Monochromatic C_4

Exposition by William Gasarch

December 31, 2024

When Do You Get A Mono C_4?

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1) $R(C_4) = 18$.

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$R(C_4)$ is the least n such that $\forall\ \mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]\ \exists$ mono C_4.
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1) $R(C_4) = 18$.
2) $10 \le R(C_4) \le 17$.

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Answer on the next page.

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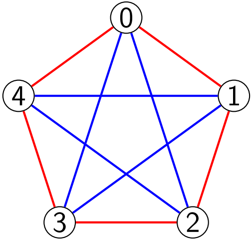

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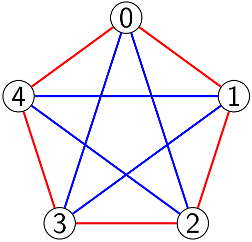

Note: There is a mono C_5 but not a mono C_4.

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Both cases will have some subcases.

Case I

$$\mathbf{COL}(T_1) = \mathbf{COL}(T_2)$$

Subcases For $\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$

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We assume $\mathrm{COL}(T_1) = \mathrm{COL}(T_2) =$ **R**.

$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$ and T_1, T_2 Share an Edge

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Can assume the **red** triangles are $1, 2, 3$ and $1, 2, 4$.

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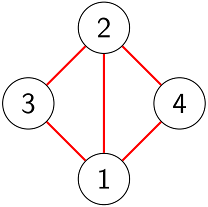

COL(T_1) = COL(T_2) and T_1, T_2 Share an Edge

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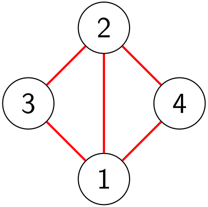

***C_4* :** $1 - 4 - 2 - 3 - 1$. DONE.

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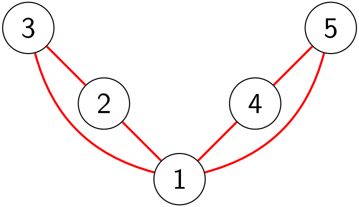

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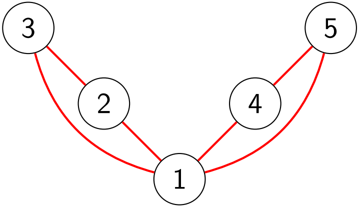

If $\mathrm{COL}(2,4) =$ **R** then C_4 **:** $2 - 4 - 1 - 3 - 2$.

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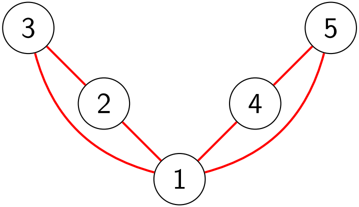

If $\mathrm{COL}(2,4) = \mathbf{R}$ then C_4 **:** $2 - 4 - 1 - 3 - 2$.
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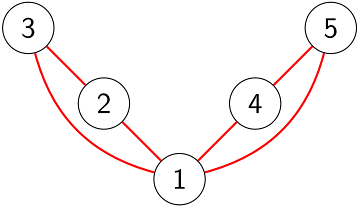

If $\mathrm{COL}(2,4) =$ **R** then C_4 **:** $2-4-1-3-2$.
If $\mathrm{COL}(2,5) =$ **R** then C_4 **:** $2-5-1-3-2$.
If $\mathrm{COL}(3,4) =$ **R** then C_4 **:** $3-4-1-2-3$.

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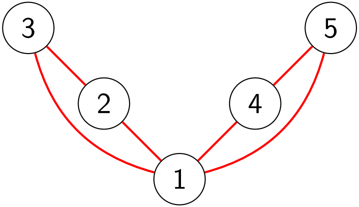

If $\mathrm{COL}(2,4) =$ **R** then $C_4 : 2 - 4 - 1 - 3 - 2$.
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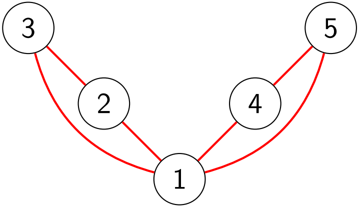
$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



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Next slide considers the case

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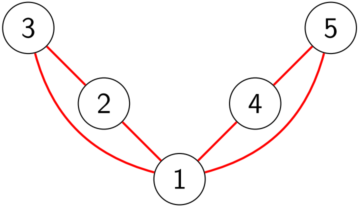

If $\mathrm{COL}(2,4) = \mathbf{R}$ then $C_4 : 2-4-1-3-2$.
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$$\mathrm{COL}(2,4) = \mathrm{COL}(2,5) = \mathrm{COL}(3,4) = \mathrm{COL}(3,5) = \mathbf{B}.$$

$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

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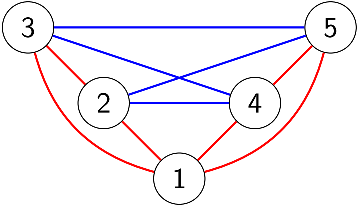

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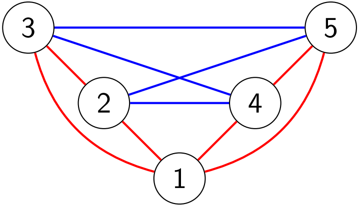

C_4 **:** $2 - 4 - 3 - 5 - 2$. DONE.

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We assume $T_1 = \{1, 2, 3\}$ and $T_2 = \{4, 5, 6\}$.

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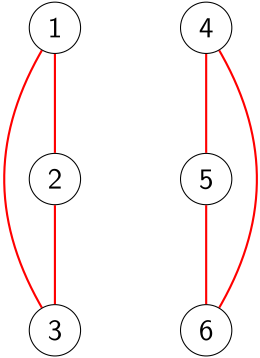

$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$, T_1, T_2 Disj, 2 R

Subcase A: $\exists \geq 2$ **red** edges between T_1 and T_2.

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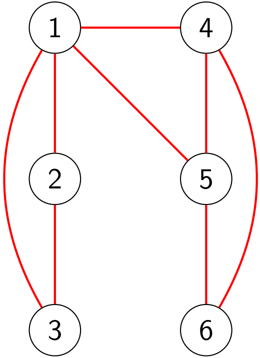

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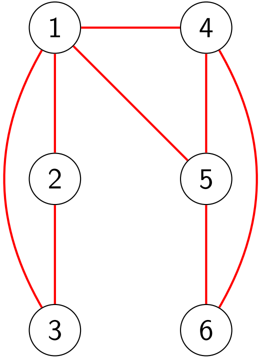

C_4 **:** $1 - 4 - 6 - 5 - 1$

$\mathbf{COL}(T_1) = \mathbf{COL}(T_2)$, T_1, T_2 Disj, ≥ 2**R**

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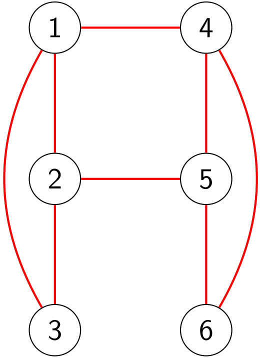

$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$, T_1, T_2 Disj, ≥ 2**R**

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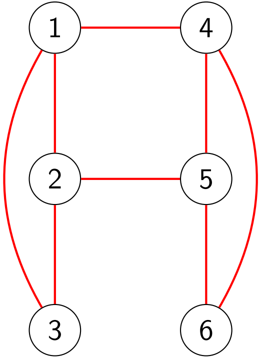

C_4 **:** $1 - 4 - 5 - 2 - 1$

$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$, T_1, T_2 Disj, ≤ 1R

Subcase B: There is ≤ 1 **red** edges between T_1 and T_2.

$\mathrm{COL}(T_1) = \mathrm{COL}(T_2)$, T_1, T_2 Disj, ≤ 1R

Subcase B: There is ≤ 1 **red** edges between T_1 and T_2.

Then there are ≥ 8 **blue** edges between T_1 and T_2.

$\mathbf{COL}(T_1) = \mathbf{COL}(T_2)$, T_1, T_2 Disj, ≤ 1R

Subcase B: There is ≤ 1 **red** edges between T_1 and T_2.

Then there are ≥ 8 **blue** edges between T_1 and T_2.

Easy Exercise: Show there is a C_4.

Case II

$$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$$

Subcases For $\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$

There are two subcases:

Subcases For $\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$

There are two subcases:

1. T_1 and T_2 share a vertex.

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We will assume $\mathrm{COL}(T_1) = $ **R** and $\mathrm{COL}(T_2) = $ **B**.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

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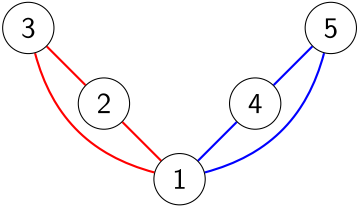

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\mathrm{COL}(3,4) = \mathrm{COL}(3,5) =$ **B**.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

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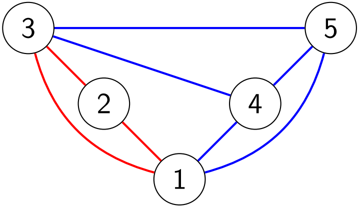
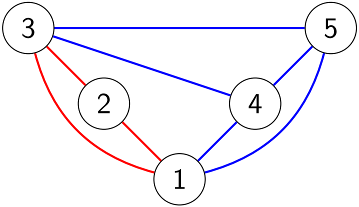


$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

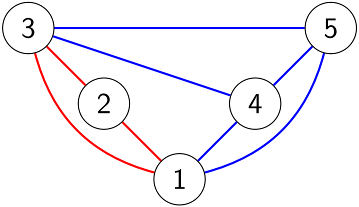
Subcase: $\mathrm{COL}(3,4) = \mathrm{COL}(3,5) = \mathbf{B}$.



C_4 : $3 - 5 - 4 - 1 - 3$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\mathrm{COL}(3,4) = \mathrm{COL}(3,5) =$ **B**.

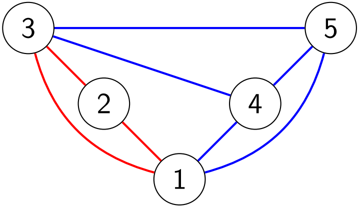


C_4 **:** $3 - 5 - 4 - 1 - 3$.

$\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **B** is similar.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\mathrm{COL}(3,4) = \mathrm{COL}(3,5) =$ **B**.

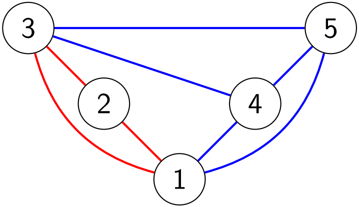


C_4 **:** $3 - 5 - 4 - 1 - 3$.

$\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **B** is similar.
$\mathrm{COL}(5,3) = \mathrm{COL}(5,2) =$ **R** is similar.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\mathrm{COL}(3,4) = \mathrm{COL}(3,5) =$ **B**.



C_4 **:** $3 - 5 - 4 - 1 - 3$.

$\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **B** is similar.
$\mathrm{COL}(5,3) = \mathrm{COL}(5,2) =$ **R** is similar.
$\mathrm{COL}(4,3) = \mathrm{COL}(4,2) =$ **R** is similar.

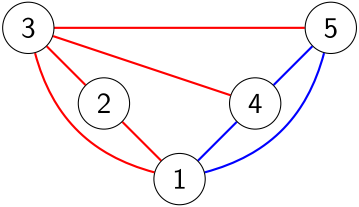
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) =$ **R**.

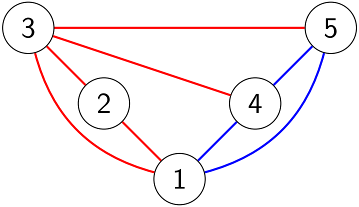
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) =$ **R**.



$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

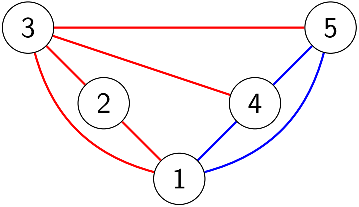
Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) = $ **R**.



If $\mathrm{COL}(2,5) = $ **R** then $C_4 : 2 - 5 - 3 - 1 - 2$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

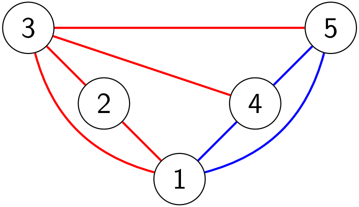
Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) =$ **R**.



If $\mathrm{COL}(2,5) =$ **R** then C_4 **:** $2 - 5 - 3 - 1 - 2$.
If $\mathrm{COL}(2,4) =$ **R** then C_4 **:** $2 - 4 - 3 - 1 - 2$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

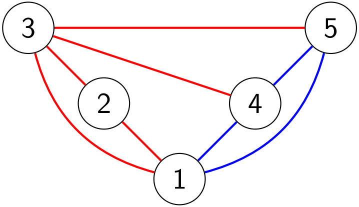
Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) = $ **R**.



If $\mathrm{COL}(2,5) = $ **R** then C_4 **:** $2 - 5 - 3 - 1 - 2$.
If $\mathrm{COL}(2,4) = $ **R** then C_4 **:** $2 - 4 - 3 - 1 - 2$.
Hence $\mathrm{COL}(2,5) = \mathrm{COL}(2,4) = $ **B** so C_5 **:** $2 - 5 - 1 - 4 - 2$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

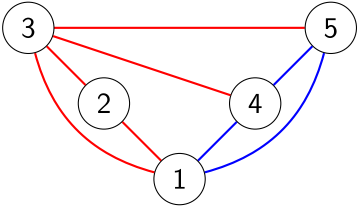
Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) =$ **R**.



If $\mathrm{COL}(2,5) =$ **R** then ***C_4* :** $2 - 5 - 3 - 1 - 2$.
If $\mathrm{COL}(2,4) =$ **R** then ***C_4* :** $2 - 4 - 3 - 1 - 2$.
Hence $\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **B** so ***C_5* :** $2 - 5 - 1 - 4 - 2$.
$\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **R** is similar.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

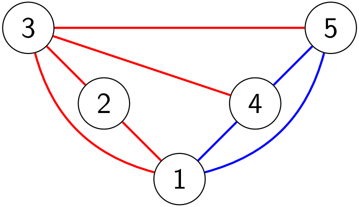
Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) = \mathbf{R}$.



If $\mathrm{COL}(2,5) = \mathbf{R}$ then $C_4 : 2-5-3-1-2$.
If $\mathrm{COL}(2,4) = \mathbf{R}$ then $C_4 : 2-4-3-1-2$.
Hence $\mathrm{COL}(2,5) = \mathrm{COL}(2,4) = \mathbf{B}$ so $C_5 : 2-5-1-4-2$.
$\mathrm{COL}(2,5) = \mathrm{COL}(2,4) = \mathbf{R}$ is similar.
$\mathrm{COL}(5,3) = \mathrm{COL}(5,2) = \mathbf{B}$ is similar.

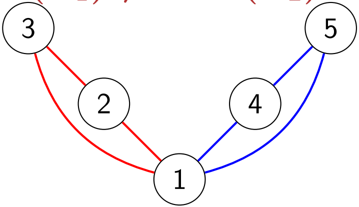
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\mathrm{COL}(3,5) = \mathrm{COL}(3,4) =$ **R**.

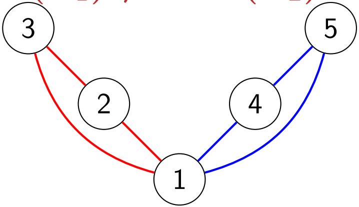


If $\mathrm{COL}(2,5) =$ **R** then C_4 **:** $2-5-3-1-2$.
If $\mathrm{COL}(2,4) =$ **R** then C_4 **:** $2-4-3-1-2$.
Hence $\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **B** so C_5 **:** $2-5-1-4-2$.
$\mathrm{COL}(2,5) = \mathrm{COL}(2,4) =$ **R** is similar.
$\mathrm{COL}(5,3) = \mathrm{COL}(5,2) =$ **B** is similar.
$\mathrm{COL}(4,3) = \mathrm{COL}(4,2) =$ **B** is similar.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

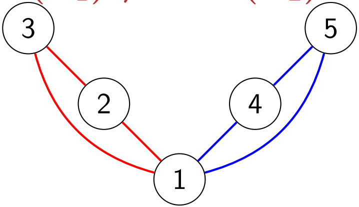


$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

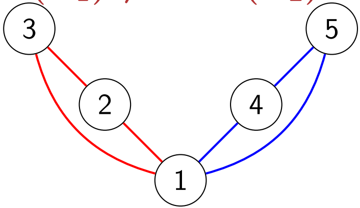
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

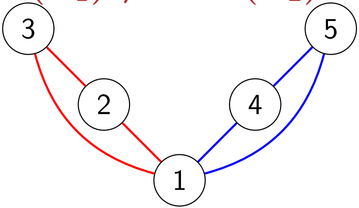


The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$

$\mathrm{COL}(2,5) \neq \mathrm{COL}(2,4)$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



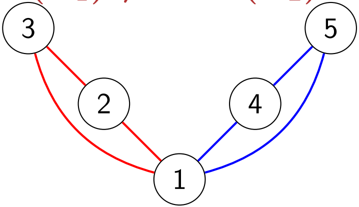
The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$

$\mathrm{COL}(2,5) \neq \mathrm{COL}(2,4)$

$\mathrm{COL}(5,3) \neq \mathrm{COL}(5,4)$

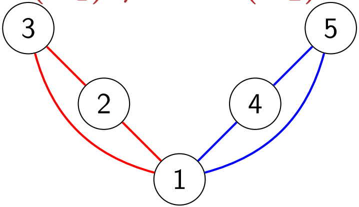
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$
$\mathrm{COL}(2,5) \neq \mathrm{COL}(2,4)$
$\mathrm{COL}(5,3) \neq \mathrm{COL}(5,4)$
$\mathrm{COL}(4,3) \neq \mathrm{COL}(4,2)$

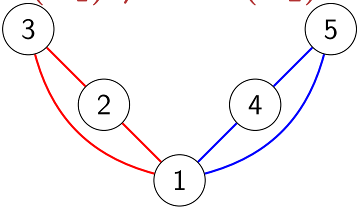
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$
$\mathrm{COL}(2,5) \neq \mathrm{COL}(2,4)$
$\mathrm{COL}(5,3) \neq \mathrm{COL}(5,4)$
$\mathrm{COL}(4,3) \neq \mathrm{COL}(4,2)$
Only two colorings to check:

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

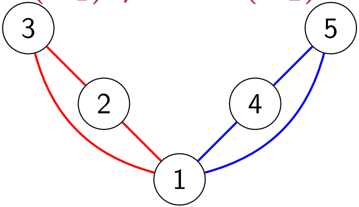


The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$
$\mathrm{COL}(2,5) \neq \mathrm{COL}(2,4)$
$\mathrm{COL}(5,3) \neq \mathrm{COL}(5,4)$
$\mathrm{COL}(4,3) \neq \mathrm{COL}(4,2)$
Only two colorings to check:

$\mathrm{COL}(3,5) = \textbf{R} \rightarrow \mathrm{COL}(3,4) = \textbf{B} \rightarrow \mathrm{COL}(4,2) = \textbf{R} \rightarrow \mathrm{COL}(2,5) = \textbf{B}$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$\mathrm{COL}(3,5) \neq \mathrm{COL}(3,4)$
$\mathrm{COL}(2,5) \neq \mathrm{COL}(2,4)$
$\mathrm{COL}(5,3) \neq \mathrm{COL}(5,4)$
$\mathrm{COL}(4,3) \neq \mathrm{COL}(4,2)$

Only two colorings to check:

$$\mathrm{COL}(3,5) = \mathbf{R} \rightarrow \mathrm{COL}(3,4) = \mathbf{B} \rightarrow \mathrm{COL}(4,2) = \mathbf{R} \rightarrow \mathrm{COL}(2,5) = \mathbf{B}$$

$$\mathrm{COL}(3,5) = \mathbf{B} \rightarrow \mathrm{COL}(3,4) = \mathbf{R} \rightarrow \mathrm{COL}(4,2) = \mathbf{B} \rightarrow \mathrm{COL}(2,5) = \mathbf{R}$$

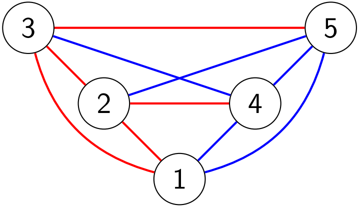
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$$\mathrm{COL}(3,5) = \textbf{R} \rightarrow \mathrm{COL}(3,4) = \textbf{B} \rightarrow \mathrm{COL}(4,2) = \textbf{R} \rightarrow \mathrm{COL}(2,5) = \textbf{B}$$

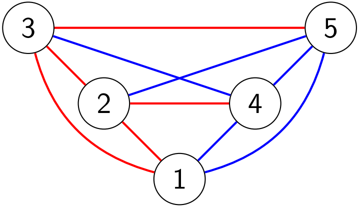
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(3,5) = \textbf{R} \rightarrow \mathrm{COL}(3,4) = \textbf{B} \rightarrow \mathrm{COL}(4,2) = \textbf{R} \rightarrow \mathrm{COL}(2,5) = \textbf{B}$



$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

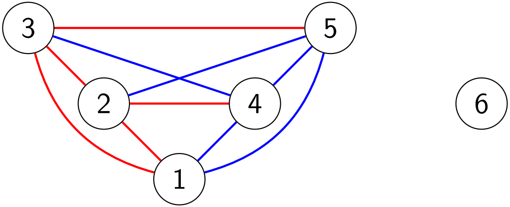
$\mathrm{COL}(3,5) = \mathbf{R} \rightarrow \mathrm{COL}(3,4) = \mathbf{B} \rightarrow \mathrm{COL}(4,2) = \mathbf{R} \rightarrow \mathrm{COL}(2,5) = \mathbf{B}$



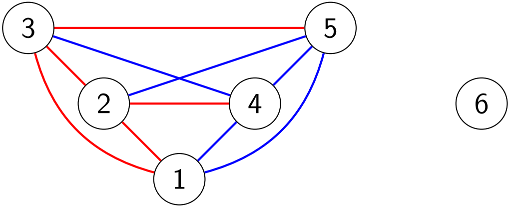
We now need to look at vertex 6.

$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

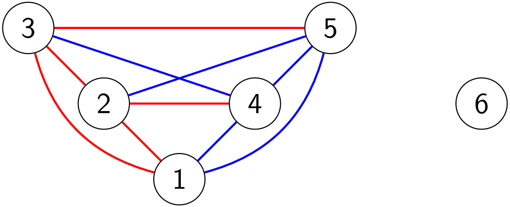


$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 **:** $6 - 1 - 3 - 2 - 6$

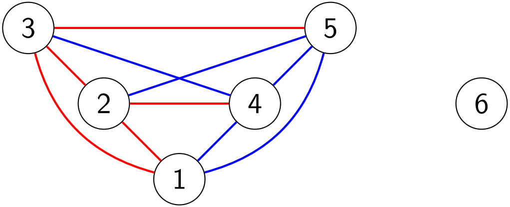
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 **:** $6 - 1 - 3 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 **:** $6 - 1 - 5 - 2 - 6$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

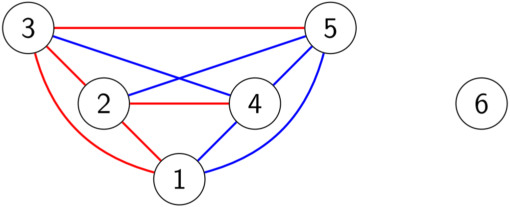


If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 **:** $6 - 1 - 3 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 **:** $6 - 1 - 5 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 **:** $6 - 1 - 2 - 3 - 6$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



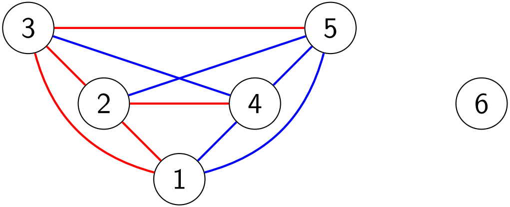
If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 : $6 - 1 - 3 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 : $6 - 1 - 5 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 : $6 - 1 - 2 - 3 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **B** then C_4 : $6 - 1 - 4 - 3 - 6$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 : $6 - 1 - 3 - 2 - 6$

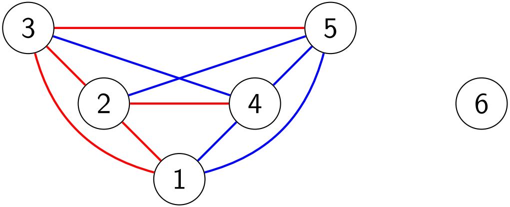
If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 : $6 - 1 - 5 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 : $6 - 1 - 2 - 3 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **B** then C_4 : $6 - 1 - 4 - 3 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **R** then C_4 : $6 - 1 - 2 - 4 - 6$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 **:** $6-1-3-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 **:** $6-1-5-2-6$

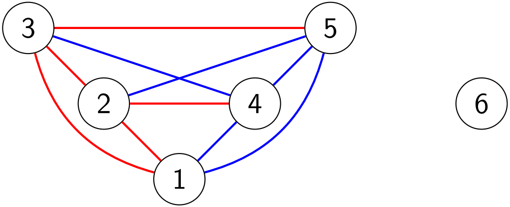
If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 **:** $6-1-2-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **B** then C_4 **:** $6-1-4-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **R** then C_4 **:** $6-1-2-4-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **B** then C_4 **:** $6-1-5-4-6$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 **:** $6-1-3-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 **:** $6-1-5-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 **:** $6-1-2-3-6$

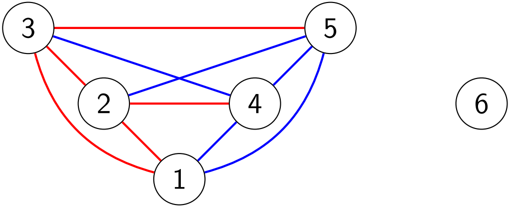
If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **B** then C_4 **:** $6-1-4-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **R** then C_4 **:** $6-1-2-4-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **B** then C_4 **:** $6-1-5-4-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) =$ **R** then C_4 **:** $6-1-3-5-6$

COL(T_1) $\neq$ COL(T_2) and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 : $6 - 1 - 3 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 : $6 - 1 - 5 - 2 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 : $6 - 1 - 2 - 3 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **B** then C_4 : $6 - 1 - 4 - 3 - 6$

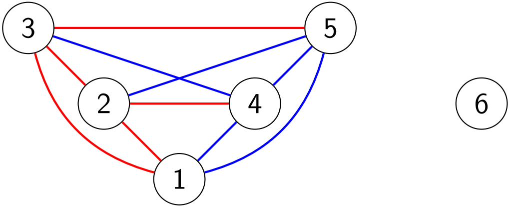
If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **R** then C_4 : $6 - 1 - 2 - 4 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **B** then C_4 : $6 - 1 - 5 - 4 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) =$ **R** then C_4 : $6 - 1 - 3 - 5 - 6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) =$ **B** then C_4 : $6 - 1 - 4 - 5 - 6$

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **R** then C_4 : $6-1-3-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) =$ **B** then C_4 : $6-1-5-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **R** then C_4 : $6-1-2-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) =$ **B** then C_4 : $6-1-4-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **R** then C_4 : $6-1-2-4-6$

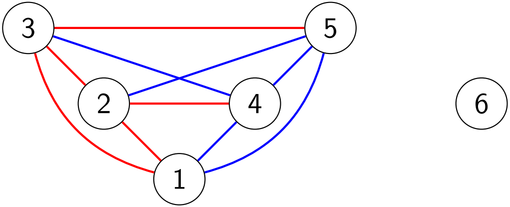
If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) =$ **B** then C_4 : $6-1-5-4-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) =$ **R** then C_4 : $6-1-3-5-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) =$ **B** then C_4 : $6-1-4-5-6$

$\mathrm{COL}(1,6) =$ **R** : $(\forall 2 \leq i \leq 5)[\mathrm{COL}(i,6) =$ **B**$]$. **$6-5-1-4-6$**

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) = \mathbf{R}$ then $C_4 : 6-1-3-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(2,6) = \mathbf{B}$ then $C_4 : 6-1-5-2-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) = \mathbf{R}$ then $C_4 : 6-1-2-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(3,6) = \mathbf{B}$ then $C_4 : 6-1-4-3-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) = \mathbf{R}$ then $C_4 : 6-1-2-4-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(4,6) = \mathbf{B}$ then $C_4 : 6-1-5-4-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) = \mathbf{R}$ then $C_4 : 6-1-3-5-6$

If $\mathrm{COL}(1,6) = \mathrm{COL}(5,6) = \mathbf{B}$ then $C_4 : 6-1-4-5-6$

$\mathrm{COL}(1,6) = \mathbf{R} : (\forall 2 \le i \le 5)[\mathrm{COL}(i,6) = \mathbf{B}]$. $\mathbf{6-5-1-4-6}$

$\mathrm{COL}(1,6) = \mathbf{B} : (\forall 2 \le i \le 5)[\mathrm{COL}(i,6) = \mathbf{R}]$. $\mathbf{6-1-2-3-6}$

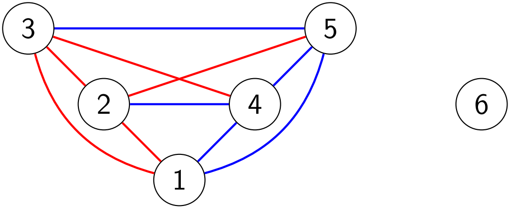
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$$\mathrm{COL}(3,5) = \textbf{B} \rightarrow \mathrm{COL}(3,4) = \textbf{R} \rightarrow \mathrm{COL}(4,2) = \textbf{B} \rightarrow \mathrm{COL}(2,5) = \textbf{R}$$

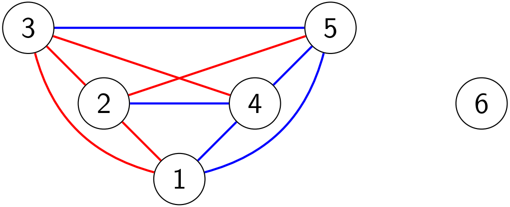
$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(3,5) = \mathbf{B} \rightarrow \mathrm{COL}(3,4) = \mathbf{R} \rightarrow \mathrm{COL}(4,2) = \mathbf{B} \rightarrow \mathrm{COL}(2,5) = \mathbf{R}$



$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\mathrm{COL}(3,5) = \mathbf{B} \rightarrow \mathrm{COL}(3,4) = \mathbf{R} \rightarrow \mathrm{COL}(4,2) = \mathbf{B} \rightarrow \mathrm{COL}(2,5) = \mathbf{R}$



We leave this case to the reader.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$ and T_1, T_2 Are Disj

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We assume $T_1 = \{1, 2, 3\}$ and $T_2 = \{4, 5, 6\}$.

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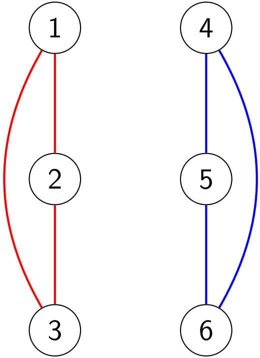
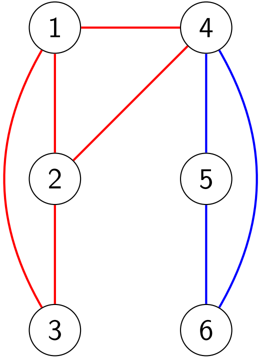


$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \deg_{\mathbf{R}}(v) \geq 2$

Subcase A: $\exists v$, Right Side, $\deg_{\mathbf{R}}(v) \geq 2$.

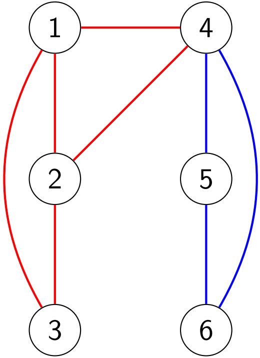
$\mathbf{COL}(T_1) \ne \mathbf{COL}(T_2)$, T_1, T_2 **Disj,** $\exists v, \deg_{\mathbf{R}}(v) \ge 2$

Subcase A: $\exists v$, Right Side, $\deg_{\mathbf{R}}(v) \ge 2$.



$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \deg_{\mathbf{R}}(v) \geq 2$

Subcase A: $\exists v$, Right Side, $\deg_{\mathbf{R}}(v) \geq 2$.



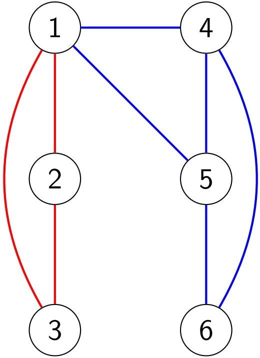
C_4 **:** $4 - 2 - 3 - 1 - 4$.

$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \deg_{\mathbf{B}}(v) \geq 2$

Subcase B: $\exists v$, Left Side, $\deg_{\mathbf{B}}(v) \geq 2$.

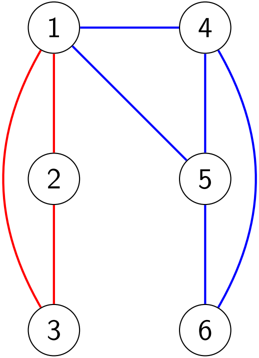
$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \deg_{\mathbf{B}}(v) \geq 2$

Subcase B: $\exists v$, Left Side, $\deg_{\mathbf{B}}(v) \geq 2$.



$\mathbf{COL}(T_1) \neq \mathbf{COL}(T_2)$, T_1, T_2 **Disj,** $\exists v, \deg_{\mathbf{B}}(v) \geq 2$

Subcase B: $\exists v$, Left Side, $\deg_{\mathbf{B}}(v) \geq 2$.



C_4 **:** $4 - 6 - 5 - 1 - 4$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:
$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and
$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and
$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Left}} \deg_{\mathbf{R}}(v) \leq 3$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and
$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Left}} \deg_{\mathbf{R}}(v) \leq 3$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Right}} \deg_{\mathbf{B}}(v) \leq 3$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:
$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and
$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.
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Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Right}} \deg_{\mathbf{B}}(v) \leq 3$.
Numb edges between T_1 and T_2 is $\leq 3 + 3 = 6$.

$\mathrm{COL}(T_1) \neq \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:
$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and
$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Left}} \deg_{\mathbf{R}}(v) \leq 3$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Right}} \deg_{\mathbf{B}}(v) \leq 3$.
Numb edges between T_1 and T_2 is $\leq 3 + 3 = 6$.
But there are 9 edges between T_1, T_2.

$\mathrm{COL}(T_1) \ne \mathrm{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:
$\forall v$, left node, $\deg_{\mathbf{R}}(v) \ge 1$ and
$\forall v$, right node, $\deg_{\mathbf{B}}(v) \ge 1$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Left}} \deg_{\mathbf{R}}(v) \le 3$.
Numb **edges** bet. $T_1, T_2 = \sum_{v \in \mathrm{Right}} \deg_{\mathbf{B}}(v) \le 3$.
Numb edges between T_1 and T_2 is $\le 3 + 3 = 6$.
But there are 9 edges between T_1, T_2.
So can't occur.