Finite Ramsey Theorem For 3-Hypergraph: Better Bounds

Exposition by William Gasarch

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We will do much better.

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$|H_{s+1}| \geq \frac{1}{2^{s+(s-1)+\cdots+1}}|H_1| \sim \frac{|H_1|}{2^{s^2/2}} = \frac{n}{2^{s^2/2}}.$

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How Big Is H_s?

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We will later see how big we need n to be.

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Hence H is homog for COL.

How Big Does N Have to Be. Part II

We need

$$k \leq \log s/2$$

$$2k \leq \log s$$

$s \geq 2^{2k}$
$n \geq 2^{s^2/2} \geq 2^{2^{4k}}$
SO we can take $n = 2^{2^{4k}}$.