Application of PVDW: Constructing Graphs with High Chromatic Number and High Girth

December 31, 2024

Credit Where Credit is Due

The results are by **Paul O'Donnell**.

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I reviewed this book in my Book Review Column:
`https://www.cs.umd.edu/~gasarch/bookrev/40-3.pdf`

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Application of Pigeonhole: Constructing Graphs with High Chromatic Number and Girth 6

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Base $c = 2$. Use C_6, the cycle on 6 vertices.

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Ind Step We construct G_c on next slide.

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We prove it works in the next few slides.

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Assume inductively that $\chi(G_{c-1}) = c - 1$.

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The vertices in A must be a diff color than the $c - 1$ colors used on the vertices of G_{c-1}^{A}. Hence the coloring must use $\geq c$ colors.
Contradiction. Done!

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$g(G_c) \leq 6$

Inductively G^A_{c-1} has a cycle of size 6. Hence G_c does.

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Let C be a cycle in G_c. We show $|C| \geq 6$.

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0) If C has 0 base vertices then C is a cycle in G_{c-1}^{A}, so $|C| \geq 6$.

1) If C has 1 base vertex v then v has two edges coming out of it, to $G_{c-1}^{A_1}$ and $G_{c-1}^{A_2}$. **GOTO White Board!**

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Cycle goes from v to $G_{c-1}^{A_1}$ then leaves $G_{c-1}^{A_1}$ and *has to goto a base vertex that is not* v.
This is impossible. So this case can't happen.

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This is impossible. So this case can't happen.

2) Can it use exactly 2 base vertices, say 1,2. Yes.
GOTO WHITE BOARD
B1 is Base vertex 1, B2 is Base vertex 2.
C1 is 1 in a copy of G_c, C2 is 2 in that copy.
D1 is 1 in a copy of G_c, D2 is 2 in that copy.

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Assume inductively that $g(G_{c-1}) = 6$.
Let C be a cycle in G_c. We show $|C| \ge 6$.

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Shortest cycle: $(B1, C1, C2, B2, D2, D1, B1)$. Len 6.

Cases 3,4,. . .

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3) Can it use exactly 3 base vertices. Say 1,2,3. Yes.

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4) **Note** If cycle uses $x \geq 2$ base vertices then shortest cycle is length $3x$. (Will use this later)
GOTO WHITE BOARD

Upshot

We have
$\chi(G_c) = c$
$g(G_c) = 6.$
So we are done.

Their Motivation, but Not Ours

Discuss Chromatic Number of the Plane **GOTO BLACKBOARD**

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Soifer, O'Donnell, Others(?) wanted to find G that had $\chi(G) = G$, $g(G) = g$, G is unit graph in the plane.

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Our interest Some of the constructions used VDW and PVDW!

Known: $(\forall c)(\exists G)[\chi(G) = c$ and $\ldots]$

$g(G)$	Math	who
6	PHP	Folklore
9	VDW, Messy	O'Donnell
12	PVDW & Hard Number Theory	O'Donnell
g	Hard Hypergraph	Erdos-Hajnal-O'Donnell

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We will do it the Gasarch Way!

Base Points and Cycles

Recall that we said earlier:

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- Fewer sets A so that for all A_1, A_2, $|A_1 \cap A_2| \leq 1$.
- Enough sets A so that can do the $\chi(G_c) \geq c$ proof.

We will use k-AP's

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Is there an m such that they **cannot** intersect in two places?
Next Slide

Want m so they Cannot Intersect in Two Places?

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If $m = 2$ then $\frac{w-y}{x-z} \in \{\frac{1}{4}, 1, 4\}$.
Solution $w = 4,\ y = 3,\ x = 4,\ z = 0,\ d_1 = 2,\ d_2 = 1.$

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If $m = 3$ then $\frac{w-y}{x-z} = 1$, so $d_1^m = d_2^m$, so $d_1 = d_2$. No solution.

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Upshot If A_1, A_2 are two 5-APs with different differences, both cubes, then $|A_1 \cap A_2| \leq 1$.

A Lemma and a Thm

Lemma Let $k \geq 3$. $(\exists m)$ such that the the following holds:
For all $\alpha, \beta \in \{1, \ldots, k\}$ there is **no** (d_1, d_2) with $d_1 \neq d_2$ such that

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Thm Let $k \geq 3$. $(\exists m = m(k))$ such that the following holds:
If A_1 is a k-AP with diff d_1^m and A_2 is a k-AP with diff d_2^m, with $d_1 \neq d_2$, then $|A_1 \cap A_2| \leq 1$.

Our Set of k-APs

Given k let $m = m(k)$. Let $D = \{d^m \colon d \geq 1\}$.

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Good news If A_1 and A_2 are k-APs with diffs in D, then $|A_1 \cap A_2| \leq 1$.

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Example $k = 5$. $d = 4$.

$$|\{1, 5, 9, 13, 17\} \cap \{13, 17, 21, 25, 29\}| = 2$$

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What to do Next Slide.

We Can Use the Following

Note that the following do not intersect in ≥ 2 places:
(1, 5, 9, 13, 17)
(2, 6, 10, 14, 18)
(3, 7, 11, 15, 19)
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Do we need to stop here? No.

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(22, 26, 30, 34, 38)
(23, 27, 31, 35, 39)
(24, 28, 32, 36, 40)

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So can start with any $a \equiv 1, 2, 3, 4 \pmod{20}$.

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More generally we can do the following for k-APs and $d \in D$.

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With this restriction on a, all k-APs with diff d intersect ≤ 1.

Easy to prove, but we won't do that.

Final Upshot for k-APs

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Let $S(k)$ be all k-APs such that

- Difference is $d^m \in D$.
- Starting point is $a \equiv 1, \ldots, d \pmod{kd^m}$.

Lemma If A_1 and A_2 are in $S(k)$ then $|A_1 \cap A_2| \leq 1$.

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Lemma on Starting Points

Start Lemma Consider the numbers

$$a, a + d, \dots, a + (k - 1)d.$$

One of them is $\equiv 1, \dots, d \pmod{kd}$.
Pf View $\{1, \dots, kd\}$ in chunks as follows:

$$\{1, \dots, d\}, \{d + 1, \dots, 2d\}, \cdots, \{(k - 1)d + 1, \dots, kd\}$$

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Note We will be applying this with $k = M_{c-1}$ and $d = d^m$.

$\chi(G_c) = c$, $g(G) = 9$

Thm For all $c \geq 3$ there exists graph G_c such that

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Base $c = 3$. Use C_9, the cycle on 9 vertices.

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Ind Step We construct G_c on next slide.

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We prove it works in the next few slides.

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Done!

$\chi(G_c) \ge c$ (This uses PVDW!)

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Want to obtain an M_{c-1}-AP in $S(M_{c-1})$ that is same color

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Set $\square = 2M_{c-1}$. (Could have made it $2M_{c-1}-1$ but bad for slides.)

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So we have a mono $A \in S(M_{c-1})$. Look at G_{c-1}^A.

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Hence $\chi(G_c) \ge c$. **Done**

$g(G_c) \geq 9$: Familiar Cases

Assume inductively that $g(G_{c-1}) = 9$.
Let C be a cycle in G_c. We show $|C| \geq 9$.
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Cycle goes from v to $G_{c-1}^{A_1}$ then leaves $G_{c-1}^{A_1}$ and *has to goto a base vertex that is not v*.
This is impossible. So this case can't happen.

$g(G_c) \geq 9$: The New Case

3) C has 2 base points u, v.

GOTO WHITE BOARD

Will show that u, v must be in the same $A \in S(M_{k-1})$.

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4) C has ≥ 3 base points. Can show that C has length ≥ 9.
Touched on this earlier in the proof for $\chi(G_c) = c$, $g(G_c) = 6$.

Application of VDW: Constructing Graphs with High Chromatic Number and Girth 12

December 31, 2024

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The same construction I did for $g(G_c) = 9$ actually shows $g(G_c) = 12$ but uses harder Number Theory.