Math/Combinatorics (2025), **xx**:xx 1–5

CAMBRIDGE
UNIVERSITY PRESS

Lecture Note

Ramsey Theory

Tianrui Sheng[1]

[1]AMSC Ph.D., University of Maryland, College Park, 20740, Maryland, United States of America.

Keywords: Ramsey theory

Abstract
This is the lecture note of 2025 Spring CMSC 752 Ramsey Theory by Professor William Gasarch.

Contents

1. Week 1: 01/28/2025, 01/30/2025

Abstract: In this week, we introduce existence in Ramsey theorem. The proofs are mainly constructions.

1.1. Introduction of Ramsey Theorem

The classical result of Ramsey theorem is:

Theorem 1.1. *$\forall$ 2-coloring of the edges of K_6 there is a mono triangle.*

In the future, we use the following language to study Ramsey theory.

Definition 1.1. Let $a, n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$.

- $[n] = \{1, \ldots, n\}$.
- $\binom{A}{a}$ is the set of all a-sized subsets of A.

Corollary 1.1. *For all functions*

$$COL : \binom{[6]}{2} \to [2]$$

there exists $A \subseteq [6]$, $|A| = 3$, such that

$$COL : \binom{A}{2} \text{ is constant.}$$

Theorem 1.2. *For all functions*

$$COL : \binom{[18]}{2} \to [2]$$

there exists $A \subseteq [18]$, $|A| = 4$, *such that*

$$COL : \binom{A}{2} \text{ is constant.}$$

Theorem 1.3 (Ramsey's Theorem for 2-col Graphs)**.** $\forall k\ \exists n = R(k)$ *such that the following happens: For all*

$$COL : \binom{[n]}{2} \to [2]$$

there exists $A \subseteq [n]$, $|A| = k$, *such that*

$$COL : \binom{A}{2} \text{ is constant.}$$

Another statement is the following:

Definition 1.2. Let $k \in \mathbb{N}$.

- An Arithmetic Sequence of Length k, denoted k-AP, is a set of k numbers that are equally spaced.
- If there is a coloring of $\mathbb{N}$ then a mono k-AP is a k-AP where all of the numbers are the same color.

Theorem 1.4. *For all 2-colorings of* $[9]$ *there exists a mono* 3*-AP.*

Theorem 1.5. *For all 2-colorings of* $[35]$ *there exists a mono* 4*-AP.*

Theorem 1.6 (Van der Waerden's Theorem for 2-coloring of $\mathbb{N}$)**.** *For all* k *there exists* $W = W(k)$ *such that the following happens: For all*

$$COL : [W] \to [2]$$

there exists a mono k*-AP.*

1.2. Infinite Ramsey Theorem For Graphs

Definition 1.3 (Homogeneous)**.** Let COL : $\binom{A}{2} \to [2]$. A set $H \subseteq A$ is homogeneous if COL restricted to $\binom{H}{2}$ is constant.

Theorem 1.7 (Infinite Ramsey Theorem)**.** *For all* $COL : \binom{\mathbb{N}}{2} \to [2]$, *there exists an infinite homogeneous set.*

Proof.

$$H_1 = \mathbb{N}$$

$$x_1 = 1$$

$$c_1 = \begin{cases} R & \text{if } |\{y \in H_1 : \text{COL}(x_1, y) = R\}| = \infty, \\ B & \text{otherwise.} \end{cases}$$

$$H_2 = \{x_1\} \cup \{y \in H_1 : \text{COL}(x_1, y) = c_1\}$$

$$x_2 = \text{the least element of } H_2 - \{x_1\}.$$

$$c_2 = \begin{cases} R & \text{if } |\{y \in H_2 : \text{COL}(x_2, y) = R\}| = \infty, \\ B & \text{otherwise.} \end{cases}$$

Assume H_s, x_s, c_s are defined.

$$H_{s+1} = \{x_1, \ldots, x_s\} \cup \{y \in H_s : \mathrm{COL}(x_s, y) = c_s\}$$

$$x_{s+1} = \text{the least element of } H_{s+1} - \{x_1, \ldots, x_s\}.$$

$$c_{s+1} = \begin{cases} R & \text{if } |\{y \in H_{s+1} : \mathrm{COL}(x_{s+1}, y) = R\}| = \infty, \\ B & \text{otherwise.} \end{cases}$$

$$X = \{x_1, x_2, \ldots\}$$

Color x_s by c_s. W.l.o.g. assume $H := \{y \in X : \mathrm{COL(y)} = R\}$ s.t. $|H| = \infty$. Then H is an infinite homogeneous set. □

Remark 1.1. This is not true when replace $\mathbb{N}$ by $\mathbb{Z}$ and require $H \equiv \mathbb{Z}$. The relative result is Theorem 1.9.

Counterexample. negative-negative = blue; negative-positive=blue; positive-positive=red. □

Corollary 1.2. $\forall c \in \mathbb{N}$, *for all COL* : $\binom{\mathbb{N}}{2} \to [c]$, *there exists an infinite homogeneous set.*

Here we introduce the concept of hypergraphs. The above results are about 2-hypergraphs.

Definition 1.4 (Hypergraph)**.** An a-hypergraph is (V, E), where $E \subset \binom{V}{a}$.

Proof of Theorem 1.3 by Theorem 1.7. Assume the statement is false. Then

$$(\exists k)(\forall n)(\exists \mathrm{COL} : \binom{[n]}{2} \to [2] \text{ with no homogeneous set of size } k).$$

Let COL_i, $i = 0, 1, \ldots$ be the coloring s.t.

$$\mathrm{COL}_i : \binom{[k+i]}{2} \to [2] \text{ with no homogeneous set of size } k.$$

Let $T = \{\mathrm{COL}_i\}_{i \in \mathbb{N}}$.

Let $e_1, e_2, \ldots$ be a list of every element of $\binom{\mathbb{N}}{2}$.

$$\mathrm{COL}(e_1) = R \text{ if } |\{y : \mathrm{COL}_y(e_1) = R| = \infty, B \text{ otherwise.}$$

Delete $\{\mathrm{COL} \in T : \mathrm{COL}(e_1) \neq R\}$ from T. Still denote as T.

By induction, do the coloring for $\{e_i\}$. Then we get a coloring COL of $\binom{\mathbb{N}}{2}$. By Theorem 1.7, there is an infinite homogeneous set $H = \{x_1 < x_2 < \ldots\}$. Then $\mathrm{COL}|_{\binom{\{x_1,\ldots,x_k\}}{2}} = \mathrm{COL}_j$ for some j and $\{x_1, \ldots, x_k\}$ is a homogeneous set of COL_j with size k. □

1.3. Theory on $\mathbb{Z}$

For relaxing the requirement of the theory, we define

Definition 1.5 (m-homogeneous)**.** Let COL : $\binom{A}{2} \to [c]$. Let $m \in \mathbb{N}$. Let $H \subseteq A$. H is m-homogeneous if COL restricted to $\binom{H}{2}$ takes on $\leq m$ values.

Here c is some large fixed number.

Definition 1.6. If L_1 and L_2 are both linearly ordered sets then $L_1 \equiv L_2$ means that $\exists$ an order preserving bijection between L_1 and L_2.

The questions we care about is to find m, s.t.

- $$\forall \text{COL} : \binom{\mathbb{Z}}{2} \to [c] \quad \exists \text{ infinite set } H \subseteq \mathbb{Z},\ H \equiv \mathbb{Z},\ H \text{ is } m\text{-homogeneous}.$$

- $$\exists \text{COL} : \binom{\mathbb{Z}}{2} \to [c] \quad \text{no } (c-1)\text{-homogeneous set } H, \text{ s.t. } H \equiv \mathbb{Z}.$$

The answer for the second problem is

Theorem 1.8. $\exists COL : \binom{\mathbb{Z}}{2} \to [4]$ *with no infinite* 3*-homogeneous* $H \equiv \mathbb{Z}$.

Proof. We will take $\mathbb{Z}$ to be the following set:

$$\{\ldots, -6, -4, -2\} \cup \{1, 3, 5, \ldots\}.$$

We define $\text{COL}(x, y) : \binom{\mathbb{Z}}{2}$. We assume $|x| < |y|$.

$$\text{COL}(x, y) = \begin{cases} 1 & \text{if } x, y \geq 1 \\ 2 & \text{if } x, y \leq -1 \\ 3 & \text{if } x \leq -1,\ y \geq 1 \\ 4 & \text{if } y \leq -1,\ x \geq 1 \end{cases}$$

There is no 3-homogeneous $H \equiv \mathbb{Z}$. Suppose that we have such H, then there are more than 2 positive nodes and more than 2 negative nodes, which means color 1, 2 are necessary. Since both sides are infinite, there exists $x, y \in H, x < 0, y > 0$, s.t. $|x| < |y|$, i.e. color 3 is necessary. Similar to color 4. □

The answer for the first problem is

Theorem 1.9. $\forall COL : \binom{\mathbb{Z}}{2} \to [c], \quad \exists$ *a* 4*-homogeneous* $H \equiv \mathbb{Z}$.

Before proving Theorem 1.9, we need a lemma on bipartite graph.

Definition 1.7 (Bipartite Graph)**.** A Bipartite Graph is (L, R, E) where the vertices are $L \cup R$ and $E \subseteq L \times R$ (so no edges within L or within R). L stands for Left, R stands for Right.

Definition 1.8. Let $n, m \in \mathbb{N}$. The complete (n, m)-bipartite graph, denoted $K_{n,m}$, is the bipartite graph $([n], [m], [n] \times [m])$.

Remark 1.2. In particular, $K_{\mathbb{N},\mathbb{N}}$ is the bipartite graph $(\mathbb{N}, \mathbb{N}, \mathbb{N} \times \mathbb{N})$.
A coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ is a coloring of $\mathbb{N} \times \mathbb{N}$.

Then we introduce the theory on $K_{\mathbb{N},\mathbb{N}}$.

Definition 1.9. Let $\text{COL} : \mathbb{N} \times \mathbb{N} \to [c]$. Let $m \in \mathbb{N}$. $H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$ is an infinite m-homogeneous if COL restricted to $H_1 \times H_2$ takes on $\leq m$ values, and H_1, H_2 both infinite.

Theorem 1.10. $\exists COL : \mathbb{N} \times \mathbb{N} \to [2]$ *with no infinite* 1*-homogeneous* $H_1 \times H_2$.

Proof. We use $\text{EVEN}^{+} = \{2, 4, 6, ...\} \times \text{ODD}^{+}$ instead of $\mathbb{N} \times \mathbb{N}$. We define $\text{COL}(x, y) : \mathbb{N} \times \mathbb{N} \longrightarrow [2]$.

$$\text{COL}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 2 & \text{if } x > y \end{cases}$$

□

Theorem 1.11. *$\forall$ COL: $\mathbb{N} \times \mathbb{N} \to [c]$, $\exists H_1 \times H_2$, it is an infinite 2-homogeneous.*

Proof. Assume left side is labeled as $X = \{x_i\}_{i \in \mathbb{N}}$ and right side is labeled as $Y = \{y_i\}_{i \in \mathbb{N}}$.

Let $\tilde{x}_1 = x_1$. There is color c_1, s.t. $|\{y \in Y : \text{COL}(\tilde{x}_1, y) = c_1\}| = \infty$. Delete $y \in Y$ such that $\text{COL}(\tilde{x}_1, y) \neq c_1$, still denote as Y.

Let $\tilde{y}_1 =$ the least element of Y. There is color d_1, s.t. $|\{x \in X : \text{COL}(x, \tilde{y}_1) = d_1\}| = \infty$. Delete $x \in X$ such that $\text{COL}(x, \tilde{y}_1) \neq d_1$, still denote as X.

Use induction to continue the construction. Color $x_i \in X$ by c_i and $y_j \in Y$ by d_j. There is a color c, s.t. $|\{x \in X : \text{COL(x)} = c\}| = \infty$, let $H_1 = \{x \in X : \text{COL(x)} = c\}$. Similarly, define H_2. Then $H_1 \times H_2$ is an infinite 2-homogeneous set. □

Proof of Theorem 1.9. Use Theorem 1.7 on $\binom{\mathbb{N}}{2}$ to find color c_1.

Use Theorem 1.7 on $\binom{-\mathbb{N}}{2}$ to find color c_2.

Use Theorem 1.11 to connect $\mathbb{N}$ and $-\mathbb{N}$ with color c_3, c_4. □