1 Erdős-Rado Proof of 3-hypergraph Ramsey

Def 1.1 $R_a(k)$ is the least number n such that, for all COL: $\binom{[n]}{a} \to [2]$ there exists a homog set of size k.

In our proof in class that $R_3(k)$ exists the construction used 2-ary Ramsey many times, and then 1-ary Ramsey once. Each use of 2-ary Ramsey cuts the set from size X to size $\log X$. The one use of 1-ary Ramsey cuts the set from size X to size $\frac{X}{2}$. All of those uses of 2-ary Ramsey force us to start with a very big set.

To be able to start with a smaller set we will do a different construction. This one will use 1-ary Ramsey many times, but 2-ary Ramsey only once.

Theorem 1.2 For almost all k, $R_3(k) \le 2^{2^{4k}}$.

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{R_2(k)}.$$

Here is the intuition: Let $x_1 = 1$. Let $x_2 = 2$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be a 1-homogeneous for COL^* of size at least $\frac{n-2}{2}$. Let $COL^{**}(x_1, x_2)$ be the color of V_1 . Let x_3 be the least vertex left (bigger than x_2).

The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[COL_1^*(y) = COL(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[COL_2^*(y) = COL(x_2, x_3, y)]$$

Let V_2 be a 1-homogeneous for COL_1^* of size $\frac{|V_1|-1}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2 . Restrict COL_2^* to elements of V_2 , though still call it COL_2^* . We reuse the variable name V_2 to be a 1-homogeneous for COL_2^* of size at least $\frac{|V_2|}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2 . Let x_4 be the least element of V_2 . Repeat the process.

We describe the construction formally.

CONSTRUCTION

$$x_1 = 1$$
 $V_1 = [n] - \{x_1\}$

Let $2 \le i \le R_2(k)$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and

$$\operatorname{COL}^{**}: \begin{pmatrix} \{x_1, \dots, x_{i-1}\} \\ 2 \end{pmatrix} \to [2]$$

are defined.

 $x_i =$ the least element of V_{i-1}

 $V_i = V_{i-1} - \{x_i\}$ (We will change this set without changing its name).

We define $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, ..., $COL^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets V_i . We will keep the variable name V_i throughout.

For j = 1 to i - 1

- 1. $COL^*: V_i \to [2]$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.
- 2. Let V_i be redefined as the largest 1-homogeneous set for COL^* . Note that $|V_i|$ decreases by at most half.
- 3. $COL^{**}(x_i, x_i)$ is the color of V_i .

KEY: For all $1 \le i_1 < i_2 \le i$, for all $y \in V_i$, $COL(x_{i_1}, x_{i_2}, y) = COL^{**}(x_{i_1}, x_{i_2})$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction can be carried out for $R_2(k)$ stages. For now assume the construction ends.

We have vertices

$$X = \{x_1, x_2, \dots, x_{R_2(k)}\}\$$

and a 2-coloring COL^{**} of $\binom{X}{2}$. By the definition of $R_2(k)$ there exists a Homog-for- COL^{**} :

$$H = \{x_{i_1}, \dots, x_{i_k}\}.$$

Let the color of this homogenous set be 1. We show that H is a homogenous set for COL. Let $1 \le i_1 < i_2 < i_3 \le k$. We show that $COL(x_{i_1}, x_{i_2}, x_{i_3}) = 1$.

By the definition of COL** for all $y \in V_{i_2}$, COL $(x_{i_1}, x_{i_2}, y) = \text{COL}^{**}(x_{i_1}, x_{i_2}) = 1$. In particular COL $(x_{i_1}, x_{i_2}, x_{i_3}) = 1$.

We now see how large n must be so that the construction be carried out. Note that in stage i $|V_i|$ be decreases by at most half, i times. Hence $|V_{i+1}| \ge \frac{|V_i|}{2^i}$.

Therefore

$$|V_i| \ge \frac{|V_1|}{2^{1+2+\dots+(i-1)}} \ge \frac{n-1}{2^{(i-1)^2}}.$$

We want $|V_{R_2(k)}| \ge 1$. It suffice so take $n = 2^{R_2(k)^2}$.

We know that $R_2(k) \leq 2^{2k}$. Hence

$$R_3(k) \le 2^{R_2(k)^2} \le 2^{2^{4k}}.$$