

Schur's Thm + FLT implies Primes Infinite

December 31, 2024

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3. Look at domains where the number of primes is finite and see where the standard proof fails, and where the EG-proof fails.

Background Needed For EG-Proof

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1. $[n] = \{1, 2, \dots, n\}$.
2. $\binom{A}{k}$ is the set of all subsets of A of size k .

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Say its $a < b < c$

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So let $S(c) = R(3; c)$ (homog set 3, colors c).

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In modern terminology:

$$(\forall n \geq 3)(\forall x, y, z \in \mathbb{N} - \{0\})[x^n + y^n \neq z^n].$$

This has come to be known as **Fermat's Last Theorem**.

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- 2) Andrew Wiles proved FLT in the early 1990s with techniques far beyond what Fermat could have known.

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$$(p_1^{x_1} \cdots p_L^{x_L})^4 + (p_1^{y_1} \cdots p_L^{y_L})^4 = (p_1^{z_1} \cdots p_L^{z_L})^4$$

This violates FLT for $n = 4$.

How to Ask the Question of Primes Infinite

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2) \mathbb{Q} , \mathbb{R} , \mathbb{C} .

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On these slides **infinite** will mean **infinite up to units**.

The Normal Proof that Primes are Infinite and Where it Falls Apart

December 31, 2024

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Thm The set of primes in \mathbb{Z} is infinite.

Assume not. Let $\{p_1, \dots, p_n\}$ be all of the primes in \mathbb{Z} .

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$$Mp = p_1 \cdots p_n + 1. \text{ Take this mod } p.$$
$$0 \equiv p_1 \cdots p_n + 1 \pmod{p}.$$
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Upshot The proof that \mathbb{Z} has an infinite number of primes uses that, for all $p_1 \cdots p_n + 1$ is never a unit.

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We actually have a list of primes: $\{2\}$.

$N = 2 + 1 = 3$ which is a unit.

So similar to why the proof fails for \mathbb{Q} .

\mathbb{A}^1 has a Finite Number of Primes

$$\mathbb{A}^1 = \{a \in \mathbb{C} : (\exists f(x) \in \mathbb{Z}[x] \text{ lead coeff } 1))[p(a) = 0]\}.$$

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There are no primes. See next slide.

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Lets revisit the proof.

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This is where the proof breaks down! In $\mathbb{A}\mathbb{I}$ you can keep going down and never get to a prime.

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Atomic Domains

Def An **Atomic Integral Domain** is an integral domain such that every element of $\mathbb{D} - (\mathbb{U} \cup \{0\})$ can be written (not necessarily uniquely) as $up_1^{x_1} \cdots p_m^{x_m}$ where u is a unit and all of the p_i 's are irreducible.

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Upshot The proof that \mathbb{Z} has an infinite number of primes used that \mathbb{Z} is atomic.

The EG-Proof that Primes are Infinite and Where it Falls Apart

December 31, 2024

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2) In our proof we used mod 4. Lets keep it n for now and try to pick some n that will work.

Where Does EG-Proof Fail for \mathbb{Q} ?

Thm The number of primes in \mathbb{Q} is infinite (attempt).
Assume, BWOC, that the primes are finite. p_1, \dots, p_L .
We define a coloring on $N \subseteq \mathbb{Q}$ as follows.

Let $\text{COL}: \mathbb{N} \rightarrow \{0, 1, 2, 3\}^L$ be the following coloring:

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2) In our proof we used mod 4. Lets keep it n for now and try to pick some n that will work.

We define the coloring as follows:

$$\text{COL}(up_1^{a_1} \cdots p_L^{a_L}) = (a_1 \pmod{n}, \dots, a_L \pmod{n})$$

Where Does EG-Proof Fail for \mathbb{Q} ? (cont)

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Not True Fix n . Let $u_x = u_y = \frac{1}{2}$, $u_z = 1$, $X = Y = Z = 1$.

$$u_x X^n + u_y Y^n = u_z Z^n$$

Becomes

$$\frac{1}{2}1^n + \frac{1}{2}1^n = 1 \times 1^n$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

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Its because the following **variant** of FLT is false for \mathbb{Q} :

There exists $n \in \mathbb{N}$ such that the following has no solution:

$$u_x X^n + u_y Y^n = u_z Z^n$$

where $u_x, u_y, u_z \in \mathbb{U}$ and $X, Y, Z \in \mathbb{Q}$.

Project TO-DO List

December 31, 2024

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2. Read in Gasarch's paper the **Sanity Check** which has more domains with a finite number of primes.
3. Read the other papers on the website of Ramsey-Primes paper. Some of the papers are difficult so try to just figure out the proof for \mathbb{Z} or \mathbb{N} , and then see where it fails for \mathbb{Q} and \mathbb{Q}_2 . (I think they all fail for $\mathbb{A}I$ because $\mathbb{A}I$ is not atomic, though check that.)