

**There is no 3-coloring of  
 $15 \times 15$**

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{15,15}$ .

$$15 \times 15 = 225$$

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There is a rectangle free set  $X$ ,  $|X| \geq \frac{225}{3} = 75$ .

# Our Plan

For  $1 \leq i \leq 15$

let  $x_i$  be the number of elements of  $X$  in the  $i$ th column.

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$$\sum_{i=1}^{15} \binom{x_i}{2}.$$

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**Plan** The number of pairs of  $\{1, \dots, 15\}$  is  $\binom{15}{2} = 105$ .

We will find a lower bound  $L$  on  $\sum_{i=1}^{15} \binom{x_i}{2}$ .

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**Plan** The number of pairs of  $\{1, \dots, 15\}$  is  $\binom{15}{2} = 105$ .

We will find a lower bound  $L$  on  $\sum_{i=1}^{15} \binom{x_i}{2}$ .

We will show  $L > 105$ , hence some four elements of  $X$  form a rectangle.

# Inequality

Want to show that  $\sum_{i=1}^{15} \binom{x_i}{2} \geq 106$ .



# Inequality

Want to show that  $\sum_{i=1}^{15} \binom{x_i}{2} \geq 106$ .

Want to find MIN of

$$\sum_{i=1}^{15} \binom{x_i}{2}$$

relative to the constraint

$$\sum_{i=1}^{15} x_i = 75.$$

# Well Known Theorem

Over the REALS:

$$\sum_{i=1}^{15} \frac{x_i(x_i-1)}{2}$$

relative to the constraint

$$\sum_{i=1}^{15} x_i = 75.$$

is MINIMIZED if all of the  $x_i$ s are equal.

We take  $x_i = 75/15 = 5$ .

$$\sum_{i=1}^{15} \frac{x_i(x_i-1)}{2} \geq \sum_{i=1}^{15} \frac{5 \times 4}{2} = 15 \times 10 = 150.$$

## Recap and Finish

The number of vertical pairs is  $\binom{15}{2} = 105$

The number of vertical pairs of points in  $X$  is

$$\geq \sum_{i=1}^{15} \frac{x_i(x_i - 1)}{2} \geq \sum_{i=1}^{15} \frac{5 \times 4}{2} = 15 \times 10 = 150.$$

## Recap and Finish

The number of vertical pairs is  $\binom{15}{2} = 105$

The number of vertical pairs of points in  $X$  is

$$\geq \sum_{i=1}^{15} \frac{x_i(x_i - 1)}{2} \geq \sum_{i=1}^{15} \frac{5 \times 4}{2} = 15 \times 10 = 150.$$

Hence some vertical pair of points occurs twice, so  $X$  is not rectangle free.

**There is no 3-coloring of  
 $14 \times 14$**

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{14,14}$ .

$$14 \times 14 = 196$$

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Assume there is a 3-coloring of  $G_{14,14}$ .

$$14 \times 14 = 196$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{196}{3} \rceil = 66$ .

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{14,14}$ .

$$14 \times 14 = 196$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{196}{3} \rceil = 66$ .

For  $1 \leq i \leq 14$

let  $x_i$  be the number of elements of  $X$  in the  $i$ th column.

Need

$$\sum_{i=1}^{14} \binom{x_i}{2} \geq \binom{14}{2} = 91.$$



## MIN the sum

Over the REALS:

$$\sum_{i=1}^{14} \frac{x_i(x_i-1)}{2}$$

relative to the constraint

$$\sum_{i=1}^{14} x_i = 66.$$

is MINIMIZED if all of the  $x_i$ 's are equal.

We take  $x_i = \frac{66}{14} = \frac{33}{7}$ .

We note that  $\frac{1}{2} \times \frac{33}{7} \left( \frac{33}{7} - 1 \right) = \frac{429}{49}$

$$\sum_{i=1}^{14} \frac{x_i(x_i-1)}{2} \geq \sum_{i=1}^{14} \frac{429}{49} = \frac{858}{7} = 122 + \frac{6}{7} > 91$$

DONE.

**There is no 3-coloring of  
 $13 \times 13$**

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{13,13}$ .

$$13 \times 13 = 169$$

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{13,13}$ .

$$13 \times 13 = 169$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{169}{3} \rceil = 57$ .

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{13,13}$ .

$$13 \times 13 = 169$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{169}{3} \rceil = 57$ .

For  $1 \leq i \leq 13$

let  $x_i$  be the number of elements of  $X$  in the  $i$ th column.

Need

$$\sum_{i=1}^{13} \binom{x_i}{2} \geq \binom{13}{2} = 78.$$

## MIN the sum

Over the REALS:

$$\sum_{i=1}^{13} \frac{x_i(x_i-1)}{2}$$

relative to the constraint

$$\sum_{i=1}^{13} x_i = 57.$$

is MINIMIZED if all of the  $x_i$ 's are equal.

We take  $x_i = \frac{57}{13}$

We note that  $\frac{1}{2} \times \frac{57}{13} \left( \frac{57}{13} - 1 \right) = \frac{1254}{169}$

$$\sum_{i=1}^{13} \frac{x_i(x_i-1)}{2} \geq 13 \times \frac{1254}{169} = 96+ > 78$$

DONE.

**There is no 3-coloring of  
 $12 \times 12$**

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{12,12}$ .

$$12 \times 12 = 144$$



# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{12,12}$ .

$$12 \times 12 = 144$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{144}{3} \rceil = 48$ .

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{12,12}$ .

$$12 \times 12 = 144$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{144}{3} \rceil = 48$ .

For  $1 \leq i \leq 12$

let  $x_i$  be the number of elements of  $X$  in the  $i$ th column.

Need

$$\sum_{i=1}^{12} \binom{x_i}{2} \geq \binom{12}{2} = 66.$$

## MIN the sum

Over the REALS:

$$\sum_{i=1}^{12} \frac{x_i(x_i-1)}{2}$$

relative to the constraint

$$\sum_{i=1}^{12} x_i = 48.$$

is MINIMIZED if all of the  $x_i$ 's are equal.

We take  $x_i = \frac{48}{12} = 4$

We note that  $\frac{1}{2} \times 4 \times 3 = 6$ .

$$\sum_{i=1}^{12} \frac{x_i(x_i-1)}{2} \geq 12 \times 6 = 72 > 66$$

DONE.

**There is no 3-coloring of  
 $11 \times 11$**

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{11,11}$ .

$$11 \times 11 = 121$$

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Assume there is a 3-coloring of  $G_{11,11}$ .

$$11 \times 11 = 121$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{121}{3} \rceil = 41$ .

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{11,11}$ .

$$11 \times 11 = 121$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{121}{3} \rceil = 41$ .

For  $1 \leq i \leq 11$

let  $x_i$  be the number of elements of  $X$  in the  $i$ th column.

Need

$$\sum_{i=1}^{11} \binom{x_i}{2} \geq \binom{11}{2} = 55.$$

## MIN the sum

Over the REALS:

$$\sum_{i=1}^{11} \frac{x_i(x_i-1)}{2}$$

relative to the constraint

$$\sum_{i=1}^{11} x_i = 41.$$

is MINIMIZED if all of the  $x_i$ 's are equal.

We take  $x_i = \frac{41}{11}$

We note that  $\frac{1}{2} \times \frac{41}{11} \left( \frac{41}{11} - 1 \right) = \frac{615}{121}$ .

$$\sum_{i=1}^{11} \frac{x_i(x_i-1)}{2} \geq 11 \frac{615}{121} = 55+ > 55$$

DONE.



**There is no 3-coloring of  
 $10 \times 10$**

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{10,10}$ .

$$10 \times 10 = 100$$

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$$10 \times 10 = 100$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{100}{3} \rceil = 34$ .

# Rectangle Free Sets

Assume there is a 3-coloring of  $G_{10,10}$ .

$$10 \times 10 = 100$$

There is a rectangle free set  $X$ ,  $|X| \geq \lceil \frac{100}{3} \rceil = 34$ .

For  $1 \leq i \leq 10$

let  $x_i$  be the number of elements of  $X$  in the  $i$ th column.

Need

$$\sum_{i=1}^{10} \binom{x_i}{2} \geq \binom{10}{2} = 45.$$

## MIN the sum

Over the REALS:

$$\sum_{i=1}^{10} \frac{x_i(x_i-1)}{2}$$

relative to the constraint

$$\sum_{i=1}^{10} x_i = 34.$$

is MINIMIZED if all of the  $x_i$ 's are equal.

$$\text{We take } x_i = \frac{34}{10} = \frac{17}{5}$$

$$\text{We note that } \frac{1}{2} \times \frac{17}{5} \left( \frac{17}{5} - 1 \right) = \frac{102}{25}.$$

$$\sum_{i=1}^{10} \frac{x_i(x_i-1)}{2} \geq 10 \frac{102}{25} = 41+ > 45$$

THAT LAST LINE IS FALSE! So DO NOT have Proof that  $G_{10,10}$  is NOT 3-col.