CMSC 752 Homework 9
Morally Due Tue April 8, 2025
Dead Cat April 10

1. (50 points) In class we proved the following:

   Thm Let $A \subseteq \mathsf{R}$, $|A| = n$. Let L, R be such that for all $x \in A$, $L \le x \le R$. There exists a real $a \in [L, R]$ such that the following set has size roughly $n/3$:

   $$B_a = \{x \in A \colon \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

   The proof is nonconstructive, so we don't actually find the a. In this problem we find the a.

   We will consider the following subcase and restatement of the theorem.

   Thm Let $n, N \in \mathsf{N}$. Let $A \subseteq \{1, \ldots, N\}$ such that $|A| = n$. There exists $1 \le a \le N$ such that

   $$B_a = \{x \in A \colon \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

   has cardinality $\sim \frac{n}{3}$

And now (finally) for the question.

(a) (Nothing to hand in for this step.) Write a program that will, on input $n, N \in \mathsf{N}$ (with $n < N$) and $a \in [1, N]$ (a rational) do the following.

   i. Produce a set $A \subseteq \{1, \ldots, N\}$ as follows:
      For $i = 1$ to N pick i to be in A with probability $\frac{n}{N}$.
      Note that A will be of size close to n.

   ii. Find the size of

$$B_a = \{x \in A\colon \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

   For future reference let the output of this program be denoted $\mathbf{f(n, N, a)}$.
   (When debugging your code you should also output B_a itself.)

(b) (Nothing to hand in for this step.)
Write a program that will, given n, find
$f(n, n^2, \frac{1}{n})$, $f(n, n^2, \frac{2}{n})$, $f(n, n^2, \frac{3}{n})$, ..., $f(n, n^2, \frac{n-1}{n})$.
Our interest is when $f(n, n^2, a)$ is large. In particular, we are curious when it is

- between $n/4$ and $n/3$
- over $n/3$

With that in mind, output a table of the following form (the numbers are made up and we only go 5 rows).
$n = 36$

a	$f(36, 1296, a)$	$9 \le f(36, 1296, a) \le 11$?	$f(36, 1296, a) \ge 12$?
$\frac{1}{36}$	13	Y	Y
$\frac{2}{36}$	7	N	N
$\frac{3}{36}$	10	Y	N
$\frac{4}{36}$	5	N	N
$\frac{5}{36}$	18	Y	Y

Call this program PROG.

(c) (This part you don't hand in.) Write a program that, given n, runs the program in the last part BUT only outputs the rows with the top 10 sizes of B_a.

(d) (20 points) (This part you hand in.) Run PROG(100) and hand in the table. What percent of the rows had sum free sets of size between 25 and 33? Above 34?

(e) (20 points) Run PROG(200) but don't hand in the table. What percent of the rows had sum free sets of size between 50 and 66? Above 67?

(f) (10 points) Based on the answers to the last two question, make a conjecture about what happens as n increases (e.g., the fraction of rows that are sum free sets of size above $n/3$ increases).

(g) (On your own) Play with the program by varying n and N. Raise questions and get empirical evidence for them. The key question is: *do most sets have a sum-free set larger than* $n/3$ *and if so how much larger?*

2. (50 points) Prove of disprove:
   $\forall \mathrm{COL}\colon \binom{Q}{2} \to [2],\ \exists H \textit{ homog},\ H \equiv \mathsf{Q}.$

3. (Extra Credit)

   (a) STATE YOUR NAME.

   (b) Prove of disprove:
   $\forall \mathrm{COL}\colon \mathsf{R} \to [2],\ \exists H \textit{ homog},\ H \equiv \mathsf{R}.$