BILL, RECORD LECTURE!!!!

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Probabilistic Method Proof For Distinct Diff Sets

Exposition by William Gasarch

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2) This method is very powerful and is used a lot.

3) We will use it to prove that there are large distinct diff sets.

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Can we do better?
STUDENTS break into small groups and try to either do better OR show that you best you can do is $O(\log n)$.

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KEY: If the prob is strictly greater than 0 then there must be SOME set of a elements where all of the diffs are distinct.

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We only need to show that the prob is LESS THAN 1.

Review a Little Bit of Combinatorics

The number of ways to CHOOSE y elements out of x elements is

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}.$$

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Two ways to create a set with a diff repeated:

Way One:

- Pick $x < y$. There are $\binom{n}{2} \le n^2$ ways to do that.
- Pick diff d such that $x + d \ne y$, $x + d \le n$, $y + d \le n$. Can do $\le n$ ways. Put $x, y, x + d, y + d$ into A.
- Pick $a - 4$ more elements out of the $n - 4$ left.

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Way Two: Pick $x < y$. Let $d = y - x$ (so we do NOT pick d). Put $x, y = x + d, y + d$ into A. Pick $a - 3$ more elements out of the $n - 3$ left.

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$$\leq \frac{32a^4}{n}$$ Need some Elem Algebra and uses $n \geq 5$.

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UPSHOT: For all $n \ge 5$ there exists a all-diff-distinct subset of $\{1, \ldots, n\}$ of size roughly $n^{1/4}$.

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- Caveat: If the Prob Proof has high prob of getting the object, then seems constructive. If all you prove is nonzero, than maybe not.

Actually Can Do Better

- With a maximal set argument can do $\Omega(n^{1/3})$.
- Better is known: $\Omega(n^{1/2})$ which is optimal.
  (That is a result by Kolmos-Sulyok-Szemeredi from 1975)