A Small NFA for $\{a^i : i \neq 1000\}$

YOU"VE BEEN PUNKED!

The last slide of the last talk said that

$$L = \{a^i : i \neq 1000\}$$

requires an NFA of size ~ 1000.

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requires an NFA of size ~ 1000.

And that the proof used Ramsey Theory.

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Actually L can be done with a **smaller** NFA.

How Small?

How Small is the NFA for L

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VOTE. Let s be numb states in smallest NFA for L that Bill knows.

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2. $400 \le s \le 699$

How Small is the NFA for L

VOTE. Let s be numb states in smallest NFA for L that Bill knows.

1. $700 \le s \le 900$
2. $400 \le s \le 699$
3. $100 \le s \le 399$

How Small is the NFA for L

VOTE. Let s be numb states in smallest NFA for L that Bill knows.

1. $700 \le s \le 900$
2. $400 \le s \le 699$
3. $100 \le s \le 399$
4. $s \le 99$

$L = \{a^n : n \neq 1000\}$

Answer There is an NFA for L with 70 states.

$L = \{a^n : n \neq 1000\}$

Answer There is an NFA for L with 70 states.
This will take a few slides.

$L = \{a^n : n \neq 1000\}$

Answer There is an NFA for L with 70 states.
This will take a few slides.
And there will be an **important moral to the story**.

Overall Method

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Two NFA's:

Overall Method

Two NFA's:

NFA A:

Overall Method

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.

Overall Method

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NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.

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Two NFA's:

NFA A:

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- We have no comment on what it does on words ≤ 999.

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NFA B:

Overall Method

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.
- We have no comment on what it does on words ≤ 999.

NFA B:

- Does NOT accept a^{1000}.

Overall Method

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.
- We have no comment on what it does on words ≤ 999.

NFA B:

- Does NOT accept a^{1000}.
- Accepts all words shorter than 1000.

Overall Method

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.
- We have no comment on what it does on words ≤ 999.

NFA B:

- Does NOT accept a^{1000}.
- Accepts all words shorter than 1000.
- We have no comment on what it does on words ≥ 1001.

Overall Method

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.
- We have no comment on what it does on words ≤ 999.

NFA B:

- Does NOT accept a^{1000}.
- Accepts all words shorter than 1000.
- We have no comment on what it does on words ≥ 1001.

Create the union of NFA's A and B.

NFA A

Sums of 32's and 33's

Thm

Sums of 32's and 33's

Thm

1. For all $n \geq 992$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y$.

Sums of 32's and 33's

Thm

1. For all $n \geq 992$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y$.
2. There does not exist $x, y \in \mathbb{N}$ such that $991 = 32x + 33y$.

Sums of 32's and 33's

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2. There does not exist $x, y \in \mathbb{N}$ such that $991 = 32x + 33y$.

Write down this theorem! Will prove on next few slides and you need to know what I am proving.

Sums of 32's and 33's

Thm

1. For all $n \geq 992$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y$.
2. There does not exist $x, y \in \mathbb{N}$ such that $991 = 32x + 33y$.

Write down this theorem! Will prove on next few slides and you need to know what I am proving.
We will prove this by induction.
Base Case $992 = 32 \times 31 + 33 \times 0$.

$(\forall n \geq 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \geq 993$ and $(\exists x', y')[n - 1 = 32x' + 33y']$.

$(\forall n \geq 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \geq 993$ and $(\exists x', y')[n-1 = 32x' + 33y']$.
Intuition Want to swap coins in and out to increase by 1. Can swap out a 32-coin and put in a 33-coin

$(\forall n \geq 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \geq 993$ and
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Intuition Want to swap coins in and out to increase by 1. Can swap out a 32-coin and put in a 33-coin if I HAVE a 32-coin.

$(\forall n \ge 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \ge 993$ and $(\exists x', y')[n - 1 = 32x' + 33y']$.
Intuition Want to swap coins in and out to increase by 1. Can swap out a 32-coin and put in a 33-coin if I HAVE a 32-coin.
Case 1 $x' \ge 1$. Then $n = 32(x' - 1) + 33(y' + 1)$.

$(\forall n \ge 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \ge 993$ and $(\exists x', y')[n - 1 = 32x' + 33y']$.
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Case 1 $x' \ge 1$. Then $n = 32(x' - 1) + 33(y' + 1)$.
Intuition What to do if $x' = 0$. Need to remove some 33's and add some 32's. Use that $32 \times 32 - 31 \times 33 = 1024 - 1023 = 1$. Can swap out 31 33-coins and put in 32 32-coins

$(\forall n \geq 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \geq 993$ and $(\exists x', y')[n-1 = 32x' + 33y']$.
Intuition Want to swap coins in and out to increase by 1. Can swap out a 32-coin and put in a 33-coin if I HAVE a 32-coin.
Case 1 $x' \geq 1$. Then $n = 32(x'-1) + 33(y'+1)$.
Intuition What to do if $x' = 0$. Need to remove some 33's and add some 32's. Use that $32 \times 32 - 31 \times 33 = 1024 - 1023 = 1$. Can swap out 31 33-coins and put in 32 32-coinsif I HAVE 31 33-coins.

$(\forall n \geq 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \geq 993$ and $(\exists x', y')[n - 1 = 32x' + 33y']$.
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Case 2 $y' \geq 31$. Then $n = 32(x' + 32) + 33(y' - 31)$.

$(\forall n \geq 992)(\exists x, y \in N)[n = 32x + 33y]$

Inductive Hypothesis $n \geq 993$ and $(\exists x', y')[n - 1 = 32x' + 33y']$.

Intuition Want to swap coins in and out to increase by 1. Can swap out a 32-coin and put in a 33-coin if I HAVE a 32-coin.

Case 1 $x' \geq 1$. Then $n = 32(x' - 1) + 33(y' + 1)$.

Intuition What to do if $x' = 0$. Need to remove some 33's and add some 32's. Use that $32 \times 32 - 31 \times 33 = 1024 - 1023 = 1$. Can swap out 31 33-coins and put in 32 32-coinsif I HAVE 31 33-coins.

Case 2 $y' \geq 31$. Then $n = 32(x' + 32) + 33(y' - 31)$.

Case 3 $x' \leq 0$ and $y' \leq 30$. Then $n - 1 = 32x' + 33y' \leq 33 \times 30 = 990 < 993$, so cannot occur.

There is no $x, y \in N$ with $991 = 32x + 33y$

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Pf by contradiction.

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Pf by contradiction.

Assume there exists $x, y \in \mathbb{N}$ such that

$$991 = 32x + 33y$$

There is no $x, y \in N$ with $991 = 32x + 33y$

Pf by contradiction.
Assume there exists $x, y \in \mathbb{N}$ such that

$$991 = 32x + 33y$$

Then

$$991 \equiv 32x + 33y \pmod{32}$$

There is no $x, y \in N$ with $991 = 32x + 33y$

Pf by contradiction.

Assume there exists $x, y \in \mathbb{N}$ such that

$$991 = 32x + 33y$$

Then

$$991 \equiv 32x + 33y \pmod{32}$$

$$31 \equiv 0x + 1y \pmod{32}$$

There is no $x, y \in N$ with $991 = 32x + 33y$

Pf by contradiction.
Assume there exists $x, y \in \mathbb{N}$ such that

$$991 = 32x + 33y$$

Then

$$991 \equiv 32x + 33y \pmod{32}$$

$$31 \equiv 0x + 1y \pmod{32}$$

$$31 \equiv y \pmod{32} \text{ So } y \geq 31$$

There is no $x, y \in N$ with $991 = 32x + 33y$

Pf by contradiction.
Assume there exists $x, y \in \mathbb{N}$ such that

$$991 = 32x + 33y$$

Then

$$991 \equiv 32x + 33y \pmod{32}$$

$$31 \equiv 0x + 1y \pmod{32}$$

$$31 \equiv y \pmod{32} \text{ So } y \geq 31$$

$991 = 32x + 33y \geq 32x + 33 \times 31 \geq 1023$ **Contradiction!**

Sums of 32's and 33's and ONE 9

Thm

1) For all $n \geq 1001$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y + 9$.

Sums of 32's and 33's and ONE 9

Thm

1) For all $n \geq 1001$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y + 9$.

2) There does not exist $x, y \in \mathbb{N}$ such that $1000 = 32x + 33y + 9$.

Sums of 32's and 33's and ONE 9

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Sums of 32's and 33's and ONE 9

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Pf

1) If $n \geq 1001$ then $n - 9 \geq 992$ so by prior Thm

$$(\exists x, y \in \mathbb{N})[n - 9 = 32x + 33y]$$

Sums of 32's and 33's and ONE 9

Thm

1) For all $n \geq 1001$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y + 9$.

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$$(\exists x, y \in \mathbb{N})[n - 9 = 32x + 33y]$$

$$(\exists x, y \in \mathbb{N})[n = 32x + 33y + 9]$$

Sums of 32's and 33's and ONE 9

Thm

1) For all $n \geq 1001$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y + 9$.

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$$(\exists x, y \in \mathbb{N})[n - 9 = 32x + 33y]$$

$$(\exists x, y \in \mathbb{N})[n = 32x + 33y + 9]$$

2) Assume, by way of contradiction,

$$(\exists x, y)[1000 = 32x + 33y + 9]$$

Sums of 32's and 33's and ONE 9

Thm

1) For all $n \geq 1001$ there exists $x, y \in \mathbb{N}$ such that $n = 32x + 33y + 9$.

2) There does not exist $x, y \in \mathbb{N}$ such that $1000 = 32x + 33y + 9$.

Pf

1) If $n \geq 1001$ then $n - 9 \geq 992$ so by prior Thm

$$(\exists x, y \in \mathbb{N})[n - 9 = 32x + 33y]$$

$$(\exists x, y \in \mathbb{N})[n = 32x + 33y + 9]$$

2) Assume, by way of contradiction,

$$(\exists x, y)[1000 = 32x + 33y + 9]$$

$$(\exists x, y)[992 = 32x + 33y]$$

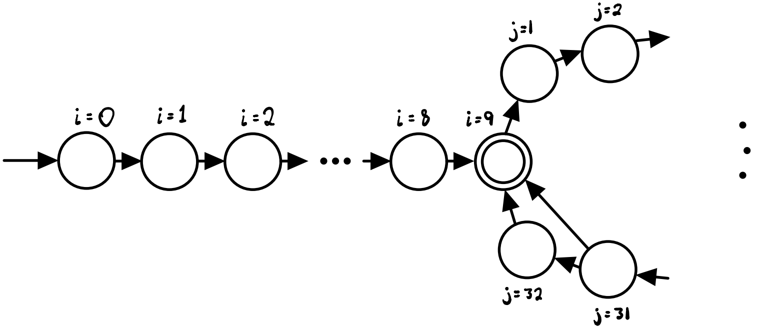
This contradicts prior Thm.

NFA A

Idea Start state, then 8 states, then a loop of size 33 with a shortcut at 32.

NFA A

Idea Start state, then 8 states, then a loop of size 33 with a shortcut at 32.



Number of States for $\{a^i : i \geq 1001\}$

Number of States for $\{a^i : i \geq 1001\}$

1. Start state

Number of States for $\{a^i : i \geq 1001\}$

1. Start state
2. A chain of 9 states including the start state.

Number of States for $\{a^i : i \geq 1001\}$

1. Start state
2. A chain of 9 states including the start state.
3. A loop of 33 states. The shortcut on 32 does not affect the number of states.

Number of States for $\{a^i : i \geq 1001\}$

1. Start state
2. A chain of 9 states including the start state.
3. A loop of 33 states. The shortcut on 32 does not affect the number of states.

Total number of states: $9 + 33 = 42$.

NFA B

Still Need NFA B

Still Need NFA B

Idea

Still Need NFA B

Idea

$1000 \equiv 0 \pmod 2$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 2\}$.
2-state DFA.

Still Need NFA B

Idea

$1000 \equiv 0 \pmod 2$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 2\}$.
2-state DFA.

$1000 \equiv 1 \pmod 3$ SO want to accept $\{a^i : i \not\equiv 1 \pmod 3\}$.
3-state DFA.

Still Need NFA B

Idea

$1000 \equiv 0 \pmod 2$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 2\}$.
2-state DFA.

$1000 \equiv 1 \pmod 3$ SO want to accept $\{a^i : i \not\equiv 1 \pmod 3\}$.
3-state DFA.

$1000 \equiv 0 \pmod 5$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 5\}$.
5-state DFA.

Still Need NFA B

Idea

$1000 \equiv 0 \pmod 2$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 2\}$.
2-state DFA.

$1000 \equiv 1 \pmod 3$ SO want to accept $\{a^i : i \not\equiv 1 \pmod 3\}$.
3-state DFA.

$1000 \equiv 0 \pmod 5$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 5\}$.
5-state DFA.

$1000 \equiv 6 \pmod 7$ SO want to accept $\{a^i : i \not\equiv 6 \pmod 7\}$.
7-state DFA.

Still Need NFA B

Idea

$1000 \equiv 0 \pmod 2$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 2\}$.
2-state DFA.

$1000 \equiv 1 \pmod 3$ SO want to accept $\{a^i : i \not\equiv 1 \pmod 3\}$.
3-state DFA.

$1000 \equiv 0 \pmod 5$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 5\}$.
5-state DFA.

$1000 \equiv 6 \pmod 7$ SO want to accept $\{a^i : i \not\equiv 6 \pmod 7\}$.
7-state DFA.

$1000 \equiv 10 \pmod{11}$ SO want to accept $\{a^i : i \not\equiv 10 \pmod{11}\}$.
11-state DFA.

Still Need NFA B

Idea

$1000 \equiv 0 \pmod 2$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 2\}$.
2-state DFA.

$1000 \equiv 1 \pmod 3$ SO want to accept $\{a^i : i \not\equiv 1 \pmod 3\}$.
3-state DFA.

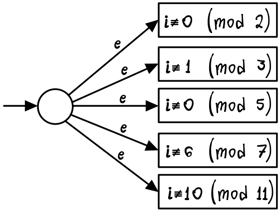
$1000 \equiv 0 \pmod 5$ SO want to accept $\{a^i : i \not\equiv 0 \pmod 5\}$.
5-state DFA.

$1000 \equiv 6 \pmod 7$ SO want to accept $\{a^i : i \not\equiv 6 \pmod 7\}$.
7-state DFA.

$1000 \equiv 10 \pmod{11}$ SO want to accept $\{a^i : i \not\equiv 10 \pmod{11}\}$.
11-state DFA.
Could go on to 13,17, etc. But we will see we can stop here.

Machine B

Machine B


e
e
e
e
e
$i \neq 0 \pmod 2$
$i \neq 1 \pmod 3$
$i \neq 0 \pmod 5$
$i \neq 6 \pmod 7$
$i \neq 10 \pmod{11}$

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

Thm Let M be the NFA from the last slide.
$M(a^{1000})$ is rejected. This is obvious.
For all $0 \le i \le 999$, $M(a^i)$ is accepted.

NFA for $\{a^i : i \leq 999\}$ AND More, but NOT a^{1000}

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$M(a^{1000})$ is rejected. This is obvious.
For all $0 \leq i \leq 999$, $M(a^i)$ is accepted.
Pf We show that if $M(a^i)$ is rejected then $i \geq 1000$. Assume $M(a^i)$ rejected. Then

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

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For all $0 \le i \le 999$, $M(a^i)$ is accepted.
Pf We show that if $M(a^i)$ is rejected then $i \ge 1000$. Assume $M(a^i)$ rejected. Then
$i \equiv 0 \pmod 2$
$i \equiv 1 \pmod 3$
$i \equiv 0 \pmod 5$
$i \equiv 6 \pmod 7$
$i \equiv 10 \pmod{11}$

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

Thm Let M be the NFA from the last slide.
$M(a^{1000})$ is rejected. This is obvious.
For all $0 \le i \le 999$, $M(a^i)$ is accepted.
Pf We show that if $M(a^i)$ is rejected then $i \ge 1000$. Assume $M(a^i)$ rejected. Then
$i \equiv 0 \pmod 2$
$i \equiv 1 \pmod 3$
$i \equiv 0 \pmod 5$
$i \equiv 6 \pmod 7$
$i \equiv 10 \pmod{11}$
Continued on next slide

NFA for $\{a^i : i \leq 999\}$ AND More, but NOT a^{1000}

NFA for $\{a^i : i \leq 999\}$ AND More, but NOT a^{1000}

$i \equiv 0 \pmod{2}$
$i \equiv 1 \pmod{3}$

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

$i \equiv 0 \pmod 2$
$i \equiv 1 \pmod 3$
Hence $i \equiv 4 \pmod 6$.

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

$i \equiv 0 \pmod 2$
$i \equiv 1 \pmod 3$
Hence $i \equiv 4 \pmod 6$.

$i \equiv 0 \pmod 5$
$i \equiv 6 \pmod 7$

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

$i \equiv 0 \pmod 2$
$i \equiv 1 \pmod 3$
Hence $i \equiv 4 \pmod 6$.

$i \equiv 0 \pmod 5$
$i \equiv 6 \pmod 7$

Hence $i \equiv 20 \pmod{35}$.

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

$i \equiv 0 \pmod{2}$
$i \equiv 1 \pmod{3}$
Hence $i \equiv 4 \pmod{6}$.

$i \equiv 0 \pmod{5}$
$i \equiv 6 \pmod{7}$

Hence $i \equiv 20 \pmod{35}$.

$i \equiv 10 \pmod{11}$

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

$i \equiv 0 \pmod 2$
$i \equiv 1 \pmod 3$
Hence $i \equiv 4 \pmod 6$.

$i \equiv 0 \pmod 5$
$i \equiv 6 \pmod 7$

Hence $i \equiv 20 \pmod{35}$.

$i \equiv 10 \pmod{11}$

So we have
$i \equiv 4 \pmod 6$
$i \equiv 20 \pmod{35}$
$i \equiv 10 \pmod{11}$.
Continued on next slide

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}?

From:
$i \equiv 4 \pmod 6$
$i \equiv 20 \pmod{35}$
$i \equiv 10 \pmod{11}$.
One can show
$i \equiv 1000 \pmod{6 \times 35 \times 11}$

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}?

From:
$i \equiv 4 \pmod{6}$
$i \equiv 20 \pmod{35}$
$i \equiv 10 \pmod{11}$.
One can show
$i \equiv 1000 \pmod{6 \times 35 \times 11}$
So
$i \equiv 1000 \pmod{2310}$
Hence $i \ge 1000$.

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}?

From:
$i \equiv 4 \pmod 6$
$i \equiv 20 \pmod{35}$
$i \equiv 10 \pmod{11}$.
One can show
$i \equiv 1000 \pmod{6 \times 35 \times 11}$
So
$i \equiv 1000 \pmod{2310}$
Hence $i \ge 1000$.
Recap If a^i is rejected then $i \ge 1000$.
Hence If $i \le 999$ then a^i is accepted.

How Many States for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}?

$2 + 3 + 5 + 7 + 11 = 28$ states.
Plus the start state, so 29.

NFA for $\{a^i : i \neq 1000\}$

NFA for $\{a^i : i \neq 1000\}$

1. We have an NFA on 42 states that accepts $\{a^i : i \geq 1001\}$ This includes the start state.

NFA for $\{a^i : i \neq 1000\}$

1. We have an NFA on 42 states that accepts $\{a^i : i \geq 1001\}$ This includes the start state.
2. We have an NFA on 29 states that accepts $\{a^i : i \leq 999\}$ and other stuff, but NOT a^{1000}. This includes the start state.

NFA for $\{a^i : i \neq 1000\}$

1. We have an NFA on 42 states that accepts $\{a^i : i \geq 1001\}$ This includes the start state.
2. We have an NFA on 29 states that accepts $\{a^i : i \leq 999\}$ and other stuff, but NOT a^{1000}. This includes the start state.

Take NFA of union using e-transitions for an NFA and do not count start state twice, so have

$$42 + 29 - 1 = 70 \text{ states.}$$

Interesting Problem, Profound Moral

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1. In the Springs of 2015, 2016, 2017, 2018, 2019, 2020, and 2021, Gasarch has given this problem to the students in CMSC 452.

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Interesting Problem, Profound Moral

1. In the Springs of 2015, 2016, 2017, 2018, 2019, 2020, and 2021, Gasarch has given this problem to the students in CMSC 452.
2. Every year almost everyone thinks **The NFA requires $\sim n$ states**.
3. Why is this? They did not know the trick.

Interesting Problem, Profound Moral

1. In the Springs of 2015, 2016, 2017, 2018, 2019, 2020, and 2021, Gasarch has given this problem to the students in CMSC 452.
2. Every year almost everyone thinks **The NFA requires $\sim n$ states**.
3. Why is this? They did not know the trick.
4. **Moral Lesson** Lower bounds are hard! You have to rule out that someone does not have a very clever trick that you just had not thought of.

This was NOT a lecture on Size of NFAs

You thought this was a lecture on sizes of NFAs.

This was NOT a lecture on Size of NFAs

You thought this was a lecture on sizes of NFAs.

It was not.

This was NOT a lecture on Size of NFAs

You thought this was a lecture on sizes of NFAs.

It was not.

- This is a lecture on NP-completeness.

This was NOT a lecture on Size of NFAs

You thought this was a lecture on sizes of NFAs.

It was not.

- This is a lecture on NP-completeness.
- Just because you cannot think of an algorithm for SAT in P does not mean that there is not one.

This was NOT a lecture on Size of NFAs

You thought this was a lecture on sizes of NFAs.

It was not.

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- **Upshot** Lower bounds are hard to prove since they must rule out techniques you have not thought of.
- Respect the difficulty of lower bounds!

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Bill knows an NFA with ≤ 69 states.
There is an NFA for L with 59 states.

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See next slide.

The 59-state NFA for L

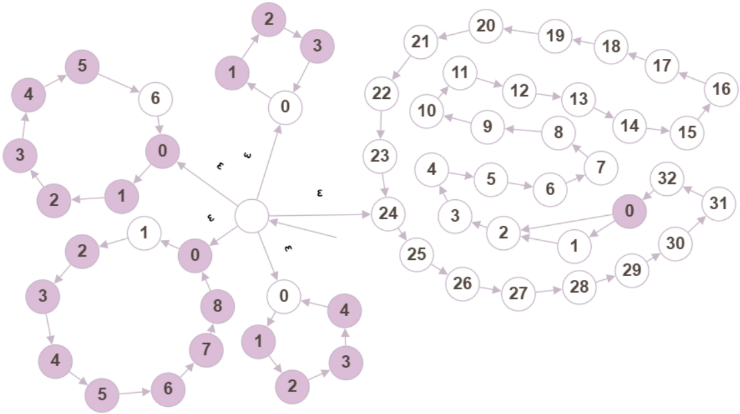

Figure: 59 State NFA for L

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   This would save us 8 states, because we still need a distinct start state.

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1. No, 59 is optimal
2. Yes, but not by much
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Answer: Unknown to science.

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Frobenius Thm (aka The Chicken McNugget Thm)

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t will be $\le \sqrt{n}$ (usually much less).
So the total number of states for this part is $\sim 2\sqrt{n}$.

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Thm Let $n \in \mathbb{N}$. Let $q_1, \ldots, q_k$ be rel prime such that $\prod_{i=1}^{k} q_i \ge n$. Then the set of all i such that
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So can use this to get NFA for $\{a^i : i \leq n-1\}$ (and other stuff but not a^n) with $\leq (\log n)^2 \log \log n$ states.

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Paper by Gasarch-Metz-Xu-Shen-Zbarsky.

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Is this interesting and/or important?

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Determinism versus Nondeterminism.