Euclidean Ramsey Theory: Area

Exposition by William Gasarch

December 31, 2024

Mono Triangles

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We will prove the following:
Thm $\forall$ finite colorings of $\mathbb{R}^2$ $\exists$ a mono triangle with area 1.

The Two Color Case

The 2-Color Case

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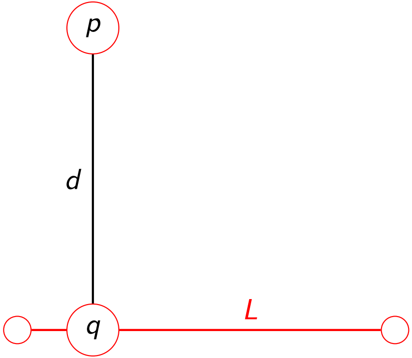

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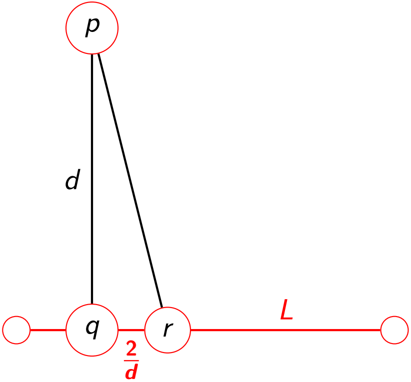

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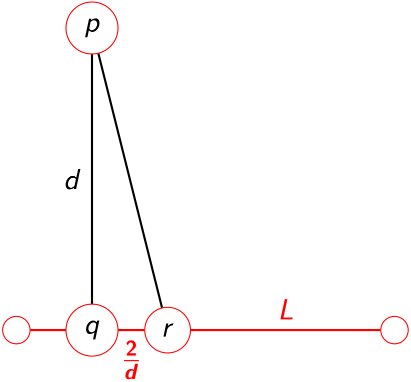

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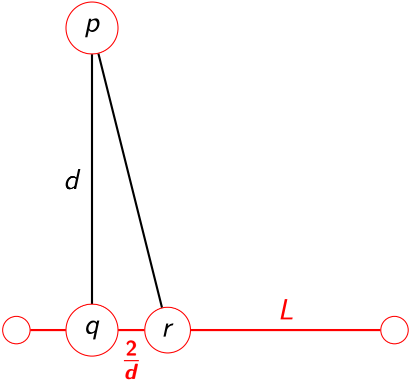

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Case 1 DONE.

Case 2,3,4

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Case 4: $\exists$ a horiz. line L which is all **B**, but every p not on L is **R**.

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Case 4: $\exists$ a horiz. line L which is all **B**, but every p not on L is **R**.

So whats left? See next slide.

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We continue on next slide.

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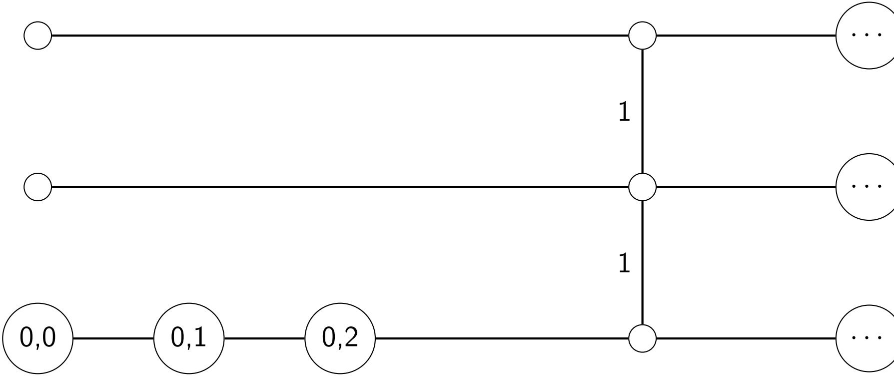

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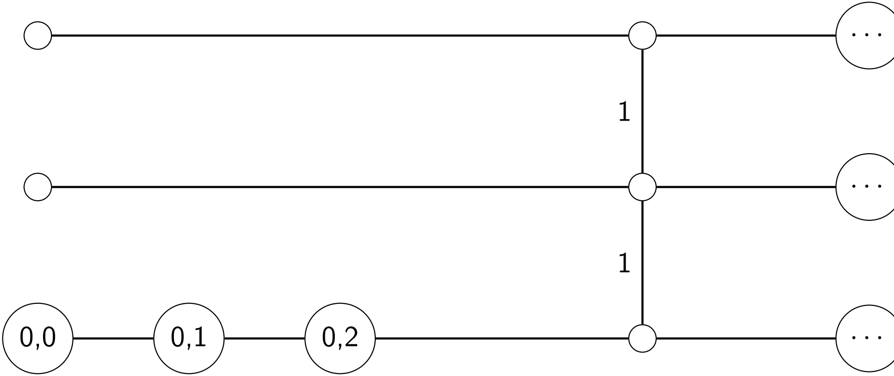

Two of $(0,0)$, $(0,1)$, $(0,2)$ are the same color, say **R**.

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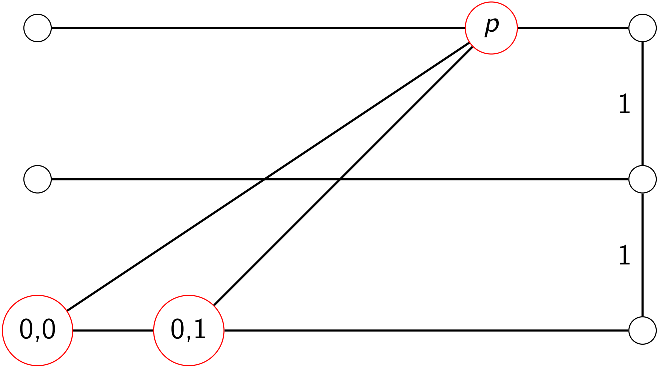

Area of $(0,0),(0,1),p$ is $\frac{1}{2}\times 1\times 2=1$.

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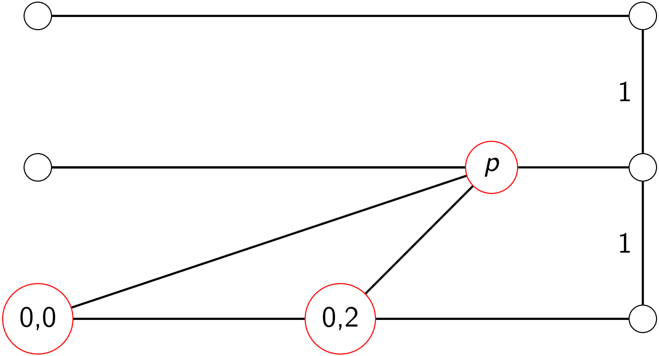

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Define

$$\mathrm{COL}' \colon [W(k,c)] \to \{\{\mathbf{R}\}, \{\mathbf{B}\}, \{\mathbf{G}\}, \{\mathbf{R}, \mathbf{B}\}, \{\mathbf{R}, \mathbf{G}\}, \{\mathbf{B}, \mathbf{G}\}, \{\mathbf{R}, \mathbf{B}, \mathbf{G}\}\}$$

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as follows:
$\mathrm{COL}'(i) =$ the set of colors used by COL on the line $y = i$.

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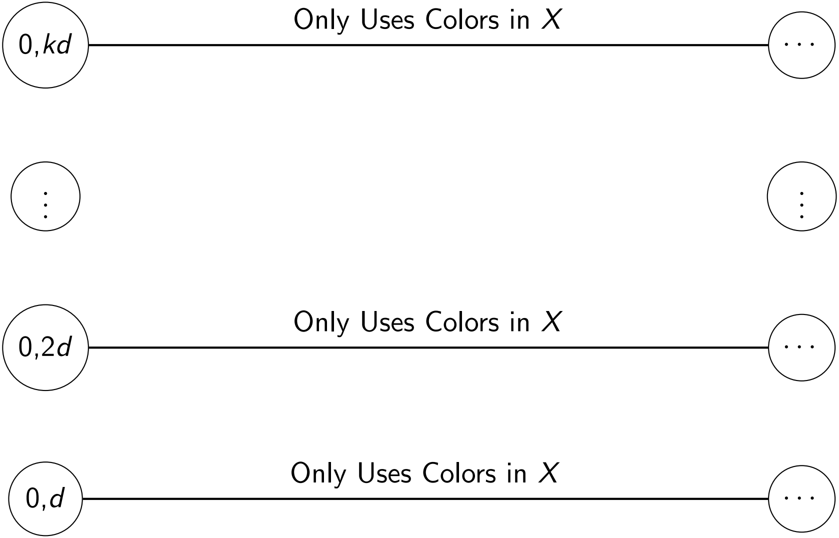

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Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d)$.

Two of them are the same color. Assume **R**.

Case 2.1: $|X| = 2$. $\mathbf{COL}(0, d) = \mathbf{COL}(\frac{1}{3d}, d) = \mathbf{R}$

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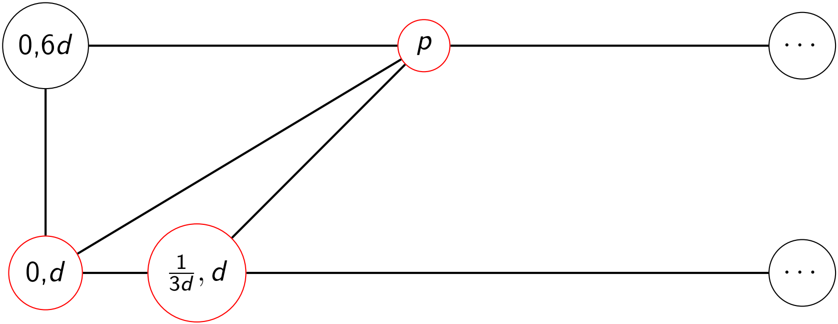

Area of triangle $((0, d), (\frac{1}{3d}, d), p)$ is $\frac{1}{2} \times \frac{1}{3d} \times 6d = 1$.

Case 2.2: $|X| = 2$. $\mathrm{COL}(0, d) = \mathrm{COL}(\frac{2}{3d}) = \mathbf{R}$

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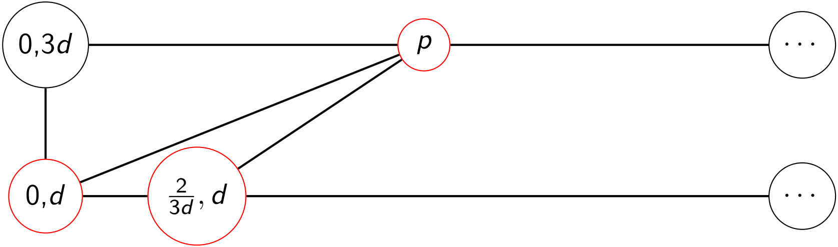

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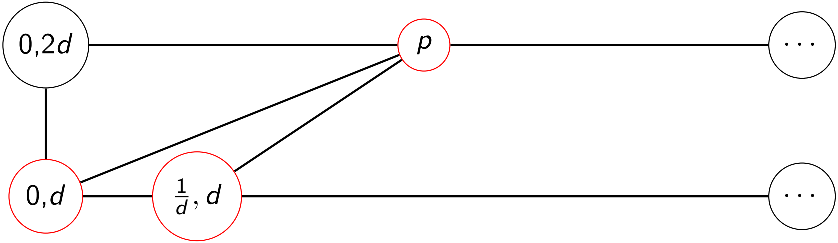

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2. We need $6d$, so AP of length 7. $k = 7$.

Generalize

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Key is to find the right parameters for VDW.