BILL, RECORD LECTURE!!!!

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Ramsey on ω^2

Results by Joanna Boyland, William Gasarch, Nathan Hurtig, Robert Rust

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Def If L_1, L_2 are linearly ordered sets then $L_1 \equiv L_2$ means there is an order preserving bijection between L_1 and L_2.

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Thm $\exists \mathrm{COL}\colon \binom{\omega^2}{2} \to [4]$ Such that there is no 3-homog $H \equiv \omega^2$.

There is Always a 4-Homog Set

Theorem About Coloring ω^2

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Edges between copies of ω are called **external**.

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We assume all Internal edges are **R**.

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Within any copy of ω all of the edges are **R**.

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  (a) H is 4-homog relative to COL, and
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Let (**B**, **G**, **Y**) be the color of the homog set.

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We continue this on the next slide.

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If $h_i < h_j < h_k < h_\ell$ then $\mathrm{COL}'(h_i < h_j < h_k < h_\ell) = (\mathbf{B}, \mathbf{G}, \mathbf{Y})$ so
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Next slide has a thinned out version of H'' that suffices.

The Homog Set

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So we are done!

There is a Coloring With No 3-Homog Set

There is a Coloring With No 3-Homog

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Thm $\exists \mathrm{COL}\colon \binom{\omega^2}{2} \to [4]$ such that there is no 3-homog $H \equiv \omega^2$.

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The proof that there is no 3-homog set is left to the reader.