BILL, RECORD LECTURE!!!!

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The Axiom of Choice and its Equivalences

Exposition by William Gasarch

March 29, 2025

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We will use DBE thm on the graph $(\mathbb{R}^2, E)$ where

$$E = \{(p, q) : |p - q| = 1\}$$

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To show some f exists needs the axiom of choice.

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Can $\mathbb{R}$ be well ordered? Discuss.

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$\mathbb{R}$ can be well ordered. Is that strange?

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Odd? Do these two odd facts make your doubt WOP?

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And who can tell about Zorn's Lemma?