a few bounds for the more general $Ax^2 + Bx$ case; if such configurations exist, it seems the numbers involved are much larger. See Appendix C for some of the upper bounds we have. We note two things about these upper bounds:

1. The largest upper bound on $W(x^2 + Bx; 4)$ that we found was when $B = 0$. Note that these are just the upper bounds we found. It is not clear how the real values compares.
2. For $W(2x^2 + Bx; 4)$ and $W(3x^2 + Bx; 4)$ the B for which we could find an upper bound seem scattered and arbitrary. For example, we were not able to find an upper bound for any of $W(2x^2 + Bx; 4)$ for $0 \leq B \leq 56$, but were able to for 57. And then not again until $B = 95$. Again, this may be a limit to our methods and not a statement about the actual values of $W(2x^2 + Bx; 4)$.

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A. SOME EXACT VALUES OF $W(AX^2 + BX; 2)$

We present a table of $W(p(x); 2)$ for $p(x) = ax^2 + bx$ for $0 \le a \le 10$ and $-10 \le b \le 10$.
The values for $a, b \ge 0$ were obtained by using our formulas for an upper bound and then searching for a 2-coloring for the lower bound.

		a										
		0	1	2	3	4	5	6	7	8	9	10
	-10	21	1	1	9	9	1	25	11	13	17	1
	-9	19	1	9	1	7	5	7	37	15	1	23
	-8	17	1	1	7	1	7	9	13	1	21	25
	-7	15	1	7	5	5	25	11	1	19	61	29
	-6	13	1	1	1	5	9	1	17	21	25	73
	-5	11	1	5	13	7	1	15	49	25	29	31
	-4	9	1	1	5	1	13	17	23	25	33	37
	-3	7	1	3	1	11	37	19	25	31	73	41
	-2	5	1	1	9	13	19	49	29	33	39	41
	-1	3	1	7	25	17	21	27	61	37	41	47
b	0	1	5	9	13	17	21	25	29	33	37	41
	1	3	13	13	17	23	49	33	37	43	85	53
	2	5	11	25	21	25	31	33	41	45	51	97
	3	7	13	19	37	29	33	37	73	49	49	59
	4	9	17	21	27	49	37	41	47	49	57	61
	5	11	25	25	29	35	61	45	49	55	97	61
	6	13	23	25	31	37	43	73	53	57	61	65
	7	15	25	31	49	41	45	51	85	61	65	71
	8	17	29	33	39	41	49	53	59	97	69	73
	9	19	37	37	37	47	73	55	61	67	109	77
	10	21	35	49	45	49	51	57	65	69	75	121

The numbers tend to increase with increasing a and $|b|$. Some of the diagonals have patterns which likely can be used to make conjectures that are almost surely true. For example:

$$(\forall a \ge 0)[W(ax^2 - (a-1)x; 2) = 2a + 3].$$

B. SOME EXACT VALUES OF $W(AX^2 + BX; 3)$

We present a table of $W(p(x); 3)$ for $p(x) = ax^2 + bx$ for $0 \le a \le 5$ and $-5 \le b \le 5$.
The values were obtained by computer.

		a					
		0	1	2	3	4	5
	-5	16	1	64	61	217	1
	-4	13	1	1	91	1	289
	-3	10	1	10	1	135	171
	-2	7	1	1	68	97	171
	-1	4	1	49	105	190	183
b	0	1	29	57	85	113	141
	1	4	73	76	65	156	253
	2	7	64	145	123	151	?
	3	10	37	95	217	?	?
	4	13	65	127	?	289	?
	5	16	55	?	109	?	361

C. SOME UPPER BOUNDS ON $W(AX^2 + BX; 4)$

We give bounds for $W(g; 4)$ where g is of the form $Ax^2 + Bx$. Only bounds for coprime coefficients (A, B) are presented. Each row of the table gives g, x, y, z, w (as in Figure 4), and the bound. We give four such tables.

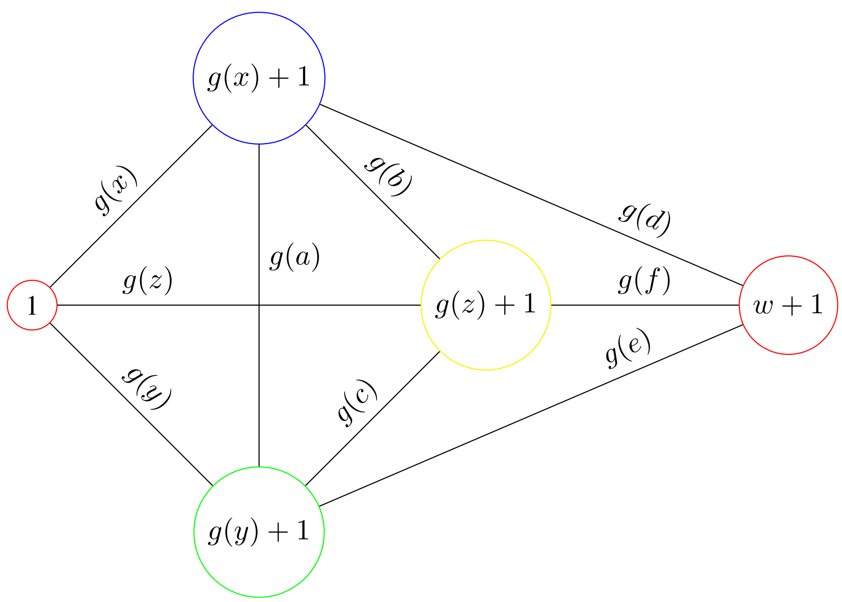

Figure 4. In any $(g(x); 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + w)$

Table for $x^2 + Bx$ where $0 \leq B \leq 20$.

g	x	y	z	w	$W(g(x); 4) \leq$
x^2	10,608	13,108	16,133	290,085,289	84,149,474,894,213,522
$x^2 + x$	299	302	327	113,262	12,828,393,907
$x^2 + 2x$	91	127	211	257,463	66,287,711,296
$x^2 + 3x$	35	43	53	3,308	10,952,789
$x^2 + 4x$	80	84	92	10,197	104,019,598
$x^2 + 5x$	70	81	100	11,250	126,618,751
$x^2 + 6x$	70	86	106	13,232	175,165,217
$x^2 + 7x$	638	785	923	988,338	976,818,920,611
$x^2 + 8x$	160	168	184	40,788	1,663,987,249
$x^2 + 9x$	35	37	44	3,242	10,539,743
$x^2 + 10x$	144	150	165	36,075	1,301,766,376
$x^2 + 11x$	364	472	727	1,263,252	1,595,819,511,277
$x^2 + 12x$	140	172	212	52,928	2,802,008,321
$x^2 + 13x$	119	129	143	38,016	1,445,710,465
$x^2 + 14x$	66	96	135	25,395	645,261,556
$x^2 + 15x$	120	138	215	54,364	2,956,259,957
$x^2 + 16x$	75	99	141	45,177	2,041,684,162
$x^2 + 17x$	123	165	255	232,908	54,250,095,901
$x^2 + 18x$	70	74	88	12,968	168,402,449
$x^2 + 19x$	65	66	69	6,852	47,080,093
$x^2 + 20x$	84	96	115	24,261	589,081,342

Table for $x^2 + Bx$ where $1980 \leq B \leq 2000$.

g	x	y	z	w	$W(g(x); 4) \leq$
$x^2 + 1{,}980x$	1,683	2,145	2,915	25,524,829	651,567,434,640,662
$x^2 + 1{,}981x$	1,674	1,735	2,026	14,236,652	202,710,462,976,717
$x^2 + 1{,}982x$	1,248	1,495	1,731	6,882,723	47,385,517,451,716
$x^2 + 1{,}983x$	3,498	3,549	3,664	24,967,678	623,434,455,617,159
$x^2 + 1{,}984x$	860	975	2,585	12,424,497	154,392,775,905,058
$x^2 + 1{,}985x$	867	1,098	2,365	11,200,200	125,466,712,437,001
$x^2 + 1{,}986x$	1,900	2,432	2,908	19,712,552	388,623,855,480,977
$x^2 + 1{,}987x$	3,048	3,393	3,987	39,165,018	1,533,976,455,831,091
$x^2 + 1{,}988x$	508	738	1,194	6,489,996	42,132,950,192,065
$x^2 + 1{,}989x$	2,023	2,288	3,094	18,950,528	359,160,204,078,977
$x^2 + 1{,}990x$	1,364	1,610	2,100	13,163,856	173,313,300,862,177
$x^2 + 1{,}991x$	1,330	1,519	1,814	7,817,030	61,121,521,727,631
$x^2 + 1{,}992x$	975	1,065	1,871	10,120,498	102,444,639,800,021
$x^2 + 1{,}993x$	1,985	2,349	4,373	68,596,488	4,705,614,878,734,729
$x^2 + 1{,}994x$	1,246	1,350	1,716	8,551,440	73,144,177,644,961
$x^2 + 1{,}995x$	891	1,185	1,464	10,543,450	111,185,372,085,251
$x^2 + 1{,}996x$	705	995	1,793	7,390,317	54,631,536,433,222
$x^2 + 1{,}997x$	1,081	1,136	1,391	8,040,026	64,658,074,012,599
$x^2 + 1{,}998x$	1,292	1,732	3,704	39,649,768	1,572,183,322,690,289
$x^2 + 1{,}999x$	1,235	1,757	2,789	14,633,322	214,163,364,766,363
$x^2 + 2{,}000x$	184	280	984	5,592,000	31,281,648,000,001

Table for $2x^2 + Bx$ for assorted B.

g	x	y	z	w	$W(g(x);4) \leq$
$2x^2 + 57x$	3,969	4,035	4,295	38,199,155	2,918,353,062,779,886
$2x^2 + 95x$	707	758	1,008	14,365,638	412,744,475,029,699
$2x^2 + 171x$	11,907	12,105	12,885	343,792,395	236,386,480,508,171,596
$2x^2 + 285x$	2,121	2,274	3,024	129,290,742	33,432,228,781,682,599
$2x^2 + 399x$	27,783	28,245	30,065	1,871,758,595	7,006,961,222,744,427,456
$2x^2 + 455x$	3,320	3,663	4,170	39,229,128	3,077,866,816,534,009
$2x^2 + 511x$	2,772	3,367	6,282	131,899,720	34,795,139,672,913,721
$2x^2 + 627x$	43,659	44,385	47,245	4,622,097,755	5,834,090,064,188,269,204
$2x^2 + 805x$	1,210	1,303	2,920	87,446,025	15,293,684,970,651,376
$2x^2 + 855x$	5,548	7,087	13,262	530,042,423	561,890,393,545,693,524
$2x^2 + 1{,}011x$	5,164	6,568	9,889	318,517,859	202,907,575,025,443,212
$2x^2 + 1{,}153x$	12,705	12,726	12,970	352,488,525	248,496,726,932,620,576
$2x^2 + 1{,}199x$	8,245	8,710	9,748	221,108,291	97,778,017,806,722,272
$2x^2 + 1{,}295x$	14,030	14,355	22,244	1,162,712,925	2,703,804,197,637,349,126
$2x^2 + 1{,}301x$	25,622	26,105	28,172	1,638,880,116	5,371,858,201,423,377,829
$2x^2 + 1{,}365x$	9,960	10,989	12,510	353,062,152	249,306,248,279,579,689
$2x^2 + 1{,}459x$	954	1,174	1,379	58,465,486	6,836,511,407,576,467
$2x^2 + 1{,}545x$	11,298	11,815	12,860	425,440,418	361,999,755,841,475,259
$2x^2 + 1{,}685x$	10,695	10,968	11,570	289,144,125	167,209,137,251,881,876
$2x^2 + 1{,}753x$	3,586	5,236	8,232	181,967,394	66,224,583,947,144,155
$2x^2 + 1{,}851x$	50,031	51,441	55,164	6,379,649,159	7,612,882,297,751,201,408
$2x^2 + 1{,}913x$	2,261	3,366	5,324	81,424,299	13,259,988,699,966,790

Table for $3x^2 + Bx$ for assorted B.

g	x	y	z	w	$W(g(x);4) \leq$
$3x^2 + x$	42,273	42,660	43,375	5,738,872,934	6,570,267,294,984,419,923
$3x^2 + 143x$	13,244	13,332	13,442	554,651,696	922,915,590,942,221,777
$3x^2 + 172x$	4,452	4,712	5,189	88,862,311	23,689,546,233,099,656
$3x^2 + 200x$	1,896	2,204	5,004	115,177,723	39,797,746,661,938,788
$3x^2 + 235x$	11,155	11,270	11,610	583,594,418	1,021,747,471,306,964,403
$3x^2 + 274x$	9,322	11,610	16,903	1,125,018,929	3,797,003,080,080,107,670
$3x^2 + 344x$	8,904	9,424	10,378	355,449,244	379,032,617,455,054,545
$3x^2 + 361x$	3,540	4,658	7,703	397,333,094	473,620,906,200,085,443
$3x^2 + 400x$	3,792	4,408	10,008	460,710,892	636,763,762,306,663,793
$3x^2 + 407x$	2,806	3,401	6,131	122,898,626	45,312,266,837,804,411
$3x^2 + 412x$	2,077	2,829	5,839	392,773,686	462,813,667,064,838,421
$3x^2 + 520x$	7,616	9,244	12,716	515,261,395	796,483,183,467,963,476
$3x^2 + 556x$	9,400	9,408	9,451	273,674,799	224,693,838,986,259,448
$3x^2 + 592x$	15,744	16,472	17,944	994,061,387	2,964,474,711,857,432,412
$3x^2 + 643x$	50,932	51,357	52,351	8,273,167,696	2,421,731,687,255,606,001
$3x^2 + 688x$	17,808	18,848	20,756	1,421,796,976	6,064,520,901,084,553,217
$3x^2 + 725x$	3,172	3,185	3,278	34,869,750	3,647,723,675,756,251
$3x^2 + 728x$	16,744	17,360	18,928	1,174,742,491	4,140,060,615,695,188,692
$3x^2 + 797x$	2,847	3,082	3,524	148,907,272	66,520,245,642,541,737
$3x^2 + 814x$	5,612	6,802	12,262	491,594,504	724,995,869,246,944,305
$3x^2 + 932x$	1,820	2,229	2,799	37,745,311	4,274,160,686,090,016
$3x^2 + 1{,}085x$	1,190	1,344	1,540	10,401,450	324,581,771,880,751
$3x^2 + 1{,}087x$	9,800	9,909	11,434	604,108,526	1,094,841,990,223,645,791
$3x^2 + 1{,}112x$	18,800	18,816	18,902	1,094,699,196	3,595,100,206,474,645,201