TAKE HOME MIDTERM FOR CMSC 752
Morally Due April 1
Dead Cat Day April 3

Rules Same as the HW: You can get help but you must understand and hand in your own work. You can use ChatGPT if you want to hand in answers that are really bad.

There are four problems and they add up to 100 points.

1. (25 points) Let L be an ordered set.

   Let COL: $L \rightarrow [d]$.

   A set H is *c-homog (relative to* COL*)* if COL restricted to H takes on $\leq c$ values.

   (a) (15 points)

   Find c such that the following is true:

   - For all d, for all COL: $\mathsf{Q} \rightarrow [d]$ there exists a c-homog set $H \equiv \mathsf{Q}$.
   - There exists COL: $\mathsf{Q} \rightarrow [c]$ such that there are no $c-1$-homog sets $H \equiv \mathsf{Q}$.

   (b) (10 points) Let $L = \mathsf{Z} + \cdots + \mathsf{Z}$ (there are n copies of Z).

   Find c such that the following is true:

   - For all d, for all COL: $L \rightarrow [d]$ there exists a c-homog set $H \equiv L$.
   - There exists COL: $L \rightarrow [c]$ such that there are no $c-1$-homog sets $H \equiv L$.

2. (25 points) For this problem you may assume the following

   $R_a(k)$ is the least n such that every $\mathrm{COL}\colon \binom{[n]}{a} \rightarrow [2]$ has a homog set of size k.

   For this problem you may assume the following:

   $R_1(k) \le 2k$

   $R_2(k) \le 2^{2k}$

   $R_3(k) \le 2^{2^{4k}}$.

   (These are not the best known upper bounds but use them as the math will be neater.)

   Fill in the function $XXX(k)$ in the statement below so that the statement is true. Prove your statement. Try to use an $XXX(k)$ that is small.

   For all k there exists $n = XXX(k)$ such that, for all triples of colorings:

   $\mathrm{COL}_1\colon [n] \rightarrow [2]$

   $\mathrm{COL}_2\colon \binom{[n]}{2} \rightarrow [2]$

   $\mathrm{COL}_3\colon \binom{[n]}{3} \rightarrow [2]$

   there exists H of size k such that

   H is homog for COL_1 (all elements of H map to same color).

   H is homog for COL_2 (all elements of $\binom{H}{2}$ map to the same color).

   H is homog for COL_3 (all elements of $\binom{H}{3}$ map to the same color).

3. (30 points) This problem has three parts: a, b, c.

In this problem we will guide you through a proof of a theorem; however, you will need to fill in the XXX. Here is the theorem:

Theorem *Let* $c \ge 2$. *Let* $n = XXX(c)$. *For all* $\mathrm{COL}\colon \binom{[n]}{2} \to [c]$ *there exists a monochromatic* C_4. *That is, four distinct vertices* a, b, c, d *such that* $\mathrm{COL}(a,b) = \mathrm{COL}(b,c) = \mathrm{COL}(c,d) = \mathrm{COL}(d,a)$. *($XXX(c)$ will be a polynomial.)*

Proof

Plan We will find a function $E(n)$ such that any graph with n vertices and $E(n)$ edges has a C_4. We will then find an $n = XXX(c)$ such that, for all $\mathrm{COL}\colon \binom{[n]}{2} \to [c]$, some color occurs in $E(n)$ edges. That subgraph must have a C_4, which will be the desired monochromatic C_4.

Let $G = (V, E)$ where $V = [n]$. Assume G does not have C_4. (We are not going towards a contradiction. We will instead get an upper bound on $|E|$.)

Notation

$$e_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \text{ or } i = j \end{cases} \tag{1}$$

(a) (10 points) Let $a, b \in V$. Show that

$$\sum_{i=1}^{n} e_{ai} e_{ib} \le 1.$$

(b) (10 points) Assuming the result of Part 1 we have

$$\sum_{i=1}^{n} e_{ai}e_{ib} \leq 1.$$

We now take a summation over a and over $b \neq a$ on both sides.

$$\sum_{a=1}^{n} \sum_{b=1,b\neq a}^{n} \sum_{i=1}^{n} e_{ai}e_{ib} \leq \sum_{a=1}^{n} \sum_{b=1,b\neq a}^{n} 1.$$

$$\sum_{i=1}^{n}\sum_{a=1}^{n} \sum_{b=1,b\neq a}^{n} e_{ai}e_{ib} \leq n(n-1).$$

$$\sum_{i=1}^{n}\sum_{a=1}^{n} e_{ai} \sum_{b=1,b\neq a}^{n} e_{ib} \leq n(n-1).$$

Note that $\sum_{b=1,b\neq a}^{n} e_{ib} \in \{\deg(i), \deg(i)-1\}$, so

$$\sum_{b=1,b\neq a}^{n} e_{ib} \geq \deg(i) - 1.$$

Hence we have

$$\sum_{i=1}^{n}\sum_{a=1}^{n} e_{ai}(\deg(i)-1) \leq n(n-1).$$

$$\sum_{i=1}^{n}(\deg(i)-1)\sum_{a=1}^{n} e_{ai} \leq n(n-1).$$

$$(\sum_{i=1}^{n}(\deg(i)-1))\deg(i) \leq n(n-1).$$

$$(\sum_{i=1}^{n}(\deg(i)-1))\deg(i) \leq n(n-1).$$

$$\sum_{i=1}^{n}\deg(i)^2 - \sum_{i=1}^{n}\deg(i) \leq n(n-1).$$

Finally, here is the problem:
Show that

$$\sum_{i=1}^{n} \deg(i) \leq 0.5n + n\sqrt{n - 0.75}$$

Hint Use the Cauchy-Schwartz Inequality AND the Quadratic Formula.

(c) (10 points) We have

$$\sum_{i=1}^{n} \deg(i) \le 0.5n + n\sqrt{n-0.75}$$

Since $\sum_{i=1}^{n} \deg(i) = 2|E|$ we have

$$2|E| \le 0.5n + n\sqrt{n-0.75}$$

$$|E| \le 0.25n + 0.5n\sqrt{n-0.75}$$

So we have the following:

Let $G = (V, E)$.
If G has no C_4 subgraph then $|E| \le 0.25n + 0.5n\sqrt{n-0.75}$

Take the contrapositive:

Let $G = (V, E)$.
If $|E| > 0.25n + 0.5n\sqrt{n-0.75}$ then G has a C_4 subgraph.

Finally, here is the problem:
Find a polynomial $XXX(c)$ such that the following is true, and prove it:
Let $c \ge 2$. Let $n = XXX(c)$. Then for all $\mathrm{COL}\colon \binom{[n]}{2} \to [c]$ *there exists a monochromatic C_4.*

4. (20 points) Recall that $5\omega = \omega + \omega + \omega + \omega + \omega$.

   Let COL: $\binom{5\omega}{3} \rightarrow [10^{10}]$.

   A set H is *c-homog (relative to COL)* if COL restricted to $\binom{H}{3}$ takes only c colors.

   Find a number c such that the following is true, and prove both statements.

   - For all COL: $\binom{5\omega}{3} \rightarrow [10^{10}]$ there exists a c-homog set $H \equiv 5\omega$.
   - There exists COL: $\binom{5\omega}{3} \rightarrow [c]$ there is no $c-1$-homog set $H \equiv 5\omega$.

   A student asked if I allow you to use the result for $\binom{5\omega}{2}$ from the HW.

   I said YES.

   Then the student said *WAIT! That doesn't work!*

   My verdict: You can use it but it might not help.