BILL, RECORD LECTURE!!!!

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Lower Bounds on $R(k)$

Exposition by William Gasarch

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We compare our LBs to the UB 2^{2k-1} for convenience.

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$\mathrm{COL}\colon \binom{f(k)}{2} \rightarrow [2]$ such that there is no homog set of size k.

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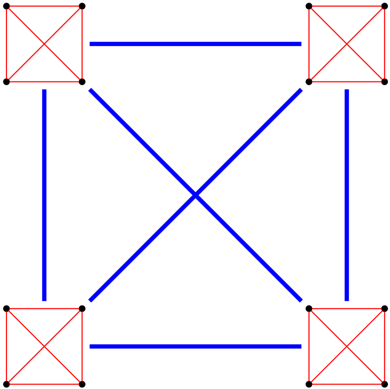
We first give an example, on the next slide.

Example: The $k = 5$ Case



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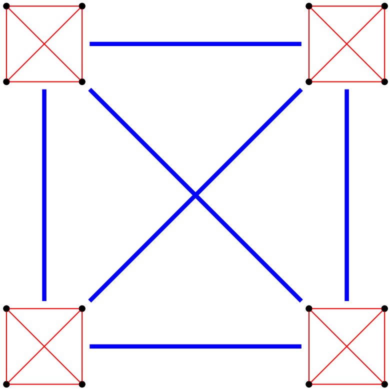

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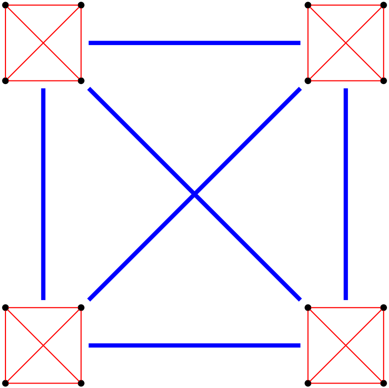

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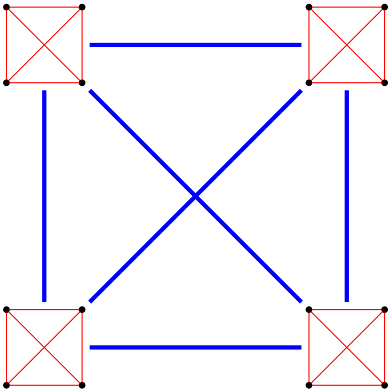

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We will do better!

We Show

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We show that G has no $\geq \frac{k-1}{2}$-clique and no $\geq \frac{k}{2}$-ind. set.

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At most 2 neighbors have 3 in it.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
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Hence $|N(\{1, 2, 3\})| \leq 6$.

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One can show that $k \leq 15$ contrary to hypothesis.

The Largest Ind. Set

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I) $\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$. (Do not confuse this **I** with the name of the ind. set which we call I.)

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I) $\{1,2,3\}$, $\{1,2,4\}$, $\ldots$, $\{1,2,17\}$. (Do not confuse this **I** with the name of the ind. set which we call I.)

II) $\{8,9,10\}$, $\{8,9,11\}$, $\{8,10,11\}$, $\{9,10,11\}$.

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Let I be a maximum sized ind. set in G. We show $|I| \leq \frac{k}{2}$.

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II) $\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.

III) $\{12, 13, 14\}$, $\{15, 16, 17\}$.

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Assume C is a type II cluster.
1) The number of vertices in C is ≤ 4.

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2) The union of the vertices in C has ≤ 4 numbers.

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2) The union of the vertices in C has ≤ 4 numbers.
3) If $v \notin C$ then for all $v' \in C$, $v \cap v' = \emptyset$.

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$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader
Let C be a cluster of type **III**.

Type III Clusters

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Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader

Let C be a cluster of type **III**.

1) If $|C| = s$ then the union of the vertices in C has $3s$ numbers.

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader
Let C be a cluster of type **III**.

1) If $|C| = s$ then the union of the vertices in C has $3s$ numbers.
2) If $v \notin C$ then for all $v' \in E$, $v \cap v' = \emptyset$. (this is by definition of a type III cluster).

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Example $k = 17$ which does not matter.
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Facts Left To The Reader
Let C be a cluster of type **III**.

1) If $|C| = s$ then the union of the vertices in C has $3s$ numbers.
2) If $v \notin C$ then for all $v' \in E$, $v \cap v' = \emptyset$. (this is by definition of a type III cluster).
3) There is at most one cluster of type **III** (if not you can union them).

No Need for Type I Clusters

Lemma Assume I is a maximum sized ind. set. Then $\exists I'$, an ind. set, $|I| = |I'|$ and I has no type I clusters.

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We do four examples of how to take a cluster of type **I** and rearrange the numbers in it to form clusters of type **II** or **III**, while not decreasing the number of vertices.

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We leave it to the reader to take our examples and make general proofs out of them (which is easy).

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\},$
$\{1, 2, 7\}, \{1, 2, 8\}, \{1, 2, 9\}, \{1, 2, 10\},$
$\{1, 2, 11\}, \{1, 2, 12\}$

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\}$

has 10 vertices.

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
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Rearrange into 3 clusters of type **II**, which is 12 vertices:

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Rearrange into 3 clusters of type **II**, which is 12 vertices:

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\},$
$\{1,2,7\}, \{1,2,8\}, \{1,2,9\}, \{1,2,10\},$
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$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

has 11 vertices.

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We cannot use the number 13. Oh well.

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

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We cannot use the number 13. Oh well.
Even so, rearranging lead to $12 > 11$ vertices.

Getting Rid of Type I Clusters of size $\equiv 2 \pmod 4$

$$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$$
$$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$$
$$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\}$$

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has 12 vertices.

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has 12 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:
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We cannot use the numbers 13 or 14. Oh well.

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We cannot use the numbers 13 or 14. Oh well.

Even so, rearranging let to $12 = 12$ vertices.

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\},$
$\{1,2,7\}, \{1,2,8\}, \{1,2,9\}, \{1,2,10\},$
$\{1,2,11\}, \{1,2,12\}, \{1,2,13\}, \{1,2,14\},$
$\{1,2,15\}$

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We rearrange this into 3 clusters of type **II** and one size-1 cluster of type **III**, which is 13 vertices.

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The number of vertices stayed the same.

Ind. Sets That have Type II,III Clusters Only

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We bound the $|I|$ if I has clusters of type **II** and **III** only.

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Next Slide Packet: **Are Better Lower Bounds Possible Given the Current State Of Our Knowledge?**.
And, as always, I ask **Will the proofs be suitable for HS students? For Ilya?**

Summary of What is Known

The following results were obtained by constructing the appropriate colorings.

Result	Comments	Paper
$R(k) \geq (k-1)^2$	Elt	Folklore, 1950s
$R(k) \geq \Omega(k^{2.3\ldots})$	Elt	[1], 1971
$R(k) \geq \Omega(k^3)$	Elt	[9], 1981, 1975
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Set Systems	[3], 1981
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Info. Theory & Lin. Alg.	[2], 1998
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Representing OR	[5], 2000
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$R(k) \geq$ Better than [3]	Extractors	[6], 2017,
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Refs at end of slide packet. Some include pointers to the papers.

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1) Want HS proof that $R(k) \ge \Omega(k^4)$. Or higher degree.
2) Want easier proofs of the results in the table beyond $\Omega(k^3)$.

First Slide of References

1) H. Abbott. Lower bounds on some Ramsey numbers. *Discrete Mathematics*, 2:289–293, 1971.

2) N. Alon. The Shannon capacity of a union. *Combinatorica*, 18(3):301–310, 1998.
`https://doi.org/10.1007/PL00009824`.

3) P. Frankl and R. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, 1:357–368, 1981.
`https://www.cs.umd.edu/~gasarch/TOPICS/CRT/FW.pdf`

Second Slide of References

4) P. Gopalan. Constructing Ramsey graphs from Boolean function representations. *Combinatorica*, 34(2):173–206, 2014. https://doi.org/10.1007/s00493-014-2367-1.

5) V. Grolmusz. Superpolynomial size set-systems with restricted intersections mod 6 and explicit Ramsey graphs. *Combinatorica*, 20(1):71–86, 2000. https://doi.org/10.1007/s004930070032.

6) X. Li. Improved non-malleable extractors, non-malleable codes and independent source extractors. In H. Hatami, P. McKenzie, and V. King, editors, *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017*, pages 1144–1156. ACM, 2017. https://doi.org/10.1145/3055399.3055486.

Third and Last Slide of References

7) X. Li. Non-malleable extractors and non-malleable codes: Partially optimal constructions. In A. Shpilka, editor, *34th Computational Complexity Conference, CCC 2019, July 18-20, 2019, New Brunswick, NJ, USA*, volume 137 of *LIPIcs*, pages 28:1–28:49. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
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`https://doi.org/10.1109/FOCS57990.2023.00075`.

9) Z. Nagy. A constructive estimate of the Ramsey numbers (In Hungarian). *Mat. Lapok*, pages 301–302, 1975.