HW 11 CMSC 452
Morally Due TUES April 22 11:00AM
Dead-Cat Due THU April 24 11:00AM

1. (40 points) In class we did an example of taking a formula ϕ and producing a graph G and a number k such that

   ϕ is satisfiable IFF G has an ind. set of size k.

   The formula was in 3-CNF form which means it is of the form

   $C_1 \wedge \cdots \wedge C_k$ where each C_i is the $\vee$ of 3 literals.

   The reduction produced k $\triangle$s and then edges between the $\triangle$s.

   (a) (10 points) Write psuedocode for a program that takes any formula in 3-CNF form and outputs a graph G and a number k.

   Here is a HINT in that its the program with FILL IN THE BLANKS.

   Begin with

   i. Input $C_1 \wedge \cdots \vee C_k$ where
   $C_1 = (L_{11} \vee L_{12} \vee L_{13})$
   $C_2 = (L_{21} \vee L_{22} \vee L_{23})$
   $\vdots \quad \vdots \quad \vdots$
   $C_k = (L_{k1} \vee L_{k2} \vee L_{k3})$

   ii. V is YOU NEED TO DESCRIBE V.

   iii. E is YOU NEED TO DESCRIBE E.

   iv. Output (V, E).

   You do not need to prove or even mention that the program works in polynomial time. It will.

   (b) Describe the graph you get on input

   $$(\neg x \vee y \vee z) \wedge (x \vee \neg y \vee z) \wedge (w \vee \neg x \vee y) \wedge (\neg w \vee x \vee \neg y).$$

   (c) (0 points) Think about. Come up with a 3-CNF formula that is NOT in 3-SAT. Apply your algorithm to it. What does the graph look like?

2. (30 points) In this problem all numbers are written in binary. Hence the number x takes $\lg_2(x)$ bits to represent and hence is of LENGTH $\lg_2(x)$.

   In this problem all of the quantifiers range over $\{0, 1, 2, \ldots\}$.

   For $k \geq 2$ let

   $SQ_k = \{x\colon (\exists y_1, \ldots, y_k)[x = y_1^2 + \cdots + y_k^2]\}$.

   (a) (6 points)

   Show that, for all k, SQ_k is in NP.

   (Just describe the witness y and the set B.)

   (b) (6 points) Look on the web to find out what is known about the following questions: (Here and for later questions, you don't have to look on the web if you are sure you know the answer.)

   Is $SQ_2 \in$ P? Is SQ_2 NP-Complete?

   (c) (6 points) Look on the web to find out what is known about the following questions:

   Is $SQ_3 \in$ P? Is SQ_3 NP-Complete?

   (d) (6 points) Look on the web to find out what is known about the following questions:

   Is $SQ_4 \in$ P? Is SQ_4 NP-Complete?

   (e) (6 points) Look on the web to find out what is known about the following questions:

   Is $SQ_5 \in$ P? Is SQ_5 NP-Complete?

3. (30 points)

A *Graph* is a $G = (V, E)$ where V is a set and E is a set of unordered pairs of elements of V. (Note that we do not allow self-loops and the edges are undirected.)

A graph $G = (V, E)$ is k-*colorable* if there is a function

$f : V \leftarrow \{1, \ldots, k\}$

such that if $(x, y) \in E$ then $f(x) \neq f(y)$. (So two neighbors cannot have the same color.)

A graph is *Planar* if it can be drawn in the plane without crossing.

You can assume the following is true (it is!): $\{G\colon\ G \text{ is Planar}\} \in \mathrm{P}$.

For all $k \geq \mathbb{N}$ let

$A_k = \{G\colon\ G \text{ is Planar and } G \text{ is } k\text{-colorable }\}$.

(a) (6 points) Show that, for all k, the set A_k is in NP.
(Just describe the witness y and the set B.)

(b) (6 points) Look on the web to find out what is known about the following questions: (Here and for later questions, you don't have to look on the web if you are sure you know the answer.)
Is $A_2 \in \mathrm{P}$? Is A_2 NP-complete?

(c) (6 points) Look on the web to find out what is known about the following questions:
Is $A_3 \in \mathrm{P}$? Is A_3 NP-complete?

(d) (6 points) Look on the web to find out what is known about the following questions:
Is $A_4 \in \mathrm{P}$? Is A_4 NP-complete?

(e) (6 points) Look on the web to find out what is known about the following questions: Is $A_5 \in \mathrm{P}$? Is A_5 NP-complete?