fRoM tHE S ENsHIpMEnT C H RIFT oF tHE ßI AR TI 2'T - LÜ TH E R - U II VE IS I T.ST H AL L E - \YI T TE ÜBE R(2r

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.tus the 1st MathemAic Instito*

Direc teren: Prof. Dt. Os Hai mticii Kom-zR Prof. Dr. Hznxsnz G nozzsc a

Regular graphs of given waist width with a minimum number of nodes

PAUL ERDÖS und HORST SACHS*)

I. An estimation for the minimum knoteric number of rcgular craphs, which do not contain a Jfreis of length f contain

Let *G"^* be a graph with. H nodes ; node piicts of G'*' are denoted by z" ... , y, ... , **edges** are denoted by (.z" z,). Loops and bicorners are not permitted. The Ya 1 e nz (or order) v (z) of a node z sion C-'" is the number of edges that incide with z. There 8ei 6'"' a €ira}3h with the node puiicts r" . , z" iveleher contains the edge (z" z,) but no edges ('r" z,); (G"' - (z" .r) - j- (z" z,)) is obtained from G"' by omitting the edge (z" z) and adding the edge (z" z). An edge is a simple closed edge. The e n t f er n u ri g (t'der der der Abstand) e (G"': *" *r,)* sion zt xind z in 'T"" is the length of the shortest edge course connecting z, and r, (the length of an edge course or circle is the .number of banks int dai1tenzuge or circles occurring edges). The length of a shortest in *J ' occurring circles are called the T ai 11en iv ei t e of G"". Let JF (z, :) be the setall nodes that can be reached from z by an edge of the T>iinge r (we should actually *K |O"', x, r)* >eliieiberi, t un Vlies a l"-r micht. dv w Jr this bymliol w-erden so that in this case no iIißverstand'lnis can arise). <S be rlie - number of elements of a chenge 6.

/ *k,* f) is the smallest number for which exists a regular graph U " of valence F with it= / (/r, /) such that each circle of U"' has at least length f (d, li, the waist width of G"' aei 1). E6 Is it clear that lf / (z*, 1)1 is unit only / 1, 3)= 2 -- l (dv is a regular graph of yalence *k* at least 1 1 note points). for *k*> :.*, f> 3 St old the determination of / (1, i) is a difficult problem, which is solved biaher riiir for special values i cm k and f . Recently isaGuS §4] received an upper estimate for / (£, *I j*, which he considers to be very poor. At his suggestion, I succeeded in obtaining quite good lower and upper estimates for / (I, i). The following applies

Satz 1: *Für jedes* $k \geqq 2,\ l \geqq 3$ *ist*

(1) $$1 + k \sum_{t=0}^{\left[\frac{l-3}{2}\right]} (k-1)^t \leqq f(k, l) \leqq 4 \sum_{t=1}^{l-2} (k-1)^t.$$

for F - 2 Eois-ie for - 3 (1) is certainly correct, so lets assume the following F> z* and *l*> 3 for 'Iss diirfcn.

At the same time, we want to use the lower -4estimation- in tl). \\"we will riur t (z) *k* (for all nodes vr'n G""). Let z be any finoten point of P""; each fi.note point of *K x.* ' —'— can be derived from

x durch genau einen Kantenzug der Länge $\leqq \left[\frac{l-1}{2}\right]$ erreicht werden, da sonst $G^{(n)}$ einen Kreis der Länge $< l$ enthielte. Da jeder Knotenpunkt von $G^{(n)}$ eine Valenz $\geqq k$ hat, zeigt ein leichter Induktionsschluß nach r, daß für alle r mit $1 \leqq r \leqq \left[\frac{l-1}{2}\right]$

(*) $$\overline{K(x, r)} \geqq 1 + k \sum_{t=0} (k-1)^t$$

p-ilt. The lower estimate of (1) follows immediately, u enn in {L*) r - ' —'— is set.

*) Part I is essentially due to the first-named author, Part II to the second-named author.

towards we want to **prove** the upper _4ba estimate. **V'e prove** the following more precise statement :

Es sei $k \geqq 2,\ l \geqq 4,\ m \geqq 2 \sum_{t=1}^{l-2} (k-1)^t$. *Dann gibt es einen regulären* $G^{(2m)}$ *der Valenz* k, *dessen Kreise alle*

f5r prove this SRtz for festet3 i by induction according to 1. For k -- :.° tat allee tririal. C'°" set afn'h a circle of length 2w (z* in I tat clear). Let us assume that our **theorem** for €-1 is already proven**, we** then want to prove it fiiz' é. H"" @, /) in the following always denotes a regular graph of valence k and waist width i with n nodes. According to our induction assumption, a graph G °"*(1-- 1, i) exists. IT'°"' is now a graph that has the following three properties:

(I) $k - 1 \leqq v(x) \leqq k$ *für alle 2m Knotenpunkte.*

(II) *Alle Kreise haben eine Länge* $\geqq l$.

(III) Carter ellen Drep6eu, *the* (I) and (II) *I'cJriedigen,* set Ute *number* of Janice ton 6'° - "i'iziniof.

Since G'° k - 1, 1) (I) imd (II) satisfies, it is clear that G °*' e ttei4. We want to show that for G" -' u (z) k We aHo z did that alao G'- "' from the Valens£ did, and herewith everything proven.

First we want to' show that G'-'-' can contain at most **one** node of valence< k. However, since this proof **is** not **quite** simple**,** we will first show that the **regularity** of R'- " follows from this. **l'egen** (I), each point of expression should have the valence k - *I.* But this is not possible, since the number of I(notan points must be even, and for even I there would be exactly one node of uneven valence, **for** even k exactly :.* in - J aoJche nodes. therefore v (z) *k I* applies *to* all z, inid our **Sstn** is proven.

We only need to show that the. Assuming that two nodes z, and z of G *° with i (*,)< T , r (z } k {asor (z_l) - r (z) =£- 1) exist, leads to a U'1derapruc1t. fire. i claim

Zeznm I. *AU* node with r (z) £ of G™ "' are furnished iii N (z" / - 2) *IC* (y, F - -').

Ea suffices... to show this for A (.z" 1 - 2)... If ant against our assumption z 5 *K.* (s" f - i.*) and e (.z)< 1-, ao obviously satisfies Cr° "'+ l- . -) satisfies conditions (I) and (II) ((I) because of c (z,)< £, v (*)< 1 and {II) because of z q A (:r" f -:.*)). However, this is contradicted by the maximum value property (III) for d'° "', and thus Eei mu I is proven.

Lemma 2. Es sei x *ein Knotenpunkt von* $G^{(2m)}$ *mit* $v(x) < k$. *Dann gilt*

(3) $$\overline{\overline{K(x, r)}} \leqq \sum_{t=0}^{r} (k-1)^t .$$

Because of t (z)< é follows (3) fiir r I. Piir r l follows (3) from t' (z',} k by a slight in'Jul:tionsoeJiluD to r.

From Lemmit 1 and a' now follows

(4) $$\overline{\overline{K(x_1, l-2) \cup K(x_2, l-2)}} \leqq m :$$

Obviously it follows from (3) and Leaima 1

$$\overline{\overline{K(x_1, l-2) \cup K(x_2, l-2)}} = \overline{\overline{K(x_1, l-2)}} + \overline{\overline{K(x_2, l-2)}} - \overline{\overline{K(x_1, l-2) \cap K(x_2, l-2)}}$$

= -' 2 '^ -1)' -2 m
f= 0

(wegen Lemma 1 und wegen $v(x_1) < k$, $v(x_2) < k$ ist $\overline{\overline{K(x_1, l-2) \cap K(x_2, l-2)}} \geqq 2$). Also ist (4) bewiesen.

There are now z . - fffp Öthe node points contained in R (z" i - z*) w K (zt. I - z°), and y" .... , y p_p aeien the remaining node points of 9'- -". Aue **(4) tolgt**

(5) $$2m - p \geqq p.$$

Now we want to show that mindert.ena zv-ei of y are -mrbt1i1den by an edge. First follows because of Lemma I $v(y_j) = k,\ 1 \leqq j \leqq 2m - p$.

If noo zis-ei of the i/ were connected by an edge, then u iirden 1 (z* ni - p) edges 'tie ¡y, with. the z, connect. However, because of (3) and e (y) K (1 N y z* m - p) u-ürda then follow v (z,) - k for 1 m i m p, and this contradicts r (z,) 1.

We can therefore assume without limitation of generality that the edge (y" 9) occurs in G ' " .

Betrachten wir jetzt den Graphen $(G^{(2m)} - (y_1, y_2) + (x_1, y_1) + (x_2, y_2)) = \bar{G}^{(2m)}$ (die edges (z_1, y,) and (z" y) For kommen offenbar nicht in $G^{(2m)}$ vor). Ich behaupte, daß $\bar{G}^{(2m)}$ (I) und (II) befriedigt. II l this is clear. If °*' would contain a circle of length t, then one (or both) of the edges (z" B_l), (zg, y) would have to occur in this circle. Ea applies, however, because of the definition of yj

(6) $$e(G^{(2m)}; x_1, y_1) \geqq l - 1$$
$$e(G^{(2m)}; x_2, y_2) \geqq l - 1$$
$$e(G^{(2m)} - (y_1, y_2); y_1, y_2) \geqq l - 1.$$

The last inequality i-on (ß) follows arts 'ler T at cache, tlat in G'° "' no Krois of length< I occurs . Aux ţö) easily follows that * "' (II) satisfies . Daa nber "-idem¡'richt (IìI¡, da °" mehr Kariten hat all G'°"'.

So Tann G'° " hüchøteiis a node 'ter 3'alenz *k* halien, nail dun it is alle proven.

demeriunge*'

I) **Flan** ltürtnnte the r'bere .ûbschát sung ín (1) still etv'as s-sharp, wean one considers that the from the nodes K (z¡, Î - 2) *K [r . I -*) has s-on Ü° " miiidebtens Æ (z" f -2) w *K* (z;,-) - I Kanteri, a bcr that would be ziomlieh insignificant.

à) .Search the lower paragraph in II) letf rich røittrls a met.hode liir 1gei'ades 1 2q versehärfeii given by F. KżRTESZi [2]: fis z and y be zivei nodes connected by an edge aa t'T ^ *[k, * q) :* then. each K node r of'* (č, * q). whose A lutøiid c row z or y p- I rat, offeziber writ genaii onei of the vertices z, y by eirien Kantenzug der Lãngc e veI bunden, und zs ar dUFOh genau einen solchen Kanteneug, denn iIi jedem anderen Fableëbe ergëbe ein Kreie der Lânge <. From this z'ir conclude that 6J"' (1. *q) contains a tree of the kind given in .Ebb. 1 contains . Therefore łolgt:

$$n \geqq 2 + 2(k-1) + 2(k-1)^2 + \ldots + 2(k-1)^{q-1} = 2\{(k-1)^q - 1\} / (k-2),$$

SO Cldf$

$$(7)\qquad f(k, 2q) \geqq 2\,\frac{(k-1)^q - 1}{k-2}$$

and this scliranke rat um *ik* - 1)' ' gi ôßer ads (iie in (I) indicated below

$$1 + k\sum_{t=0}(k-1)^t = \{k(k-1)^{q-1} - 2\} / (k-2).$$

x y

K=4 t-6

Abb. 1

3)'we do not want the *F'ällé* 2IIsãlnme1 1- we have been told. places where you can aim for *l* *k, I)* e x¡ilizit.

If multiple edges are used, it makes sense to use Glen FslI / 2 to considered.

Let's start with the t rivial cases.

a) *k* *z*>. Obviously i6t

$$(8)\qquad f(2, l) = l,$$

the G1'aphen G ' (2. I) mind *(6i* aøf Loinorphie) clearly b' true, nöœbcli ciriee of length.

b) I - 2. Apparently

(9) / fk, â) - 2.

The grophen f-° *(k,* 2) are clearly defined, they consist of two nodes and the two nodes connected to each other.

c) $l = 3$. Wie am Anfang der Arbeit bemerkt wurde, ist

$$(10)\qquad f(k, 3) = k + 1,$$

rlie graphs fi"' * (I. 3) are unidirectional t'estîmmt, närnlit Ïi which consist of *k* *I* nodes and $\binom{k+1}{2}$ diese nodal points paa re cite interconnected want en consist of complete greplicn.

'1) 1 4. Nacli a nríintlly žiitteilnng of I . Pósa (BudaҎegt) it

(11) / (2, 4)= - I-.

Ben c sharp: V'egen (ï) is / (I:, 4) 2, and it is also *I fk,* 4) 2, for there is a Ü'° ' (č, 4), namely Glen pairs Єłraplien with the 2 ś node piinkts

. , to, ; v , ' , . . , y

unrł the č° edges

$$(x_i, y_j)\quad (i = 1, 2, \ldots, k;\quad j = 1, 2, \ldots, k).$$

dli der mind die Grnplien U^{f}"' *i k,* 4) ein'Jeutig bcatiinmt, win mam sich auf Grund *wren* Bcmerkimg 2 lcieht clear.

e) f 3. From (1) follows directly

$f(k, 5) \geqq k^2 + 1$. A. J. HOFFMAN und R. R. SINGLETON [1] haben bewiesen:

$$(12) \quad \begin{cases} f(k, 5) = k^2 + 1 & \text{für } k = 2, 3, 7 \\ f(k, 5) > k^2 + 1 & \text{für } k \neq 2, 3, 7 \text{ und } 57; \end{cases}$$

In the case of b ö7, the question has not been clarified.

f) 1 G. F. Kl RT£szi [ü] was able to show :

Ist p eine beliebige Primzahl und r eine beliebige natürliche Zahl, so gilt

$$(13) \quad f(1 + p^r, 6) = 2(1 + p^r + p^{2r}).$$

g) 7-. i' ;i j is greater for $k > 2$ steta / (£, "i) than the lower Schranke (1).

h) é= 3. here is according to \\'. T. Tr ers [3]und lJ . F. 3Io GEH 3) / (3, 1) in the cases i - 2, :3, 4, 5, Q, 7, S ; z e give for the6e \l*erte ron f the size / (3, i)'ie the bounds that result from (1) or (7):

$$\begin{array}{rl} 2 \leqq f(3, 2) = & 2 \\ 4 \leqq f(3, 3) = & 4 \leqq 8 \\ 6 \leqq f(3, 4) & 6 \leqq 24 \\ 10 \leqq f(3, 5) & 10 \leqq 56 \\ 14 \leqq f(3, 6) & 14 \leqq 120 \\ 22 \leqq f(3, 7) & 24 \leqq 248 \\ f(3, 8) & 30 \leqq 504. \end{array}$$

For f> 8 iet / (3, f) steta larger than the lower S' hranke (1) bsw. ('i).

Eß zP t 6JGh that in all cases in which we know / (£, f) (with the exception of k= 3, f =- '?), / (k, Ü is always equal to the lower bound given by (1} or . t7). If this is the case and is even, eo each Grsph G^ (ä, i) mii tt *f [k,* i) obviously consists of the tree mentioned in Remark 2 (Fig. 1) and gen isaen edges n, which i-connect nodes. whose distance in the tree is gloieh i - 1 ; it follows that then 6? " J, f) is a paired Grnph {v-which can easily be checked directly in the known cases). The following question arises :

Gibt es eine Zahl $k \geqq 3$ und eine gerade Zahl $l \geqq 6$ so, daß einer der Graphen $G^{(n)}(k, l)$ mit minimaler Knotenzahl n kein Graph ist?

II. Some features of the mini inline graphs

For the following, only the existence of grapheii f,"" (L, i) for any natural numbers *(* , i 3 and suitable it; this can be taken from 'I'eil I of this paper or, independently of this, also from [41.

tYou want to have some of your own tler of those €'rapheii Ü^ (£, /), for "-elche u au kleinatmögliche \4 ert / (k, 1) ; these graphs are referred to as a.1s 41in i m algr a phen in the following. - Ea at consistently *k* *2,* f 3.

(A) *Der Abstand zweier beliebiger Knotenpunkte eines Minimalgraphen $G^{(n)}(k, l)$ ist $\leqq l$.*

Proof: Assume that in N'^' (£, 1) two vertices z, y whose distance> I did. z is connected to the vertices z¡, ... , z" y eei to the vertices y¡, ... , y¢ are each connected by an edge. Then we consider the graph G', isBF ffUi5 G " (£, t) if we delete the node points z and y as well as all the edges emanating from them and instead use the nodes t z" y,), . , , (zi, 9) instead. 6' proves to be a G "*° (A, 1) (possibly splitting into several separate parts), i.e. n is not the minimum number of nodes - in 1\'contradiction to the *requirement.

Folgerung aus (A):

$$(14) \quad f(k, l) \leq \frac{k(k-1)^l - 2}{k - 2} \qquad (k > 2).$$

This results from (A) by means of a simple inductive closure (already used in I), t)herefore (A) the 3linimal graph no longer has nIa

$$1 + k + k(k - 1) + \ldots + k(k - 1)^{l-1} = 1 + k\{(k - 1)^l - 1\} / (k - 2) = \{k(k - 1)^l - 2\} / (k - 2)$$

Node pointc. The upper 5ehrouke thus obtained for / (L, 1) is of a similar .4rt as the upper 5ehrouke given in (l) by Eouiig an-

$$4 \sum^{l-2} (k-1)^i \left\{ \frac{(k-1)^{l-1} - 1}{k - 2} \right\}$$

but less well than these.

(B) *Es sei $k = 2h$ gerade und x ein beliebiger Knotenpunkt eines Minimalgraphen $G^{(n)}(k, l)$. Dann gilt: Die Anzahl derjenigen Kanten von $G^{(n)}(k, l)$, welche je zwei Knotenpunkte verbinden, die beide von x den* *l*

Proof: Assume that (B) is false. Then ea *io G'" k,* f) gives a node z and J' edges u" y.. . . , " both of which have end}tjyoten points of have the ab8t&nd l. The (not necessarily different) end node points of these edges are ;y" . . , y" . Let the vertex z be connected to the vertices z" ... , z t are each connected by an edge. \\'We consider the graph 9", which results from *G!"! (k, l j* if w-we connect all the edges " . , x as well as the vertex. z and the edges extending from it and instead delete the edges (z" y¡), , (z ". yt) instead. U" **proves** to be a C*" [l] (£, f), aleo tot n **not** the minimum number of nodes - in \\'contradiction to the presupposition.

From (B} we conclude

(U) As *cei k -- '2 h* qe7nde und z siit *beliebter Knaten punkt einer Mini malqrapheti H!"! t,* 1). Dnxn Uf the *Anmhl a* deiyenipen *Knoten punkte non G!"! (k,* f). *which have of x* the *Abela M l,* {I— 1) ' [1].

Proof: If u= 0, then (CJ is certainly correct; we may therefore assume u W I . Let v be the number of vertices ron 9" *(k,* t) which at a distance f— 1 from z, and q the number of edges, srelehe connect node points ron distance J and /— 1 ron z with each other. Then, because of (B)

$$q \geqq uk \quad 2(h \quad 1) \quad (u \; -1) \cdot 2h + 2,$$

on the other hand

$$q \leqq v(k \quad 1) \quad v(2h \quad 1),$$

so daß

$$(u \quad 1) \cdot 2h + 2 \leqq v \cdot (2h \quad 1).$$

If one inserts an upper bound for v into this, one obtains an upper bound for o. According to the inductive closure, which has already been used several times, it is certain that

$$v \leqq k(k \quad 1)^{l-2} \quad 2h \cdot (2h \quad 1)^{l-2},$$

so daß sich der Reihe nach ergibt:

$$(u \; -1) \cdot 2h + 2 \leqq 2h \cdot (2h - 1)^{l-2}(2h - 1),$$
$$(u \; -1) \cdot h < h \cdot (2h \quad 1)^{l-1},$$
$$u \quad 1 < (k-1)^{l-1},$$
$$u \leqq (k \quad 1)^{l-1},$$

as was claimed.

VoTgecuog off (Ü):

The upper bound (l4) can be improved in the case *k* -- 2h *ih* 1), because due to (A) and (€) a minimal graph K'*' (3 ä, f) is not more than

$$1 + k + k(k \; -1) + \ldots + k(k \quad 1)^{l-2} + (k \quad 1)^{l-1}$$
$$= 1 + k\{(k-1)^{l-1} \quad 1\} / (k \quad 2) + (k \quad 1)^{l-1}$$
$$= 2\{(k \quad 1)^{l} \quad 1\} / (k \quad 2) = \{(2h \quad 1)^{l} - 1\} / (h - 1)$$

nodes, so that ako

(15) $$f(2h, l) \leqq \frac{(2h - \text{'''} -}{h \quad 1} (h \quad I).$$

Abßr also the upper bound is w-less good aL9 the upper bound (1).

{D) *Through each& Kanlc* some *MinimalgrapMn G!"! [k, I j go* insnJesfens $\frac{k+1}{2}$ | *Kreise der Länge* $\leqq l + 1$.

Proof 6: In the cases f= 3 and f 4, we can directly testify to the correctness of the assertion and may therefore rorauaaet l> 4.

Suppose (D} aei taleeh. Then there is an edge (z, y) in *G!"! [k,* 1) through which less than $\frac{+1}{2}$ ciriae of length N f 3- 1 pass through. z is, apart from iziit y, also connected to the nodesnn z" ... , z, _ " y is not only connected to z but also to the nodes y" . . , ¡y . by one edge each; because of 1 4 8no two of the nodes z" ... , z , ; y" ... , yq_ 2 are identical or connected by an edge. The fiumeration of the z" .. . , z,_ , ; y,. ... , , y,_ , eei such that for i= 1, 2, ... , $\frac{k}{2}$ each circle containing the edges (z" z) and (r, y) and each circle containing the edges (z, y) and (y, y _,) has a length t 1 (one can certainly be numbered øo, because in the other case it could be øthoughtø that the edge (z, y) is contained in at $k-\left\lceil\frac{k}{2}\right\rceil=\left\lceil\frac{k+}{2}\right.$ least itreiaen of length ł+ l). Then for j 1, *, ... ,, that every circle, which goes through the edges (z" z), (z, y), (y, ¡y,), has a length> 1 1. We consider the graph G*, which arises from C- '-' (£, I), if we delete the nodes z and y as well as the øămtlích edges emanating from them and, instead of them, the edges (z" 9,), (z" y), ... , (zø_ t. yø _ ,). R* is Cr" -' (£, I), so n is not the miniæal number of nodes - in \Viderapruch for the prerequisite.

Folgørungaue(D):

(E)

Eİgenachaft (Ealso results from the following property:

(F) *Etc finìmal9raph G!'! lk, I) bœilzt kein 2er]ällvng8knoteupiinH.*

(A node z of a connected graph H is called a decomposition c n o te n pun k I (or &rti- cølation) of G if G is decomposed into several separate parts by deleting z and the edges leading from z).

Beweia: ,4assuming R ^' *{k, I* decompose by deleting aøa node s and the edges starting from you z into several separate parts, non of which G may have the minimum number of nodes Ii ; then. olfenbar z°żi< n. ø be in 9"" (ł, 1) with the node points q, ... zø (p< £) of Ğ each connected by a Dante. *G', G"* are two graphs isomorphic to€ , "-where the nodes p', . , zp' of 6' and z"" ... , zø" of Ğ" and the nodes q, . . , zø of Ğ may correspond. Then we consider the graph O**, which arises when Ğ' and G" are connected by the edges (z,', ø,"), (z ', zø"), ... , lz ', ø ") are connected to each other. *u*** **proves** to be a **U'-*'**, *I ,* so n is not the minimal enotenum - in contradiction to the .

\Veiter iolgt leicht a.us {D):

(G) *Durch jeden Knotenpunkt eines Minimalgraphen $G^{(n)}$ (k, l) gehen mindestens* $\left|\frac{1}{2}\left(k \quad \frac{k+,}{2}\right|\right.$ *Circles of the*

Bøw-eis: The arbitrary node z of C"" (/r, 1) has the neighboring nodes zt, ... , zt, through which want.e (z, z,) may pass exactly ct circles of length 1 '- I (i - 1, 2, ..."). I8f then c is the number of old circles of length< f -1- 1 passing through z, the following obviously applies

c,+ c+ ... -J- c,= -*c;

because of (D) is

$c_i \geqq \frac{k+1}{2}$, Consequently z*c $\left.\frac{k+1}{2}\right|$,

and this immediately results in the assertion (G).

Auaøage (R) can be made more precise by two further statements:

(H) *Es sei $k=2h$ gerade und x ein beliebiger Knotenpunkt eines Minimalgraphen $G^{(n)}$ (k, l). Dann sind mindestens $h+1$ der von x ausgehenden Kanten je in einem Kreis der Länge l enthalten.*

Durch x gehen also mindestens $\left|\frac{\ddot{}}{2}\right|+1$ *Kreise der Länge l.*

Proof: In the case of I= 3, the assertion is correct, we may therefore assume 1> 3. Suppose, (D) be fœløch. Then in G"', *I j* there is a vertex z such that mindeatrnaof the edges starting from z do not belong to a circle of length f. Let z be connected to the vertices y, ... , zp by one edge each, the numbering such that none of the edges (z, y), ... , (z, zø) is connected to a circle length i. Because of 1 3, no two of the nodes z, (i= I, ... , *k Ì* are identiøeh or connected by aB eante. We consider the graph 9º, which arises from 6'-' (č, 1) if we delete the node z 8as well as all the edges emanating from it and BtBtt their edges (y, z "), (zø, zatø), . . , *l z , zz)* **are simple.** éi0 proves sick Us ed Ğ'" " {ł,), i.e. Est ii not the minimum node count - in contradiction to the \orauaa settlement.

(J) *E8 cei k -- 2 h+ 1* ti *erade,* and *there ceier x and y two bełieòiąe, ãby* else Jfaale rerbundene *Kiiotenpuiikle eine8 l4linimalgraphen Cl!" [k, I). Datsn lBM: mirdeatenø aet of the node puucts z, y |olgende i!igeH8Gha|t . Of the 2 k edges extending from* the *node point, the* cow from the *connection Q8 rite ţx, yl edges and a minimum of* Ä -i- 1 }e tn azn area *of Łength* I etifäøtfen, *member* matt *dark (x, p] gehl.*

Durch x oder y gehen also mindestens $\left|\frac{\ddot{}}{2}\right|+1$ *Kreise der Länge l, welche die Kante (x, y) nicht enthalten.*

Proof: In the cases I - 3 and 1 4 the assertion is certainly correct, we may therefore preaudit I> 4. Suppose (J) is fal8cb. Then in O'*' (ć, 1) there are zsvei vertices z, ¡y connected by an edge with the property that in G'" (£, lJ - (z, y) both from z and from y there are at least A edges which do not belong to a circle of length I. Let z be (except with y) with the nodal points r¡, . . , z¡" y aei {except with z) with the nodes y" ... , yt are each connected by an edge, where z the numbering is such that none of the edges (z, y), ... , (r,r) ; (y, 9,), ... , (y, y) in 6r ^* (k, 1) - (z, y) belongs to a circle of length f . Because of l > 4 .no two of the node purics r¡, yy (i, j 1, 2, ... , *2 h* are identical or connected by an edge. We consider the graph Cr0°, which results from O * (1, 1), if we delete the endpoints z, y and all edges starting from them and instead use the edges

$(x_1, x_{h+1}), (x_2, x_{h+2}), \ldots, (x_h, x_{2h}); (y_1, y_{h+1}), (y_2, y_{h+2}), \ldots, (y_h, y_{2h})$ einführen. G^{00} erweist sich als ein $G^{(n-2)}(k, l)$, alao tat n not the itiirrimal node calil - in \ contradiction to the prerequisite.

Bemerkung: *Es entsteht hier die Frage, ob für geeignete Zahlen h und l ein Minimalgraph $G^{(n)}(2h+1, l)$ existiert, welcher einen Knotenpunkt besitzt, durch den kein Kreis der Länge l geht.*

I'olution from {H) and (II:

(K) *Ein Minimalgraph $G^{(n)}(k, l)$ hat die Taillenweite l.*

\Vir conclude from this:

Gatz 2: Su zt i be/iebfpsu xoffirf*chea Za7dsn k1 2, 3 ezisf?erat' 8fsfa regsdöra GropÄett der P'zh'tzz£ und der *Taillenweite l* (vgl. dazu [4]).

Ein regulärer Graph der Valenz k und der Taillenweite l minimaler Knotenzahl ist zugleich ein Minimalgraph $G^{(n)}(k, l)$ mit $n = f(k, l)$ und hat folglich, falls k gerade ist, die Eigenschaften (A) *bis* (H), *und falls k ungerade ist, die Eigenschaften* (A), (D), (E), (F), (G) *und* (J). *Seine Knotenzahl f (k, l) genügt den Ungleichungen* (1) *und – falls l gerade ist –* (7). (Siehe auch (17) am Schluß der Arbeit.)

Note: The sections (14) and (IS) can easily be changed using (D), (O) and (H) or'. (J), but we will not go into this in detail.

Bemerkung): Zwischen $f(k, 2q+1)$ und $f(k, 2q+2)$ besteht folgende Relation:*

(16) $$f(k, 2q+2) \leqq 2 f(k, 2q+1) \quad (q = 1, 2, 3, \ldots)$$

Prove: Let O - t " *(k,* 2g 1) be a minimal graph and G', O" two graphs that are iomorphic to G. G' contains the edge (e', S'), whichBor corresponds to the map (e", h") on the basis of the I6Oolorphie in d". We delete the edges (n', f") and (o", b") and new edges *{a',*"), (e",') in their place, and proceed in the same way with each pair of corresponding edges of U' and *a*". The resulting graph d has 2o nodes and is. regu1itr of valence b. its 'faill width is not less than that of 9, and since it did openly O2ll Q8aPCr graph, it does not contain a circle of odd length. From this it immediately follows that fi is a G °- (£, 2q 2), and therefore / (k, 2 2+ 2J 2 n ; this is to be proved.

Old (16) and (1) we scbtie0en:

$$f(k, 2q+2) \leqq 8 \sum^{2q-1} (k-1)^t = 8 \{ (k-1)^{2q} - 1 \} / (k-2) - 8,$$

and this SpJl bound is better than the upper bound (1) for even L

FO88It summarizes the estimates obtained (1), (7) and (18):

$$(\text{"} \quad \begin{cases} 1 \quad k \dfrac{[k \quad 1)^e - I}{k-2} \leqq f(k, 2q+1) \leqq 4 \left\{ \dfrac{(k-1)^{2q} - 1}{k-2} - 1 \right\} \\ 2 \dfrac{(k-1)^{q+1} - 1}{} \leqq f(k, 2q+2) \leqq 8 \left\{ \dfrac{(k-1)^{2q} - 1}{k - } - 1 \right\} \end{cases}$$

(y= 1, *, 3, ...)

*) Addendum during printinga. II. P.

IITERATURE

[}] }jq n an, A, J., and Srsozwrou, R. R. : On foore graphs with diameters two and three. I. B. II. Journal of Research and Development A (1960), 487 - c'04,

2) KkRM8ZI, F. : Piani finiti oieliei come riaoluzioni di un oerto prnblema di minimo. Boll. Un. Mat. Ital. (3) 13 (J980), s29-?i28 ; aietse also: hat. £.apoL 11 (1 960), 323-329 (ungariach).

[3] hie Gma, 'W'. F. : A minimal oubie graph of girth 8even. Canad. iYfeth. Bull, 8 {l8ß0j, l4fi - 152.

[4] Gaeta, IT.: Regular graphe with given girth and restricted cirouita. Jour. London Math. Soc. {in Druuk).

[ß] Tv'zzz, Yt". T. : -4- family of oubioal graphs. Proo. Cambridge Phil. soo. 43 {1947}, 4ö9 - 4?4.

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ERDE, Paul - Atathemxtlk,
Budapest, n. Z. bnlxvrstty Go1lege, London,
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Your rer. net. (Ptom. l9S8 Belle), senior assistant at the 1. Afafhein. Inst:t.. I,chs
L'eauftzegter a. ü. J\Math. xat. T'alc. of the mir. Hall