CMSC 752 Homework 6
Morally Due Tue March 11, 2025
Dead Cat March 13

1. (50 points) Show that the following is a well-quasi-ordering

   - The set is $\{a, b\}^*$, so all strings over a, b.
   - The ordering is subsequence:
     $x \preceq y$ if there is a way to remove letters from y and obtain x.
     Examples
     $aabba \preceq abbbabbbaa$
     $baaa \preceq abbbabbbaa$
     $baaaa \not\preceq abbbabbbaa$

   Obvious Hint Use a Minimal Bad Sequence Argument.

2. (50 points) Here is some background on this problem to tell you why it is interesting:

   Let $\preceq_m$ be the graph minor ordering.

   - Kruskal showed that the set of *rooted trees* under $\preceq_m$ is a wqo. I did this in class.
   - Robertson & Seymour showed that the set of *graphs* under $\preceq_m$ is a wqo. (This was very hard.)
   - Look at the set of planar graphs. Note that if G is planar and $H \preceq G$ then H is planar. Hence the set of planar graph is *closed downward under* $\preceq_m$.
   - Wagner's Theorem: A graph is planar iff it does not have K_5 or $K_{3,3}$ as a minor. We rewrite that:
     G is planar iff $(K_5 \not\preceq_m G \wedge K_{3,3} \not\preceq_m G)$.
   - Wagner's theorem can also be seen as a consequence of the Graph Minor Theorem and THIS HW assignment.

   Let $(X, \preceq)$ be a well quasi order. ADDED LATER: X is countable.

   Let $Y \subseteq X$

   Assume Y is closed downward: if $a \in Y$ and $b \preceq a$ then $b \in Y$.

   Show that there exists a finite subset of X, $\{o_1, \ldots, o_m\}$ such that

   $x \in Y$ iff $(o_1 \not\preceq x \wedge \cdots \wedge o_m \not\preceq x)$.

3. (0 points, Extra Credit)

   (Note: I do not know how to solve this problem. I do not know if it has been solved.)

   (a) Give your name since Bill is grading it and needs to know your name. (This is not the part thats hard.)

   (b) Prove the following without using Ramsey Theory:

   For all n, for all primitive recursive functions $f\colon \mathsf{N}^n \to \mathsf{N}$, there exists an infinite set $\mathsf{D} \subseteq \mathsf{N}$ such that f restricted to D^n is not onto.