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RAMSEY'S THEOREM AND RECURSION THEORY

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§1. Let N be the set of natural numbers. If $A \subseteq N$, let $[A]^n$ denote the class of all n-element subsets of A. If P is a partition of $[N]^n$ into finitely many classes $C_1, \cdots, C_p$, let $H(P)$ denote the class of those infinite sets $A \subseteq N$ such that $[A]^n \subseteq C_i$ for some i. Ramsey's theorem [8, Theorem A] asserts that $H(P)$ is nonempty for any such partition P. Our purpose here is to study what can be said about $H(P)$ when P is recursive, i.e. each C_i is recursive under a suitable coding of $[N]^n$. We show that if P is such a recursive partition of $[N]^n$, then $H(P)$ contains a set which is Π_n^0 in the arithmetical hierarchy. In the other direction we prove that for each $n \geq 2$ there is a recursive partition P of $[N]^n$ into two classes such that $H(P)$ contains no Σ_n^0 set. These results answer a question raised by Specker [12].

A *basic partition* is a partition of $[N]^2$ into two classes. In §§2, 3, and 4 we concentrate on basic partitions and in so doing prepare the way for the general results mentioned above. These are proved in §5. Our "positive" results are obtained by effectivizing proofs of Ramsey's theorem which differ from the original proof in [8]. We present these proofs (of which one is a generalization of the other) in §§4 and 5 in order to clarify the motivation of the effective versions.

We now develop some terminology and notation. If $A \subseteq N$, then $\bar{A}$ denotes $N - A$. The set A is called *bi-immune* if both A and $\bar{A}$ are immune, i.e. are infinite but have no infinite r.e. subset. If D is a finite set, $|D|$ is its cardinality. If f, g: $N \to N$, g is said to *majorize* f in case $g(n) \geq f(n)$ for all $n \in N$. If $f: N \times N \to N$ and $\lim_s f(n, s)$ exists for all $n \in N$, then *lim* f is defined to be $\{n: \lim_s f(n, s) = 0\}$.

If $A \subseteq N$, then A' denotes the jump of A and $A^{(n)}$ denotes the n-fold jump of A. We fix a recursive set denoted by O and let $\mathbf{O}, \mathbf{O}', \mathbf{O}''$ denote the degrees of O, O', O'' respectively. For sets A, B the expressions "A is recursive in B," "A is B-recursive," and "$A \leq_T B$" are used synonymously. A set $A \subseteq N$ is called Σ_n^0 [Π_n^0] if it has a definition consisting of an $(n + 1)$-place recursive predicate preceded by n number quantifiers with the leftmost quantifier existential [universal]. According to the *strong hierarchy theorem* [9, §14.5] the Σ_n^0 sets of numbers are exactly the sets r.e. in $O^{(n-1)}$ for $n > 0$. For further information on the arithmetical hierarchy the reader is referred to [9, Chapter 14].

§2. In considering effective versions of Ramsey's theorem, the first question to arise naturally is whether $H(P)$ contains a recursive set for every recursive basic partition P. To the author's knowledge, the question was first answered negatively

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by Specker, who presented his result in a talk at Manchester in 1966 and in a later paper [12]. Specker's proof was based on the existence of incomparable r.e. degrees [1]. Some time ago, Yates [14, Corollary 1.5] and the present author also obtained negative answers using, respectively, the existence of a bi-immune retraceable set ([7, Theorem 3] or [14, Theorem 1.1]) and the existence of a bi-immune semi-recursive set [2, Theorem 5.2]. We now give a proof which is simpler in that it merely uses the existence of a bi-immune set recursive in O'.

PROPOSITION 2.1. *If a set A is recursive in O', then there exists a recursive basic partition P such that every element of $H(P)$ is a subset of A or of $\bar{A}$.*

PROOF. Since $A \leq_T O'$, it follows from [11, Theorem 2] that there is a recursive function f such that $A = \lim f$. Let one class of a basic partition P consist of all pairs $\{n, s\}$ such that $n < s$ and $f(n, s) = 0$. Let the other class of P consist of all other 2-element sets. Clearly P has the desired property.

COROLLARY 2.2 (SPECKER). *There exists a recursive basic partition P such that $H(P)$ has no recursive member.*

PROOF. Let A be a bi-immune set recursive in O', and apply Proposition 2.1. (The existence of such a set A follows from any of the existence results cited at the beginning of this section or [3, Theorem 3] but may also be established by a very simple direct construction.)

Corollary 2.2 will be considerably extended in §§3 and 5. We present here a pair of minor extensions which were announced in [4]. The original proofs of these results have been simplified and unified by combining Proposition 2.1 with the following observation.

REMARK. If $A \leq_T C \leq_T O'$ and the partition P is determined from A as in Proposition 2.1, then $H(P)$ has an element of the same degree as C.

To prove the remark, assume that $A = \lim f$ and $C = \lim g$, where f, g are binary recursive functions and f determines P. Also assume that A is infinite, since otherwise A may be replaced by $\bar{A}$. We now define a function h. Assuming inductively that $h(i)$ is defined and in A for all $i < n$, define $h(n)$ to be the least member k of A sufficiently large that $(n \in C \leftrightarrow g(n, k) = 0)$ and for all $i < n$, $h(i) < k$ and $f(h(i), k) = 0$. Clearly the range of h is the desired member of $H(P)$ of the same degree as C.

COROLLARY 2.3. *If $\mathbf{a}$ is a degree and $\mathbf{O} < \mathbf{a} \leq \mathbf{O}'$, then there exists a recursive basic partition P such that $H(P)$ has a member of degree $\mathbf{a}$ but no recursive member. Also $H(P)$ contains only hyperimmune sets.*

PROOF. Let A be a set of degree $\mathbf{a}$ such that both A and $\bar{A}$ are hyperimmune [2, Theorem 5.2], and let $A = C$ in the Remark.

It can be seen from the proof of [6, Theorem 5.1] that $\mathbf{O}'$ cannot be replaced by $\mathbf{O}''$ in Corollary 2.3.

COROLLARY 2.4. *If $\{A_i\}$ is a sequence of sets uniformly recursive in O', then there exists a recursive basic partition P such that $H(P)$ has a member of degree $\mathbf{O}'$, but no A_i is in $H(P)$.*

PROOF. First obtain by direct construction a set $A \leq_T O'$ such that no infinite A_i is contained in A or $\bar{A}$. Then let C have degree $\mathbf{O}'$ in the Remark.

§3. In §2 several methods are mentioned for obtaining recursive basic partitions

P such that $H(P)$ has no recursive set. However, if P is obtained by any of these methods, $H(P)$ necessarily contains a set recursive in O'. We now prove by direct construction that this does not extend to recursive basic partitions in general.

THEOREM 3.1. *There exists a recursive basic partition P such that $H(P)$ has no member recursive in O'.*

PROOF. Shoenfield [11, Theorem 2] proved that every set recursive in O' is $\lim f$ for some binary recursive function f. His proof actually shows the existence of a *uniformly* recursive sequence of recursive functions $\{f_e\}$ such that for each $A \leq_T O'$, $A = \lim f_e$ for some e. We now write A_e for $\lim f_e$. (We consider A_e to be undefined if for some n, $\lim_s f_e(n, s)$ does not exist.)

If A_e is defined and has at least $2e + 2$ members, let D_e consist of the least $2e + 2$ members of A_e. (Otherwise let D_e be undefined.) We construct a recursive basic partition P such that for all e, P satisfies the eth *requirement*: if D_e is defined, then it is not contained in any member of $H(P)$. Clearly the theorem follows once this is done.

We now define a finite set D_e^s to approximate D_e at stage s. If there are at least $2e + 2$ numbers u such that $u < s$ and $f_e(u, s) = 0$, let D_e^s consist of the least $2e + 2$ such numbers. Otherwise, let D_e^s be undefined.

The partition P is obtained by placing each pair $\{n, s\}$ of distinct integers in exactly one of the two classes C_1, C_2. If $n < s$, the pair $\{n, s\}$ goes into $C_1 \cup C_2$ at stage s of the construction, which we now give.

Stage s ($s \geq 0$). There will be successive substages $e = 0, 1, \cdots, s$. Note that at each substage except possibly the last, at most two pairs $\{n, s\}$ ($n < s$) enter $C_1 \cup C_2$.

Substage e ($e < s$). This substage is devoted to the eth requirement. If D_e^s is undefined, do nothing. If D_e^s is defined, let F_e^s be the set of numbers n such that the pair $\{n, s\}$ is already in $C_1 \cup C_2$. Since at most two pairs entered $C_1 \cup C_2$ at each of the e previous substages, $|F_e^s| \leq 2e$. Also $|D_e^s| = 2e + 2$, so we may (effectively) choose distinct numbers j_e^s, k_e^s, each in $D_e^s - F_e^s$ and hence also less than s. Place the pair $\{j_e^s, s\}$ in C_1 and $\{k_e^s, s\}$ in C_2. This ensures that $D_e^s \cup \{s\}$ is contained in no member of $H(P)$.

Substage s. For each $n < s$, if $\{n, s\}$ is not yet in $C_1 \cup C_2$, place it in (say) C_1.

This completes the construction. Clearly the partition P determined by C_1 and C_2 is recursive. Also if D_e is defined, then D_e^s is defined and equal to D_e for all sufficiently large s. Hence $D_e \cup \{s\}$ is not contained in any member of $H(P)$, for s sufficiently large, by construction. Therefore the eth requirement is satisfied and the theorem follows.

COROLLARY 3.2. *There exists a recursive basic partition P such that $H(P)$ contains no Σ_2^0 set.*

PROOF. Let P be the partition of Theorem 3.1. Assume $H(P)$ has a Σ_2^0 member A. Then A is r.e. in O' by the strong hierarchy theorem [9, §14.5] and so A has an infinite subset B which is recursive in O'. Then $B \in H(P)$, contradicting Theorem 3.1.

In the following section we show that Corollary 3.2 is optimal with respect to the arithmetical hierarchy.

§4. We first give a proof of Ramsey's theorem for basic partitions which we learned from D. Kleitman. Using this proof as a model, we will then show that if P is a recursive basic partition, then $H(P)$ contains both a Π_2^0 set and a set A such that $A' \leq_T O''$. The former result can also be obtained by suitably effectivizing Ramsey's original proof [8, Theorem A], although the proof obtained in this way is perhaps less natural than the one given here. We do not see how to obtain the latter result starting from Ramsey's original proof. A generalization of this latter result will be the crucial tool in §5.

THEOREM 4.1 (RAMSEY). *If P is any basic partition, then $H(P)$ is nonempty.*

PROOF. Assume that each pair of numbers is labeled "red" or "blue" according to its class in the given partition P. We shall define an increasing sequence of numbers $\{a_i\}$ ($i \in N$) and each a_i will also be colored red or blue when it is defined. The coloring of the a_i's will be arranged so that if $i < j$ then the pair $\{a_i, a_j\}$ has the same color as a_i. To this end, a number c is defined to be *k-acceptable* if for each $i < k$, a_i has the same color as $\{a_i, c\}$ and $a_i < c$.

Assume inductively that a_i has been defined and colored for all $i < k$ and that the set A_k of all k-acceptable numbers is infinite. Let a_k be the least element of A_k. If a_k forms a red pair with infinitely many members of A_k, color a_k red, and otherwise color a_k blue. Clearly there are infinitely many $(k + 1)$-acceptable numbers in either case and so the induction assumption persists.

Let R be the set of red a_i's and B the set of blue a_i's. Either R or B is infinite. If R is infinite then $R \in H(P)$ because any two elements of R form a red pair by definition of "acceptability." Similarly, if B is infinite, then $B \in H(P)$.

E. P. Specker [12] and A. Manaster [private communication] each proved several years ago that if P is a recursive basic partition, then $H(P)$ contains a set recursive in O''. This can be seen directly from the above argument (or Ramsey's original proof) by noting that the only nonrecursive questions asked in it are whether certain explicitly given recursive sets are infinite. (We except of course the *single* question as to whether R is infinite.) In order to refine this result, we now use a priority argument which is similar to, but easier than, Sacks' construction of a minimal degree below $\mathbf{O}'$ [10]. Yates presents an alternative proof of this refinement in [14, Theorem 2].

REMARK. The existence of the single question mentioned above makes the proof that $H(P)$ has a member recursive in O'' *nonuniform*. This nonuniformity is essential because it can be shown that there is no recursive function f such that for all e, if e is an index of a basic recursive partition P, then $f(e)$ is an index of a reduction procedure from O'' to a member of $H(P)$. On the other hand, given such an e this argument shows that one can effectively find *two* numbers, at least one of which indexes an appropriate reduction procedure. Even this level of uniformity cannot be achieved in the next theorem however; it can be shown that there is no recursive procedure which for each index of a basic recursive partition P effectively yields finitely many numbers, at least one of which is a Π_2^0-index of a member of $H(P)$. These proofs of nonuniformity, which are not difficult, use the recursion theorem.

THEOREM 4.2. *If P is a recursive partition of all pairs of integers into p classes, then $H(P)$ contains a Π_2^0 set.*

PROOF. We first assume that $p = 2$ (i.e., P is basic) and then at the end indicate the treatment of the general case. We assume as in Theorem 4.1 that each pair of integers is labeled red or blue according to its partition class. In terms of the proof of Theorem 4.1, the method of proof here is roughly to *assume* that a_k may be colored red, thus avoiding the O'' question as to the proper color for a_k. This assumption may later prove incorrect, and when this happens the color of a_k is changed from red to blue and the part of the constructed set predicated on the false assumption is destroyed. Because of this, we must also allow the value of a_k to change, and we write a_k^s for the number used at stage s to approximate a_k. For fixed k, a_k^s will be defined and equal to a limiting value (denoted a_k) for all sufficiently large s.

Following Rogers [9, Chapter 10], we shall also speak of a_k^s as "the position of the marker Λ_k at (the beginning of) stage s". A number c is called *k-acceptable at s* if, for each $i < k$, a_i^s is defined and has the same color at s as $\{a_i^s, c\}$ and $a_i^s < c$. A number c is called *free* at s if prior to stage s it has not been the position of any marker and $c \geq s$. We now describe the construction.

Stage s ($s \geq 0$). Assume inductively that there is a number $n(s)$ such that the markers presently having a position are exactly the Λ_i for $i < n(s)$. The first case below corresponds exactly to the proof of Theorem 4.1 while the second case corresponds to rectification of a previous incorrect assumption.

Case 1. There exists a number c which is free and $n(s)$-acceptable at s. Attach $\Lambda_{n(s)}$ to the least such number c and color c red.

Case 2. Otherwise. Let $j(s)$ be the largest number j such that there exists a number which is free and j-acceptable at s. (Such numbers j exist because every number is 0-acceptable at all stages. Also $j(s) < n(s)$ since Case 1 does not apply.) Change the color of $a_{j(s)}^s$ and (if $j(s) + 1 < n(s)$) detach all markers Λ_i for $j(s) < i < n(s)$.

In either case, any unmentioned marker is left unchanged. The construction may be carried out recursively in O' because the only nonrecursive questions asked in it are whether certain explicitly given recursive sets are nonempty.

LEMMA 4.3. *For any k there is a number a_k such that Λ_k has position a_k from some stage on. The color of a_k may change only from red to blue.*

PROOF. We assume inductively that the lemma holds for all $k < n$ and prove it for $k = n$. Let s_0 be the least stage s such that for all $k < n$, Λ_k has position a_k and a_k has its eventual color at the beginning of stage s. At stage s_0, Λ_n becomes attached to a number c through Case 1 of the construction. Also Λ_n can never be detached from c after s_0 because no a_k ($k < n$) changes color after s_0. Thus $a_n = c$. Let A_n be the set of all numbers which are n-acceptable at stage s_0. If $s \geq s_0$, A_n is also the set of n-acceptable numbers at s, because no a_k ($k < n$) changes color after s_0. Hence A_n is infinite because A_n contains a number free at s for all $s \geq s_0$. If A_n contains infinitely many numbers which form a red pair with a_n, then a_n will retain its initial red color forever. Otherwise, the color of a_n will change from red to blue at some stage $s_1 > s_0$, but the color of a_n will never again change after s_1 because A_n has infinitely many numbers which form a blue pair with a_n.

LEMMA 4.4. *If $i < j$, the pair $\{a_i, a_j\}$ has the same color as the eventual color of a_i.*

PROOF. When a_j first is the position of Λ_j, a_j must be j-acceptable. The color of a_i cannot later be changed, lest a_j forever lose its marker. The lemma now follows from the definition of "j-acceptable."

We define M to be the set of all a_i's, R to be the set of all a_i's which are eventually red, and B to be the set of a_i's which are eventually blue. It follows from Lemma 4.4 that if R is infinite then $R \in H(P)$ and similarly for B. Also M is infinite because $\{a_i\}$ is a nonrepeating sequence.

LEMMA 4.5. *The sets M and R are each Π_2^0.*

PROOF. By the definition of "free" and the fact that a number which loses a marker never again bears a marker, we have, for all x,

$$x \in \overline{M} \leftrightarrow (\exists s)[s \geq x \text{ and } x \text{ has no marker at } s].$$

Because the color of an a_j can be changed only from red to blue, we have, for all x,

$$x \in \overline{R} \leftrightarrow x \in \overline{M} \quad \text{or} \quad (\exists s)[x \text{ is blue at } s].$$

In each of the above equivalences, the bracketed portion is a O'-recursive predicate of x and s. Hence $\overline{M}$ and $\overline{R}$ are each r.e. in O', so by the strong hierarchy theorem M and R are Π_2^0.

The proof is now complete for basic partitions because if R is infinite, then $R \in H(P)$ and if R is finite, then $B\ (= M - R)$ is infinite and Π_2^0, so $B \in H(P)$.

REMARK. If R is finite, one can actually show that $B \leq_T O'$. Observe that for all i, $a_i = a_i^{s+1}$, where s is the least stage such that for all $j < i$, a_j^s has (at stage s) the eventual color of a_j. But if R is finite, then the eventual color of a_j can be determined recursively, and hence a_i is a O'-recursive function of i. It is an immediate consequence of this observation that if a recursive basic partition P is determined by classes C_1, C_2 then either $H(P)$ has a member recursive in O' or there exist infinite Π_2^0 sets R_1, R_2 such that $[R_i]^2 \subseteq C_i$ for $i = 1, 2$.

By a generalization of the above observation it can be shown, using the separation principle for Π_2^0 sets, that B is Π_2^0 only if $H(P)$ has a member recursive in O'. This points up the asymmetric roles of R and B.

Consider now the more general case where the given partition P consists of p classes, represented by colors $C_1, \cdots, C_p$. The numbers are colored in the same way as before except that a number is first colored C_1 and all changes of color are from C_i to C_{i+1} for some $i < p$. The markers and colors stabilize as before. Let M_i be the set of numbers which from some stage on have a marker and have color C_i. Then $\bigcup_{i \leq j} M_i$ is Π_2^0 for all $j \leq p$. Thus if i_0 is the *least* i such that M_i is infinite, M_{i_0} is the desired Π_2^0 set in $H(P)$.

A Π_1^0 *class of functions* is one which has the form $\{f : f \in N^N \text{ and } (\forall k)R(f, k)\}$, where R is a recursive predicate of one function and one number variable [9, Chapter 15]. More intuitively, a Π_1^0 class of functions can be represented as the set of infinite paths through some recursive tree of finite sequences. The following proposition will allow us to deduce results about $H(P)$ from theorems on such classes.

PROPOSITION 4.6. *If P is a recursive basic partition, then there is a nonempty Π_1^0 class of functions $\mathscr{S}$ such that*

$$(\forall f \in \mathscr{S})(\exists A \in H(P))[A \leq_T f].$$

$\mathscr{S}$ has the additional property that there is a O'-recursive function w which majorizes all members of $\mathscr{S}$.

PROOF. The proof is closely related to the proof of Theorem 4.1. Assume again that P is specified by a red-blue coloring of pairs. By a *string* we mean a finite sequence of natural numbers. Let σ be a nonempty string $\langle a_0, \cdots, a_k\rangle$. A number c is called *σ-acceptable* if $c > a_k$ and for all $i < k$, the color of $\{a_i, c\}$ is the same as the color of $\{a_i, a_{i+1}\}$. Let $r(\sigma)$ [$b(\sigma)$] be the least number c such that c is σ-acceptable and $\{a_k, c\}$ is red [blue], if such exists. Then $r(\sigma)$, $b(\sigma)$ have recursive graphs (although not necessarily recursive domains). Define $\mathscr{S}$ to be the class of all strictly increasing functions h such that $h(0) = 0$ and

$$(\forall k)[h(k+1) = r(\langle h(0), \cdots, h(k)\rangle) \text{ or } h(k+1) = b(\langle h(0), \cdots, h(k)\rangle)].$$

Then the matrix of the above is a recursive predicate of h and k, so $\mathscr{S}$ is a Π_1^0 class of functions.

Assume $h \in \mathscr{S}$. Label $h(k)$ as *red* if $\{h(k), h(k+1)\}$ is a red pair, and otherwise *blue*. Then if $i < j$, $h(i)$ has the same color as $\{h(i), h(j)\}$. As in Theorem 4.1, either $R \in H(P)$ or $B \in H(P)$, where R is the set of red $h(i)$'s and B is the set of blue $h(i)$'s. Both R and B are recursive in h, so $H(P)$ has a member recursive in h.

The proof of Theorem 4.1 shows that $\mathscr{S}$ is nonempty. It remains to define a O'-recursive function w which majorizes every member of $\mathscr{S}$. We define w inductively. Let $w(0) = 0$. Assuming that $w(0), \cdots, w(k)$ are defined, let $w(k+1)$ be the largest of all the (finitely many) numbers of the form $r(\sigma)$ or $b(\sigma)$ where σ has the form $\langle a_0, \cdots, a_k\rangle$ and $a_i \le w(i)$ for all $i \le k$. Clearly w has the desired properties, so the proof is complete.

If the majorizing function w of Proposition 4.6 were recursive instead of only O'-recursive, then $\mathscr{S}$ would be a *recursively bounded* Π_1^0 *class*, and the results of [6] would apply to $\mathscr{S}$. However, because w is actually O'-recursive, the results of [6] must be relativized to O' to be applicable to $\mathscr{S}$. For instance, Theorem 2.1 of [6] states that every nonempty recursively bounded Π_1^0 class of functions has a member whose degree $\mathbf{a}$ satisfies $\mathbf{a}' = \mathbf{O}'$. Relativizing this and applying Proposition 4.6, we obtain

COROLLARY 4.7. *If P is a recursive basic partition, then $H(P)$ contains a set A such that $A' \le_T O''$.*

We do not know whether it is possible to extend Corollary 4.7 so that $A'' \le_T O''$ in its conclusion, but we conjecture it is not.

Similarly, Theorem 2.4 of [6] yields

COROLLARY 4.8. *If P is a recursive basic partition, then $H(P)$ contains a set A such that every function recursive in A is majorized by a O'-recursive function.*

Also Corollary 2.9 of [6] yields

COROLLARY 4.9. *If P is a recursive basic partition, then $H(P)$ has members A_1, A_2 such that every set recursive in both A_1 and A_2 is recursive in O'.*

We do not know whether it is possible to strengthen Corollary 4.9 by replacing "recursive in O'" by "recursive." In the other direction it is conceivable that there is a recursive basic P such that $H(P)$ contains only sets of degree $\ge \mathbf{O}'$.

We mention one final application of Proposition 4.6. The proof of that result shows that if P is a recursive basic partition, the corresponding class $\mathscr{S}$ may be represented as the set of infinite branches of a recursive tree T in which the branching is at most binary. If P is chosen so that $H(P)$ has no member recursive in O'

(Theorem 3.1), then T easily yields a finite-one retracing function which retraces no infinite set recursive in O'. Such a retracing function was shown to exist by Yates [13, Theorem 6]; another proof is given in [5, Theorem 4.22].

§5. We now study partitions of $[N]^n$ using the methods developed for the case $n = 2$. We take the liberty of assuming that all of our previous results hold in relativized form and even assume that induction assumptions hold in relativized form.

First we point out that the simple idea of Proposition 2.1 suffices to yield the desired extension of Theorem 3.1.

THEOREM 5.1. *If $n \geq 2$, there exists a recursive partition P of $[N]^n$ into two classes such that $H(P)$ contains no set recursive in O^{n-1} and hence no Σ_n^0 set.*

PROOF. We first need a lemma.

LEMMA 5.2. *If P_n is a O'-recursive partition of $[N]^n$ into two classes then there exists a recursive partition P_{n+1} of $[N]^{n+1}$ into two classes such that $H(P_{n+1}) \subseteq H(P_n)$.*

PROOF. For $n = 1$, this is a restatement of Proposition 2.1. The proof for arbitrary n is the same as for Proposition 2.1, except that the role of least element of a pair is taken over by the set of the least n elements of an $(n + 1)$-tuple.

Theorem 5.1 is now proved by induction on n. For $n = 2$, it is Theorem 3.1. Now assume it to be true for n. Relativizing this assumption to O', we see that there is a O'-recursive partition P_n of $[N]^n$ into two classes such that $H(P_n)$ has no set recursive in $O^{(n)}$. Then by Lemma 5.2 there is a recursive partition P_{n+1} of $[N]^{n+1}$ into two classes such that $H(P_{n+1})$ has no member recursive in $O^{(n)}$.

We now turn to the problem of showing that if P is a recursive partition of $[N]^n$ into finitely many classes, then $H(P)$ contains a Π_n^0 set. By "Ramsey's theorem for n" we mean the assertion that $H(P)$ is nonempty for every partition P of $[N]^n$ into finitely many classes. In Ramsey's original proof, Ramsey's theorem for $n + 1$ is proved essentially by iterating Ramsey's theorem for n infinitely often. It does not seem possible to obtain any sort of reasonable effectivization by this method for $n \geq 2$ because each application of Ramsey's theorem for n introduces new quantifiers. However, if $n > 1$, there is also a proof of Ramsey's theorem for $n + 1$ in which Ramsey's theorem for n is used only once. (The author is grateful to J. D. Halpern for pointing this out to him.) We now give this proof.

THEOREM 5.3 (RAMSEY). *If P is a partition of $[N]^n$ into finitely many classes, $H(P)$ is nonempty.*

PROOF. We employ induction on n. If $n = 1$, the assertion is obvious. We assume it for n and prove it for $n + 1$. If $D \in [N]^{n+1}$, let D^* be the set of the least n elements of D. Assume that the given partition P of $[N]^{n+1}$ into p classes corresponds to a function f: $[N]^{n+1} \to \{i: i < p\}$. (Thus f assigns to each $D \in [N]^{n+1}$ the index of its class.) We shall construct an infinite set A and a function g: $[A]^n \to \{i: i < p\}$ such that $f(D) = g(D^*)$ for all $D \in [A]^n$.

Let a_i be the ith member of A in natural order. Assume inductively that a_i has already been defined for all $i < k$ and $g(D)$ has been defined for all $D \in [\hat{A}_k]^n$, where $\hat{A}_k = \{a_i: i < k\}$. A number c is called *k-acceptable* if $c > \max \hat{A}_k$ and for all $D \in [\hat{A}_k]^n, f(D \cup \{c\}) = g(D)$. Assume inductively that there are infinitely many k-acceptable numbers. Define a_k to be the least k-acceptable number. If $D \in [\hat{A}_k]^{n-1}$

and $i < p$, let $S_i(D)$ be the set of k-acceptable numbers c such that $f(D \cup \{a_k, c\}) = i$. We now extend g to $[\hat{A}_k \cup \{a_k\}]^n$ so as to preserve our inductive assumptions. If $k + 1 < n$, then the extension is vacuous so assume $k + 1 \geq n$. Let the $(n - 1)$-element subsets of $\hat{A}_k$ be $D_1, \cdots, D_t$, where $t = \binom{k}{n-1}$. Define $g(D_1 \cup \{a_k\}) = i_1$, where i_1 is chosen so that $S_{i_1}(D_1)$ is infinite. (Such an i_1 exists because there are infinitely many k-acceptable numbers.) Then define $g(D_2 \cup \{a_k\}) = i_2$ where i_2 is chosen so that $S_{i_1}(D_1) \cap S_{i_2}(D_2)$ is infinite, and continue in this manner for t steps. All elements of $S_{i_1}(D_1) \cap \cdots \cap S_{i_t}(D_t)$ are $(k + 1)$-acceptable, so the induction assumption is preserved.

Let P' be the partition of $[A]^n$ defined by g. Since Ramsey's theorem for n clearly applies to partitions of $[A]^n$ just as well to partitions of $[N]^n$, we have $\varnothing \neq H(P') \subseteq H(P)$.

If P is recursive, then a direct analysis of the above argument shows that we may arrange $A \leq_T O''$. From this it follows at once by induction on n that if P_n is a recursive partition of $[N]^n$, then $H(P_n)$ contains a set recursive in $O^{(2n-2)}$. This result was proved years ago by Manaster. The following lemma, which is essentially an extension of Proposition 4.6, will allow this result to be considerably sharpened. If P is a partition of $[N]^{n+1}$, let $H^*(P)$ be the class of those infinite sets A such that for all $D \in [A]^{n+1}$, the partition class of D in P is determined by its least n elements, i.e., by D^*.

LEMMA 5.4. *If P is a recursive partition of $[N]^n$ into finitely many classes, then $H^*(P)$ contains a set A such that $A' \leq_T O''$.*

PROOF. It suffices to construct a nonempty Π^0_1 class of functions $\mathscr{S}$ which contains only increasing functions which enumerate members of $H^*(P)$ and such that there is a O'-recursive function which majorizes every member of $\mathscr{S}$. (The lemma then follows in the same way as Corollary 4.7.) Let the partition P be defined by a function f: $[N]^n \to \{i: i < p\}$. Define $\mathscr{S}$ to be the class of strictly increasing functions h such that $h(0) = 0$ and for all k there exists a (finite) function g_k: $[\{h(0), \cdots, h(k)\}]^n \to \{i: i < p\}$ such that $h(k + 1)$ is the least number $c > h(k)$ such that for all $D \in [\{h(0), \cdots, h(k), c\}]^{n+1}, f(D) = g_k(D^*)$. Then $\mathscr{S}$ is Π^0_1 and $\mathscr{S}$ is nonempty by the proof of Theorem 5.3. If $h \in \mathscr{S}$ then $g_k \subseteq g_{k+1}$ for all k and so $f(D) = g(D^*)$ for all $D \in [\text{range } h]^{n+1}$, where $g = \bigcup_k g_k$. Hence if $h \in \mathscr{S}$, then $(\text{range } h) \in H^*(P)$. Finally, there is a O'-recursive function which majorizes every member of $\mathscr{S}$ because, as in the proof of Proposition 4.6, once $h(0), \cdots, h(k)$ are determined there are only finitely many possibilities for $h(k + 1)$ over all $h \in \mathscr{S}$.

Our main result now follows easily.

THEOREM 5.5. *If P is a recursive partition of $[N]^n$ into finitely many classes, then $H(P)$ contains a Π^0_n set.*

PROOF. We prove this by induction on n. For $n = 1$ it is obvious and for $n = 2$ it is Theorem 4.2. Thus we may assume it is true for n $(n \geq 2)$ and prove it for $n + 1$. Let P be a recursive partition of $[N]^{n+1}$. Let A be a member of $H^*(P)$ such that $A' \leq_T O''$. Then P induces a natural partition P' of $[A]^n$ which is recursive in A. (The P'-class of $D \in [A]^n$ is determined by the P-class of $D \cup \{c\}$ where $c \in A$ and $c > \max D$. By assumption this is independent of the choice of c.) By induction assumption (relativized to A), $H(P')$ has a member $B \subseteq A$ which is Π^0_n in A. Then

$B \in H(P)$ so it remains to show that B is Π^0_{n+1}. Since B is Π^0_n in A, there is an A-recursive $(n + 1)$-place predicate R such that, for all x,

$$x \in B \leftrightarrow (\forall x_1) \cdots (Q_n x_n) R(x, x_1, \cdots, x_n).$$

Then $(Q_n x_n)R$ is recursive in A' and hence in O''. Applying Post's hierarchy theorem, one may replace $(Q_n x_n)R$ by a Σ^0_3 or Π^0_3 predicate, according as Q_{n-1} is $\exists$ or $\forall$. The resulting predicate is the required Π^0_{n+1} definition of B. (This makes sense because $n \geq 2$.)

We may also extend Corollary 4.7 in a similar way.

THEOREM 5.6. *If P is a recursive partition of $[N]^n$ into finitely many classes, then $H(P)$ contains a set B such that $B' \leq_T O^{(n)}$.*

PROOF. We use induction on n. For $n = 1$ it is obvious. We assume it for n and prove it for $n + 1$. Let P be a recursive partition of $[N]^{n+1}$, and let A be a member of $H^*(P)$ with $A' \leq_T O''$. By the induction assumption applied to the induced partition P' (defined as in Theorem 5.5), $H(P)$ has a member B such that $B' \leq_T A^{(n)}$. But $A^{(n)} \leq_T O^{(n+1)}$ so the induction is complete.

We have mentioned that we do not know whether there exists a recursive partition P of $[N]^2$ such that every member of $H(P)$ has degree at least $\mathbf{O}'$. On the other hand, we now construct a recursive partition P of $[N]^3$ (rather than $[N]^2$) such that every member of $H(P)$ has degree strictly above $\mathbf{O}'$. The proof will be facilitated and the statement of the result strengthened by use of the following terminology:

DEFINITION. If P is a partition of $[N]^n$ into two classes C_1 and C_2, then for $i \in \{1, 2\}$, $H_i(P)$ is defined to be the class of all infinite sets A such that $[A]^n \subseteq C_i$. (Thus $H(P) = H_1(P) \cup H_2(P)$.) The partition P is called *unbalanced* if either $H_1(P)$ or $H_2(P)$ is empty.

THEOREM 5.7. *If $n \geq 2$, there is a recursive unbalanced partition P of $[N]^{n+1}$ such that every element of $H(P)$ has degree strictly above $O^{(n-1)}$.*

PROOF. The result will be a consequence of the following three lemmas.

LEMMA 5.8. *For any recursive partition P of $[N]^n$ (into finitely many classes) there is a recursive unbalanced partition $P^\#$ of $[N]^{n+1}$ (into two classes) such that $H(P^\#) = H(P)$.*

PROOF. Let $C^\#_1$ be the class of all sets D in $[N]^{n+1}$ such that all n-element subsets of D belong to the *same* partition class of the given partition P of $[N]^n$. Let $P^\#$ be the partition of $[N]^{n+1}$ determined by the classes $C^\#_1$, $C^\#_2$, where $C^\#_2 = [N]^{n+1} - C^\#_1$. Obviously $P^\#$ is recursive if P is and $H(P) \subseteq H_1(P^\#)$. From the latter it follows that $H_2(P^\#)$ is empty. (For if A is any infinite set, A has an infinite subset $B \in H(P)$ by Ramsey's theorem for n, applied inside A. Therefore $B \in H_1(P^\#)$, so $A \notin H_2(P^\#)$ as required.) It remains to show that $H_1(P^\#) \subseteq H(P)$. Assume $A \in H_1(P^\#)$ and D, D^* are distinct n-element subsets of A. In order to conclude that $A \in H(P)$, we must show that D, D^* belong to the same class of P. Clearly there is a sequence, $D_1, \cdots, D_k$, of n-element subsets of A such that $D_1 = D$, $D_k = D^*$ and $|D_i \cup D_{i+1}| = n + 1$ for $1 \leq i < k$. (Just let $D_{i+1} = D_i \cup \{a\} - \{b\}$, where a, b are chosen from $D^* - D_i$, $D_i - D^*$ respectively.) Since $D_i \cup D_{i+1} \in [A]^{n+1} \subseteq H_1(P^\#)$ we see that D_i, D_{i+1} must belong to the same partition class of P. Thus all D_i's have the same class and so D, D^* have the same class.

REMARK. If $n \geq 2$, Lemma 5.8 and Theorem 5.1 already yield the existence of a recursive unbalanced partition $P^{\#}$ of $[N]^{n+1}$ such that no element of $H(P^{\#})$ is recursive in $O^{(n-1)}$. The next lemma will establish another special case of our theorem.

LEMMA 5.9. *If $n \geq 1$ there is a recursive unbalanced partition P^* of $[N]^{n+1}$ such that $H(P^*)$ contains only sets in which $O^{(n-1)}$ is recursive.*

PROOF. Let n be given and let f be an increasing function of degree $\mathbf{O}^{(n-1)}$ such that f is recursive in every function g which majorizes f. (Such a function f exists by [5, Theorem 4.13]. In fact if $n \geq 2$ it is easy to see inductively that if f is any increasing function which is recursive in $O^{(n-1)}$ and dominates all functions recursive in $O^{(n-2)}$, then f has the desired property.) Let $C_1 \subseteq [N]^2$ be the set of all pairs $\{n, s\}$ such $s > n$ and $s > f(n)$, and let $C_2 = [N]^2 - C_1$. If P is the partition of $[N]^2$ determined by C_1, C_2, clearly P is recursive in $O^{(n-1)}$, and P is unbalanced because $H_2(P) = \varnothing$. Also if $A \in H_1(P)$ and $a_0, a_1, \cdots$ are the elements of A in their natural order, then $a_{i+1} > f(a_i) \geq f(i)$ for all i. It follows that f (and hence $O^{(n-1)}$) is recursive in every member of $H(P)$. Since P is a $O^{(n-1)}$-recursive partition of $[N]^2$ into two classes, it follows from Lemma 5.2 (relativized to $O^{(n-2)}$) that there is a $O^{(n-2)}$-recursive partition P' of $[N]^3$ into two classes such that $H(P') \subseteq H(P)$ (and in fact $H_i(P') \subseteq H_i(P)$ for $i = 1, 2$). Iterating this observation $n - 1$ times, reducing by a jump each time, we finally arrive at a recursive partition P^* of $[N]^{n+1}$ into two classes such that $H_i(P^*) \subseteq H_i(P)$ for $i = 1, 2$. This P^* is the desired recursive unbalanced partition.

By an argument similar to the proof of Lemma 5.9 (but using the full strength of [5, Theorem 4.13]) one may show that there is an unbalanced partition P of $[N]^2$, recursive in Kleene's O, such that $H(P)$ contains only sets in which all hyperarithmetic sets are recursive.

The last lemma will conclude the argument by showing that there is a single partition P which satisfies both the conditions of the Remark after Lemma 5.8 and those of Lemma 5.9.

LEMMA 5.10. *If $P^{\#}$, P^* are each unbalanced recursive partitions of $[N]^{n+1}$, there is a recursive unbalanced partition P of $[N]^{n+1}$ such that $H(P) = H(P^{\#}) \cap H(P^*)$.*

PROOF. Suppose $P^{\#}$ is given by classes $C_1^{\#}$, $C_2^{\#}$ and P^* by classes C_1^*, C_2^* and that $H_2(P^{\#}) = H_2(P^*) = \varnothing$. Let P be the partition determined by $C_1^{\#} \cap C_1^*$, $C_2^{\#} \cup C_2^*$. Obviously $H_1(P) = H_1(P^{\#}) \cap H_1(P^*)$, so it suffices to show that $H_2(P) = \varnothing$. Suppose, for a contradiction, that $H_2(P) \neq \varnothing$ and choose $A \in H_2(P)$. Let P^0 be the partition of $[A]^n$ defined by the classes $[A]^n \cap C_2^{\#}$, $[A]^n - C_2^{\#}$. Then (since $[A]^n - C_2^{\#} \subseteq [A]^n \cap C_2^*$), $H(P^0) \subseteq H_2(P^{\#}) \cup H_2(P^*)$ so $H(P^0) = \varnothing$, contradicting Ramsey's theorem for n, applied inside A. This completes the proof of Lemma 5.10 and thus of Theorem 5.7.

Lemma 5.8 implies the truth of the "unbalanced analogue" of any *existence* theorem (in terms of $H(P)$) for recursive partitions P of $[N]^n$. (The unbalanced analogue is obtained by replacing "partition of $[N]^n$ into finitely many classes" by "unbalanced partition of $[N]^{n+1}$" throughout the statement of the result.) We do not know of a result comparable to Lemma 5.8 for showing that *universal* results about $H(P)$ imply their unbalanced analogues. (In fact, one can show that for each n there is a recursive unbalanced partition $P^{\#}$ of $[N]^{n+1}$ such that no partition P of

$[N]^n$ into finitely many classes (recursive or otherwise) satisfies $H(P\#) = H(P)$.) On the other hand, the unbalanced analogues of many positive results do hold. For instance, it is obvious that $H(P)$ contains a recursive set for any recursive partition P of $[N]^1$. One can prove the unbalanced analogue of this result by analyzing the proof of Ramsey's theorem for $n = 2$ (following either [8] or Theorem 4.1). It follows that Theorem 5.7 fails for $n = 1$. Also, Alfred Manaster has pointed out that the unbalanced analogue of Theorem 5.6 holds. This shows that Theorem 5.7 is near optimal. Below we prove a slightly stronger result.

THEOREM 5.11. *If $n \geq 1$ and P is a recursive partition of $[N]^{n+1}$ into two classes, then either $H_i(P)$ has a member A_i such that $A_i' \leq_T O^{(n+1)}$ for $i = 1, 2$ or $H(P)$ has a member A such that $A' \leq_T O^{(n)}$ (and A is recursive if $n = 1$).*

SKETCH OF PROOF. For $n = 1$, the result follows from the observation that if the set R defined in the proof of Proposition 4.6 is finite, then the set B defined there is recursive. Also if B is finite, then R is recursive. The general proof is by induction on n using Lemma 5.4.

We do not know whether the unbalanced analogue of our main result (Theorem 5.5) is true. On the other hand, it is possible to extend Theorem 5.5 by showing that if P is any recursive partition of $[N]^{n+1}$ into two classes, then either $H(P)$ has a member recursive in $O^{(n)}$ or $H_i(P)$ has a Π^0_{n+1} member for $i = 1, 2$. For $n = 1$ this is proved in the Remark just after Lemma 4.5 and the general result is proved by induction on n using Lemma 5.4.

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