Application of PVDW: Constructing Graphs with High Chromatic Number and High Girth

December 31, 2024

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https://www.cs.umd.edu/~gasarch/bookrev/40-3.pdf

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Application of Pigeonhole: Constructing Graphs with High Chromatic Number and Girth 6

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Ind Step We construct G_c on next slide.

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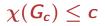
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Construction is done.

We prove it works in the next few slides.



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Assume inductively that $\chi(G_{c-1}) = c - 1$. We show $\chi(G_c) \ge c$. Assume, BWOC, $\chi(G_c) \le c - 1$. Of the *L* base vertices, there exists $\left\lfloor \frac{L}{c-1} \right\rfloor$ that are same color.

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Inductively G_{c-1}^A has a cycle of size 6. Hence G_c does.



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Assume inductively that $g(G_{c-1}) = 6$. Let C be a cycle in G_c . We show |C| > 6. 0) If C has 0 base vertices then C is a cycle in G_{c-1}^A , so $|C| \ge 6$. 1) If C has 1 base vertex v then v has two edges coming out of it, to $G_{c-1}^{A_1}$ and $G_{c-1}^{A_2}$. **GOTO White Board!** Cycle goes from v to $G_{c_1}^{A_1}$ then leaves $G_{c_1}^{A_1}$ and has to goto a base vertex that is not v. This is impossible. So this case can't happen. 2) Can it use exactly 2 base vertices, say 1,2. Yes. **GOTO WHITE BOARD** B1 is Base vertex 1, B2 is Base vertex 2. C1 is 1 in a copy of G_c , C2 is 2 in that copy. D1 is 1 in a copy of G_c , D2 is 2 in that copy. Shortest cycle: (B1, C1, C2, B2, D2, D1, B1). Len 6.

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4) Note If cycle uses $x \ge 2$ base vertices then shortest cycle is length 3x. (Will use this later) GOTO WHITE BOARD

Upshot

We have $\chi(G_c) = c$ $g(G_c) = 6$. So we are done.

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The construction we did for $\chi(G_c) = c$, $g(G_c) = 6$, credited to **Blanch Descartes**, yields such a graph when c = 4.

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Our interest Some of the constructions used VDW and PVDW!

g(G)	Math	who	
6	PHP	Folklore	
9	VDW, Messy	O'Donnell	
12	PVDW & Hard Number Theory	O'Donnell	
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Fewer sets A so that for all $A_1, A_2, |A_1 \cap A_2| \le 1$.

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We want the following:

- Fewer sets A so that for all $A_1, A_2, |A_1 \cap A_2| \leq 1$.
- Enough sets A so that can do the $\chi(G_c) \ge c$ proof.

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$$(w - y)d_{1}^{m} = (x - z)d_{2}^{m} \text{ so } \frac{w - y}{x - z} = (\frac{d_{2}}{d_{1}})^{m}$$

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If $m = 2$ then $\frac{w - y}{x - z} \in \{\frac{1}{4}, 1, 4\}$.
Solution $w = 4, y = 3, x = 4, z = 0, d_{1} = 2, d_{2} = 1$.

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A Lemma and a Thm

Lemma Let $k \ge 3$. $(\exists m)$ such that the the following holds: For all $\alpha, \beta \in \{1, ..., k\}$ there is **no** (d_1, d_2) with $d_1 \ne d_2$ such that

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$$\alpha d_1^m = \beta d_2^m.$$

Thm Let $k \ge 3$. $(\exists m = m(k))$ such that the following holds: If A_1 is a k-AP with diff d_1^m and A_2 is a k-AP with diff d_2^m , with $d_1 \ne d_2$, then $|A_1 \cap A_2| \le 1$.

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Given k let m = m(k). Let $D = \{d^m : d \ge 1\}$.

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What to do Next Slide.

We Can Use the Following

Note that the following do not intersect in ≥ 2 places: (1, 5, 9, 13, 17) (2, 6, 10, 14, 18) (3, 7, 11, 15, 19) (4, 8, 12, 16, 20) Do we need to stop here? No.

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(21, 25, 29, 33, 37)
(22, 26, 30, 34, 38)
(23, 27, 31, 35, 39)
(24, 28, 32, 36, 40)
```

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(24, 28, 32, 36, 40)
```

So can start with any $a \equiv 1, 2, 3, 4 \pmod{20}$.

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Starting Points *a*

More generally we can do the following for k-APs and $d \in D$.

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Easy to prove, but we won't do that.

Given k



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Lemma If A_1 and A_2 are in S(k) then $|A_1 \cap A_2| \le 1$.

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Start Lemma Consider the numbers

$$a, a+d, \ldots, a+(k-1)d.$$

One of them is $\equiv 1, \dots, d \pmod{kd}$. **Pf** View $\{1, \dots, kd\}$ in chunks as follows:

$$\{1,\ldots,d\}, \{d+1,\ldots,2d\}, \cdots, \{(k-1)d+1,\ldots,kd\}$$

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Note We will be applying this with $k = M_{c-1}$ and $d = d^m$.

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Thm For all $c \ge 3$ there exists graph G_c such that

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$$\chi(G_c) = c, \text{ and}$$
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Ind Step We construct G_c on next slide.

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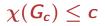
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GOTO WHITE BOARD TO LOOK AT G₄

Construction is done.

We prove it works in the next few slides.



Assume inductively that $\chi(G_{c-1}) = c - 1$.



$\chi(G_c) \leq c$

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Color each G_{c-1}^A with [c-1].

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Color all of the base vertices c.

Done!

Assume inductively that $\chi(G_{c-1}) = c - 1$. We show $\chi(G_c) \ge c$.

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Assume inductively that $\chi(G_{c-1}) = c - 1$. We show $\chi(G_c) \ge c$. Assume, BWOC, $\chi(G_c) \le c - 1$ via COL.

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Assume inductively that $\chi(G_{c-1}) = c - 1$. We show $\chi(G_c) \ge c$. Assume, BWOC, $\chi(G_c) \le c - 1$ via COL. *L* is a c - 1-colored sequence of integers.

Assume inductively that $\chi(G_{c-1}) = c - 1$. We show $\chi(G_c) \ge c$. Assume, BWOC, $\chi(G_c) \le c - 1$ via COL. L is a c - 1-colored sequence of integers. Choose $L = W(x^m, 2x^m, \dots, \Box x^m; c - 1)$ where we choose \Box later.

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 $a, a + d^m, a + 2d^m, \ldots, a + \Box d^m$ same color.

Want to obtain an M_{c-1} -AP in $S(M_{c-1})$ that is same color

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Want to obtain an M_{c-1} -AP in $S(M_{c-1})$ that is same color We are halfway there since diff is an *m*th power.

 $(\exists a, d)[a, a + d^m, a + 2d^m, \dots, a + \Box d^m \text{ same color }].$

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 $(\exists a, d)[a, a + d^m, a + 2d^m, \dots, a + \Box d^m$ same color]. Want to obtain an M_{c-1} -AP in $S(M_{c-1})$ that is same color. We are halfway there since diff that is an *m*th power. By **Start Lemma** there exists $0 \le x \le M_{c-1} - 1$ such that $a + xd^m \equiv 1, \dots, d \pmod{M_{c-1}d^m}$. If we start out sequence there we get

$$(a + xd^m, a + (x + 1)d^m, \dots, a + (M_{c-1} + x - 1)d^m) \in S(M_{c-1}).$$

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Need all of these to be $\leq \Box d^m$.

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Need all of these to be $\leq \Box d^m$. $M_{c-1} + x - 1 \leq M_{c-1} + M_{c-1} - 1 = 2M_{c-1} - 1.$

 $(\exists a, d)[a, a + d^m, a + 2d^m, \dots, a + \Box d^m$ same color]. Want to obtain an M_{c-1} -AP in $S(M_{c-1})$ that is same color. We are halfway there since diff that is an *m*th power. By **Start Lemma** there exists $0 \le x \le M_{c-1} - 1$ such that $a + xd^m \equiv 1, \dots, d \pmod{M_{c-1}d^m}$. If we start out sequence there we get

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Need all of these to be $\leq \Box d^m$.

 $M_{c-1} + x - 1 \le M_{c-1} + M_{c-1} - 1 = 2M_{c-1} - 1.$ Set $\Box = 2M_{c-1}$. (Could have made it $2M_{c-1} - 1$ but bad for slides.)

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Back to $\chi(G_c) \geq c$

We want to prove $\chi(G_c) \geq c$.



Back to $\chi(G_c) \geq c$

We want to prove $\chi(G_c) \ge c$. We assume, BWOC, that $\chi(G_c) \le c - 1$ via COL.



Back to $\chi(G_c) > c$

We want to prove $\chi(G_c) \ge c$. We assume, BWOC, that $\chi(G_c) \le c - 1$ via COL. Look at COL on the *L* base points.

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We want to prove $\chi(G_c) \ge c$. We assume, BWOC, that $\chi(G_c) \le c - 1$ via COL. Look at COL on the *L* base points. *L* is chosen to be $W(x^m, 2x^m, \dots, 2M_{c-1}x^m; c-1)$, so that there will be a mono $A \in S(M_{c-1})$.

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$g(G_c) \ge 9$: The New Case

3) *C* has 2 base points *u*, *v*. **GOTO WHITE BOARD**

Will show that u, v must be in the same $A \in S(M_{k-1})$.

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$g(G_c) \ge 9$: The New Case

3) *C* has 2 base points *u*, *v*. **GOTO WHITE BOARD**

Will show that u, v must be in the same $A \in S(M_{k-1})$. Recall that $S(M_{k-1})$ was constructed so that no two APs in it intersected in ≥ 2 points.

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4) C has \geq 3 base points. Can show that C has length \geq 9. Touched on this earlier in the proof for $\chi(G_c) = c$, $g(G_c) = 6$.

Application of VDW: **Constructing Graphs with High Chromatic Number** and Girth 12

December 31, 2024

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The same construction I did for $g(G_c) = 9$ actually shows $g(G_c) = 12$ but uses harder Number Theory.