A Slightly Better Upper Bound on $R(k)$

Exposition by William Gasarch

January 20, 2025

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We will use item (2) to improve the upper bound from item (1).

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I prefer writing

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Answer on the Next Slide!

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Result (5) is the best known.