Euclidean Ramsey Theory: Triangles

Exposition by William Gasarch

December 31, 2024

Credit Where Credit is Due

The the main thm of these slides is due to
Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

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Euclidean Ramsey Theorems I

Journal of Combinaorical Theory (A), Vol. 14, 341-363, 1973

Here is a link.
`https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/eramseyOne.pdf`

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Answer on next slide

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New Question either a mono $1-1-1$ **or** mono $2-2-2$ **or** $\cdots$.

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We prove this rather than $T_1 - T_{\sqrt{3}} - T_2$ since this makes the figures easier to draw.

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Assume by way of contradiction that there is a $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ with no mono T_2, $T_{2\sqrt{3}}$ or T_4.

There are Two **R** Points Two Apart

By Thm from last lecture $\exists$ two points, an inch apart, same color. We can assume that (0,0) and (2,0) are **R**.

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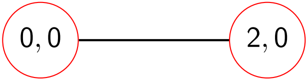

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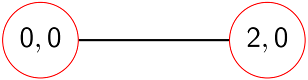

On the next slide we add four more points of interest.

Six Point of Interest

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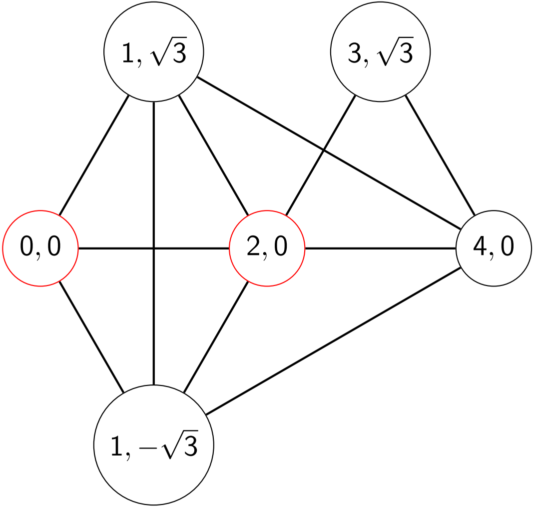

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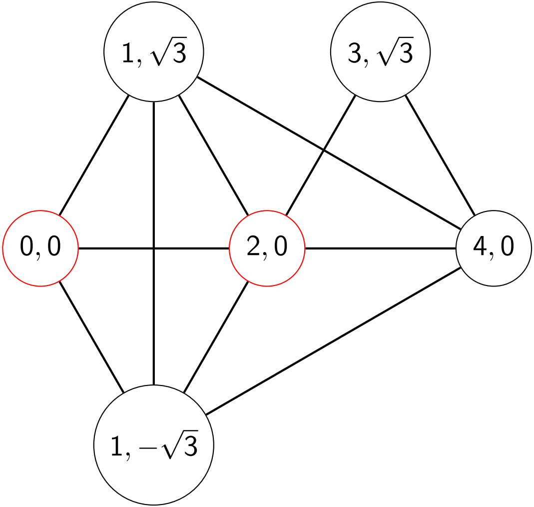

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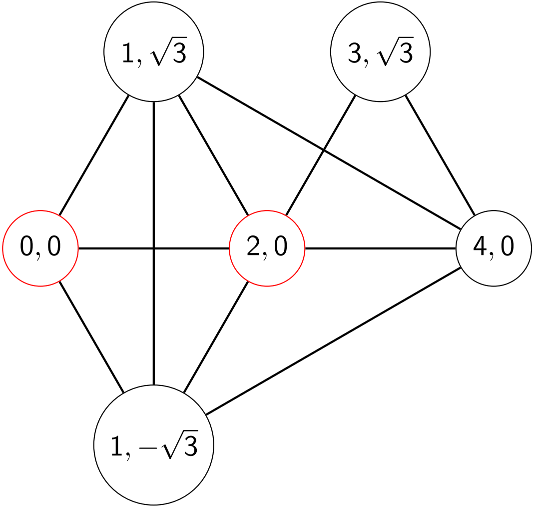

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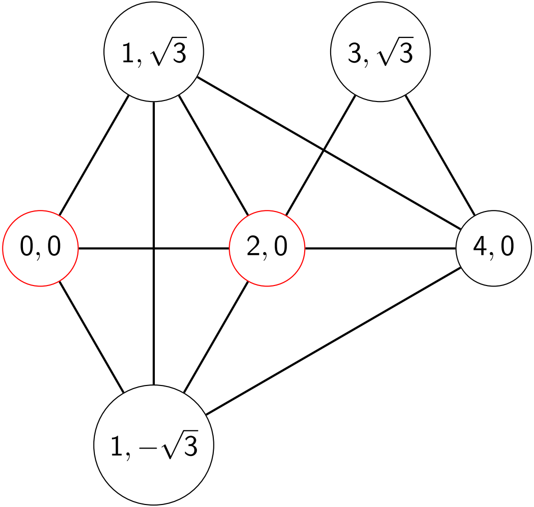

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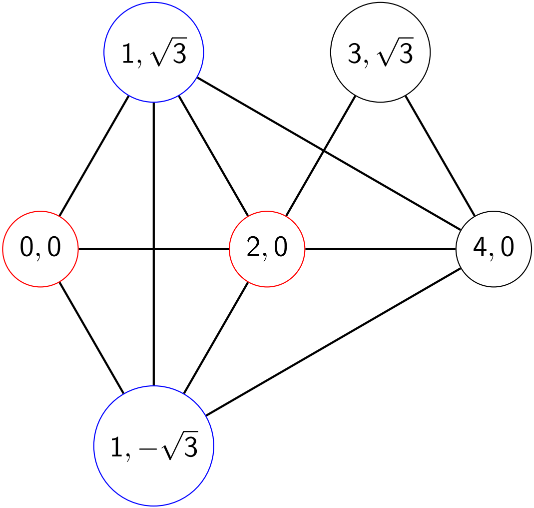
Next picture has this information.

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are B



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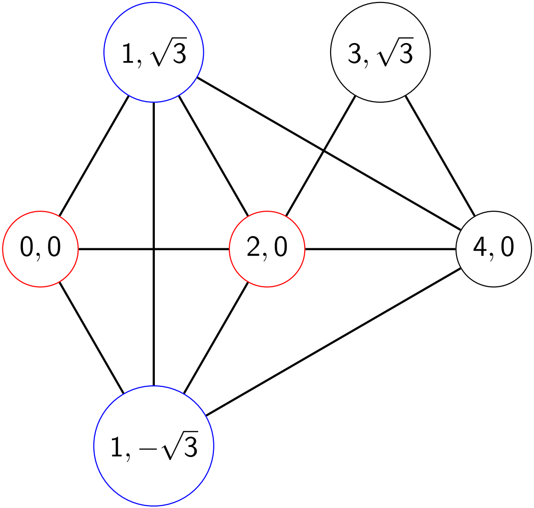

$(1, \sqrt{3}) - (1, -\sqrt{3}) - (4, 0)$ is a $T_{2\sqrt{3}}$ so $\mathrm{COL}(4, 0) =$ **R**.

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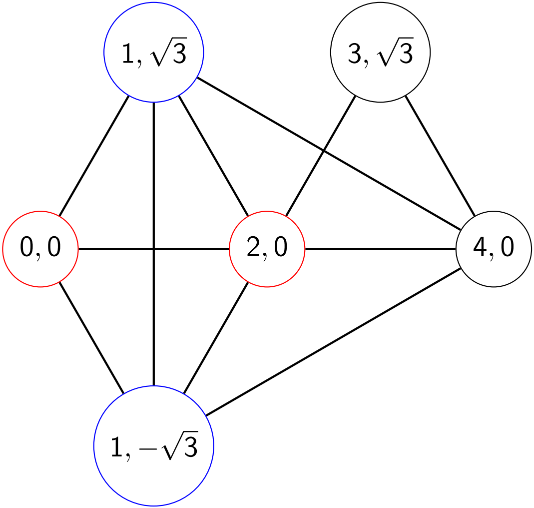

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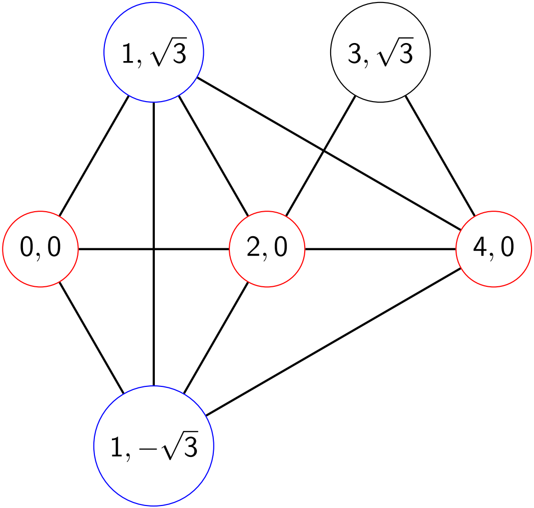

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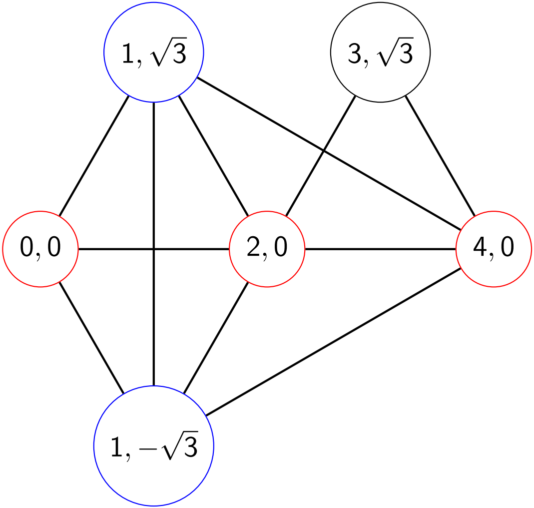

$(2, 0) - (4, 0) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(3, \sqrt{3}) =$ **B**.

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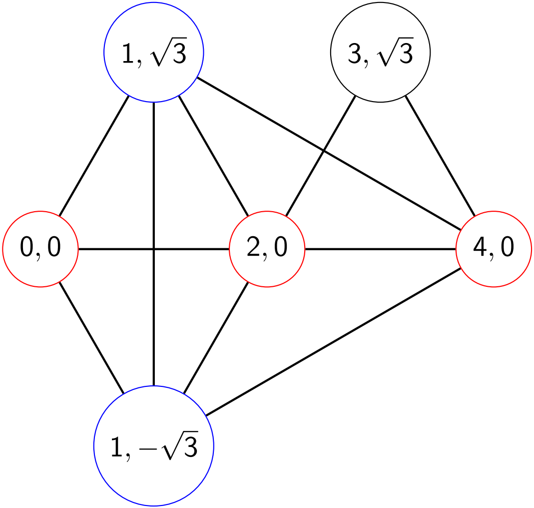

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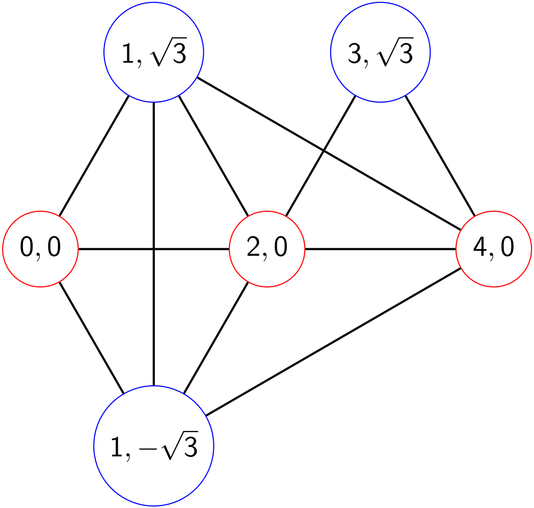

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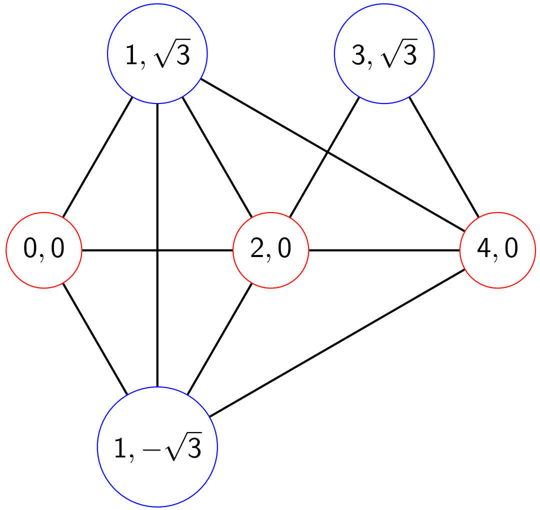
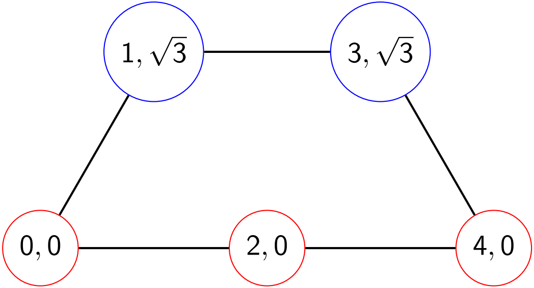


Next picture removes stuff we don't need anymore.

Where We Are Now



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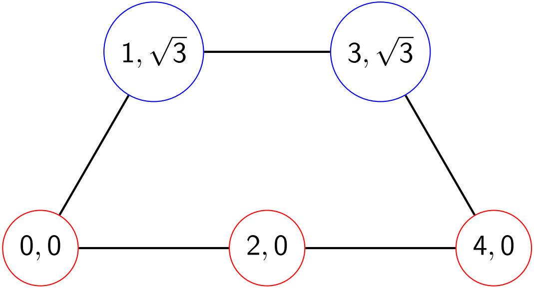
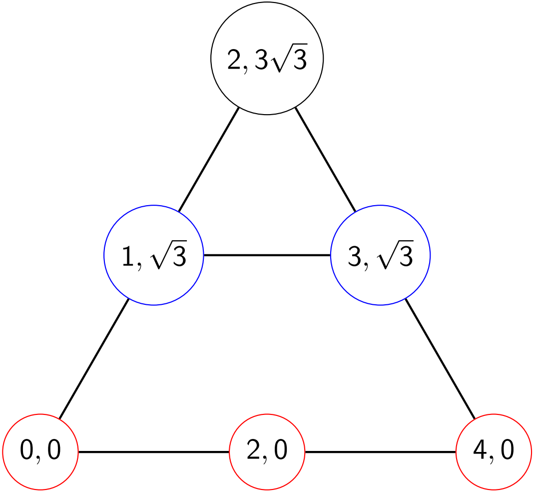


We add a the point $(2, 2\sqrt{3})$ on the next slide.

A Point That Can't be R or B



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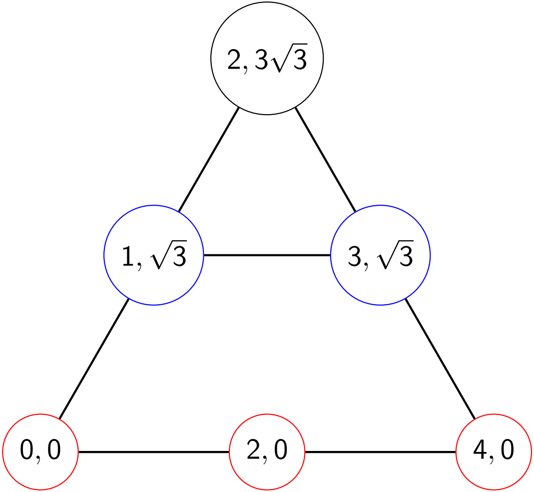

$(2, 3\sqrt{3}) - (1, \sqrt{3}) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(2, 3\sqrt{3}) \neq$ **B**.

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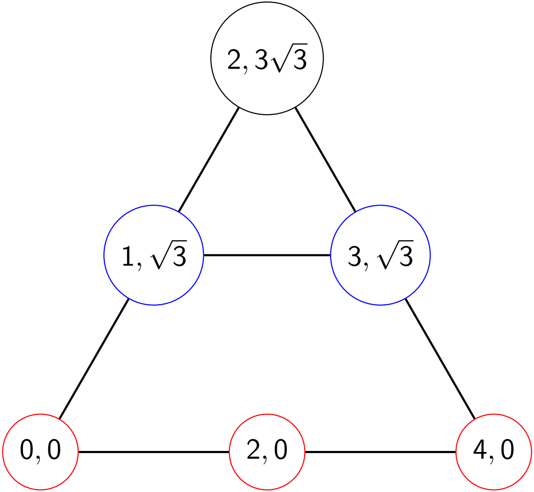

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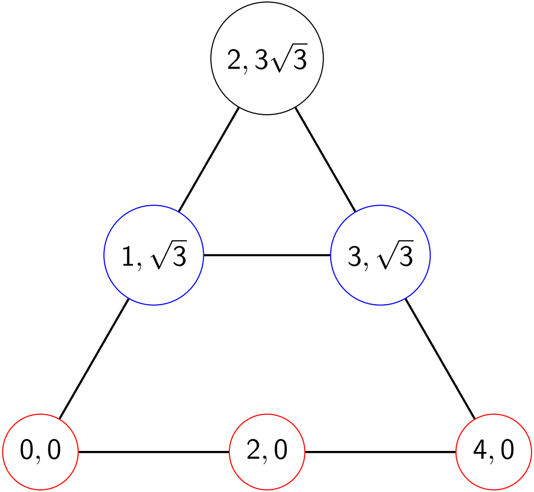

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$\mathrm{COL}(2, 3\sqrt{3}) \notin \{$**R**, **B**$\}$. Contradiction!