CFGs For Finite Unary Sets

Where this Talk Came From

This talk is based on the paper
Simulating Finite Automata with Context-Free Grammars by Domaratzki, Pighizzini, Shallit. Information Processing Letters, Volume 84, 2002,339-344.

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4) Our concern **is** the number of nonterminals (NTs).

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2) Towards the end of the talk I will tell you what the results are for the real Chomsky Normal Form.

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4) Does $\exists\ A \subseteq \{e, a, a^2, a^3, \ldots, a^n\}$ such that **every CNFG** for A has $\Omega(n^{1/3})$ NT's?

Goal: Small CFG for Subsets Of $\{e, a, a^2, a^3, \ldots, a^{124}\}$

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Key We are writing the numbers $0 \le m \le 124$ in base 5.

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Clearly $L(S) = \{a^0, \ldots, a^{124}\}$ and G has 16 NT's.

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One more point to make before we go to E_i, F_j, G_k.

An Example and a Point

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Keys For Later

1) This is the **only** way to get a^{87}

2) If the rule $S \rightarrow E_2 F_2 G_3$ is removed then the **only** string that is no longer generated is a^{87}.

E_0, E_1, E_2, E_3, E_4 for $\{e, a, a^2, a^3, a^4\}$

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$$\begin{array}{lll} E_0 \to e & \qquad & L(E_0) = \{e\} \\ E_1 \to a & \qquad & L(E_1) = \{a\} \end{array}$$

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F_0, F_1, F_2, F_3, F_4 for $\{e, a^5, a^{10}, a^{15}, a^{20}\}$

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$F_0 \rightarrow e \qquad L(F_0) = \{e\}.$

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G_0, G_1, G_2, G_3, G_4 for $\{e, a^{25}, a^{50}, a^{75}, a^{100}\}$

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G_0, G_1, G_2, G_3, G_4 for $\{e, a^{25}, a^{50}, a^{75}, a^{100}\}$

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$G_1 \to F_4 F_1 \qquad L(G_1) = \{a^{25}\}.$

G_0, G_1, G_2, G_3, G_4 for $\{e, a^{25}, a^{50}, a^{75}, a^{100}\}$

$$G_0 \to e \qquad L(G_0) = \{e\}.$$
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G_2 \to G_1 G_1 & & L(G_2) = \{a^{50}\}. \\
G_3 \to G_2 G_1 & & L(G_3) = \{a^{75}\}.
\end{array}
$$

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G_3 \to G_2 G_1 & & L(G_3) = \{a^{75}\}. \\
G_4 \to G_3 G_1 & & L(G_4) = \{a^{100}\}.
\end{array}
$$

Done and Recap

The CNFG Grammar is:

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1) The E_i's, F_j's, and G_k's as described above.

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1) The Grammar generates $\{e, a, \ldots, a^{124}\}$.

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The following are true:

1) The Grammar generates $\{e, a, \ldots, a^{124}\}$.

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Done and Recap

The CNFG Grammar is:

1) The E_i's, F_j's, and G_k's as described above.

2) For all $0 \leq i, j, k \leq 4$ the rule $S \rightarrow E_i F_j G_k$.

The following are true:

1) The Grammar generates $\{e, a, \ldots, a^{124}\}$.

2) The Grammar has 16 NT's.

3) Can modify the Grammar to omit **any** set of strings (next slide).

Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

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Let $A = \{e, a, a^2, \ldots, a^{124}\} - \{a^{20}, a^{49}, a^{87}\}$

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Let $A = \{e, a, a^2, \ldots, a^{124}\} - \{a^{20}, a^{49}, a^{87}\}$
$20 = 1 \times 0 + 5 \times 4 + 25 \times 0$. $S \to E_0 F_4 F_0 \Rightarrow a^0 a^{5\times 4} a^{25\times 0} = a^{20}$.

Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

Let $A = \{e, a, a^2, \ldots, a^{124}\} - \{a^{20}, a^{49}, a^{87}\}$
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Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

Let $A = \{e, a, a^2, \ldots, a^{124}\} - \{a^{20}, a^{49}, a^{87}\}$
$20 = 1 \times 0 + 5 \times 4 + 25 \times 0$. $S \to E_0 F_4 F_0 \Rightarrow a^0 a^{5 \times 4} a^{25 \times 0} = a^{20}$.
$49 = 1 \times 4 + 5 \times 4 + 25 \times 1$. $S \to E_4 F_4 G_1 \Rightarrow a^4 a^{5 \times 4} a^{25 \times 1} = a^{49}$.

Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

Let $A = \{e, a, a^2, \ldots, a^{124}\} - \{a^{20}, a^{49}, a^{87}\}$
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$87 = 1 \times 2 + 5 \times 2 + 25 \times 3$.

Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

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$87 = 1 \times 2 + 5 \times 2 + 25 \times 3$. $S \to E_2 F_2 G_3 \Rightarrow a^2 a^{5\times2} a^{25\times3} = a^{87}$.

Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

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To get a CNFG for A take the CNFG we described for

$$\{e, a, a^2, \ldots, a^{124}\}$$

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and **remove**

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and **remove**
$S \rightarrow E_0 F_4 G_0$
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We get a CNFG for A with 16 NTs.

Example of $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

Let $A = \{e, a, a^2, \ldots, a^{124}\} - \{a^{20}, a^{49}, a^{87}\}$
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To get a CNFG for A take the CNFG we described for

$$\{e, a, a^2, \ldots, a^{124}\}$$

and **remove**
$S \to E_0 F_4 G_0$
$S \to E_4 F_4 G_1$
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We get a CNFG for A with 16 NTs.

This trick works for **any** subset of $\{e, a, a^2, \ldots, a^{124}\}$.

Theorem about $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

Theorem about $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$

Thm For all $A \subseteq \{e, a, a^2, \ldots, a^{124}\}$ there is a CNFG with 16 NTs.

General Theorem About $A \subseteq \{e, a, a^2, \ldots, a^{t^3-1}\}$

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We will prove the following in a similar manner

General Theorem About $A \subseteq \{e, a, a^2, \ldots, a^{t^3-1}\}$

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Thm Let $t \in \mathbb{N}$. For all $A \subseteq \{e, a, a^2, \ldots, a^{t^3-1}\}$ there is a CNFG with $3t + 1$ NTs.

Small CFG for $\{e, a, a^2, \ldots, a^{t^3-1}\}$

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Plan

Small CFG for $\{\mathbf{e}, \mathbf{a}, \mathbf{a}^2, \ldots, \mathbf{a}^{t^3-1}\}$

Plan

1) Find rules and NTs $E_0, \ldots, E_{t-1}$, such that $L(E_i) = \{a^i\}$.

Small CFG for $\{e, a, a^2, \ldots, a^{t^3-1}\}$

Plan

1) Find rules and NTs $E_0, \ldots, E_{t-1}$, such that $L(E_i) = \{a^i\}$.

2) Find rules and NT's $F_0, \ldots, F_{t-1}$ such that $L(F_i) = \{a^{ti}\}$.

Small CFG for $\{e, a, a^2, \ldots, a^{t^3-1}\}$

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3) Find rules and NT's $G_0, \ldots, G_{t-1}$ such that $L(G_i) = \{a^{t^2 i}\}$.

How to Use E_i, F_j, G_k

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$\forall\ 0 \le i, j, k \le t-1$ add the rule

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$$S \rightarrow E_i F_j G_k.$$

How to Use E_i, F_j, G_k

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How to Use $\mathbf{E_i, F_j, G_k}$

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Let $0 \le n \le t^3 - 1$. We show why $S \Rightarrow a^n$.

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$\forall\ 0 \le i, j, k \le t-1$ add the rule

$$S \to E_i F_j G_k.$$

Let $0 \le n \le t^3 - 1$. We show why $S \Rightarrow a^n$.
Use base t: $n = n_0 + tn_1 + t^2 n_2$ where $n_0, n_1, n_2 \in \{0, \ldots, t-1\}$.

How to Use E_i, F_j, G_k

1) $E_0, \ldots, E_{t-1}$ such that $L(E_i) = \{a^i\}$.
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$$S \rightarrow E_{n_0} F_{n_1} G_{n_2} \Rightarrow a^{n_0} a^{tn_1} a^{t^2 n_2} = a^n.$$

How to Use E_i, F_j, G_k

1) $E_0, \ldots, E_{t-1}$ such that $L(E_i) = \{a^i\}$.
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$\forall\ 0 \le i, j, k \le t-1$ add the rule

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Let $0 \le n \le t^3 - 1$. We show why $S \Rightarrow a^n$.
Use base t: $n = n_0 + tn_1 + t^2 n_2$ where $n_0, n_1, n_2 \in \{0, \ldots, t-1\}$.

$$S \rightarrow E_{n_0} F_{n_1} G_{n_2} \Rightarrow a^{n_0} a^{tn_1} a^{t^2 n_2} = a^n.$$

Still need to describe the rules for E_i, F_j, G_k.

$E_0, \ldots, E_{t-1}$ for $\{e, a, a^2, \ldots, a^{t-1}\}$

$E_0, \ldots, E_{t-1}$ for $\{e, a, a^2, \ldots, a^{t-1}\}$

$E_0 \to e \qquad L(E_0) = \{e\}$

$E_0, \ldots, E_{t-1}$ for $\{e, a, a^2, \ldots, a^{t-1}\}$

$$
\begin{array}{ll}
E_0 \to e & \qquad L(E_0) = \{e\} \\
E_1 \to a & \qquad L(E_1) = \{a\}
\end{array}
$$

$E_0, \dots, E_{t-1}$ for $\{e, a, a^2, \dots, a^{t-1}\}$

$$
\begin{array}{ll}
E_0 \to e & L(E_0) = \{e\} \\
E_1 \to a & L(E_1) = \{a\} \\
E_2 \to E_1 E_1 & L(E_2) = \{aa\}
\end{array}
$$

$E_0, \ldots, E_{t-1}$ for $\{e, a, a^2, \ldots, a^{t-1}\}$

$$
\begin{array}{ll}
E_0 \to e & L(E_0) = \{e\} \\
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E_2 \to E_1 E_1 & L(E_2) = \{aa\} \\
E_3 \to E_2 E_1 & L(E_3) = \{aaa\}
\end{array}
$$

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$$
\begin{array}{lll}
E_0 \to e & & L(E_0) = \{e\} \\
E_1 \to a & & L(E_1) = \{a\} \\
E_2 \to E_1 E_1 & & L(E_2) = \{aa\} \\
E_3 \to E_2 E_1 & & L(E_3) = \{aaa\} \\
\vdots & \vdots & \vdots
\end{array}
$$

$E_0, \ldots, E_{t-1}$ for $\{e, a, a^2, \ldots, a^{t-1}\}$

$$
\begin{array}{lll}
E_0 \to e & & L(E_0) = \{e\} \\
E_1 \to a & & L(E_1) = \{a\} \\
E_2 \to E_1 E_1 & & L(E_2) = \{aa\} \\
E_3 \to E_2 E_1 & & L(E_3) = \{aaa\} \\
\vdots & \vdots & \vdots \\
E_{t-1} \to E_{t-2} E_1 & & L(E_{t-1}) = \{a^{t-1}\}
\end{array}
$$

$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

$$F_0 \to e \qquad L(F_0) = \{e\}.$$

$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

$F_0 \to e \qquad L(F_0) = \{e\}.$

$F_1 \to E_{t-1}E_1 \qquad L(F_1) = \{a^t\}.$

$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

$F_0 \rightarrow e \qquad L(F_0) = \{e\}.$

$F_1 \rightarrow E_{t-1}E_1 \qquad L(F_1) = \{a^t\}.$

$F_2 \rightarrow F_1F_1 \qquad L(F_2) = \{a^{2t}\}.$

$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

$$F_0 \to e \qquad L(F_0) = \{e\}.$$
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$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

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\vdots & \vdots & \vdots
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$F_0, \ldots, F_{t-1}$ for $\{e, a^t, a^{2t}, \ldots, a^{(t-1)t}\}$

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3) Can modify the Grammar to omit **any** set of strings (next slide).

Example of $A \subseteq \{e, a, a^2, \ldots, a^{t^3-1}\}$

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To omit $a^{n_0+tn_1+t^2n_2}$

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To omit many strings, remove many rules of the form $S \to$.

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Can We Do Better Than $O(n^{1/3})$ NTs?

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$\forall A \subseteq \{e, a, \ldots, a^n\}$ $\exists$ a CNFG with $O(n^{1/3-\epsilon})$ NTs, $\epsilon = \frac{1}{10^{40}}$.

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$\forall A \subseteq \{e, a, \ldots, a^n\}$ $\exists$ a CNFG with $O(n^{1/4})$ NTs.

$\exists A \subseteq \{e, a, \ldots, a^n\}$ all CNFGs have $\Omega(n^{1/3})$ NTs.

Can We Do Better Than $O(n^{1/3})$ NTs?

Thm $\exists A \subseteq \{e, a, \ldots, a^n\}$ $\forall$ CNFG have $\Omega(n^{1/3})$ NTs.

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We choose t later.

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So there are $\sim t^3 + O(t) \leq 2t^3$ possible rules.

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So there are $\leq 2^{2t^3}$ possible CFG's with t nonterminals.

Can We Do Better Than $O(n^{1/3})$ NTs?(cont)

Thm $\exists A \subseteq \{e, a, \ldots, a^n\}$ $\forall$ CNFG have $\Omega(n^{1/3})$ NTs.
There are 2^n subsets A.

Can We Do Better Than $O(n^{1/3})$ NTs?(cont)

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WE ARE DONE.