Probabilistic Temporal Databases, I: Algebra

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February 16, 1999

Abstract

Dyreson and Snodgrass have drawn attention to the fact that in many temporal database applications, there is often uncertainty present about the start time of events, the end time of events, the duration of events, etc. When the granularity of time is small (e.g. milliseconds), a statement such as "Packet p was shipped sometime during the first 5 days of January, 1998" leads to a massive amount of uncertainty $(5 \times 24 \times 60 \times 60 \times 1000)$ possibilities. As noted in [53], past attempts to deal with uncertainty in databases have been restricted to relatively small amounts of uncertainty in attributes. Dyreson and Snodgrass have taken an important first step towards solving this problem.

In this paper, we first introduce the syntax of Temporal-Probabilistic (TP) relations and then show how they can be converted to an explicit, significantly more space-consuming form called Annotated Relations. We then present a *Theoretical Annotated Temporal Algebra* (TATA). Being explicit, TATA is convenient for specifying how the algebraic operations should behave, but is impractical to use because annotated relations are overwhelmingly large.

Next, we present a *Temporal Probabilistic Algebra* (TPA). We show that our definition of the TP-Algebra provides a correct implementation of TATA despite the fact that it operates on implicit, succinct TP-relations instead of the overwhelmingly large annotated relations. Finally, we report on timings for an implementation of the TP-Algebra built on top of ODBC.

1 Introduction

The world we live in evolves dynamically over time. Furthermore, our knowledge about what is true in the world at a fixed point in time is highly uncertain. Databases that attempt to capture temporal aspects of the world encounter uncertainty in a variety of applications.

- **Scheduling:** Consider the databases maintained by a transportation provider such as CSX or Federal Express. When a package is delivered to such an organization for shipping, a tentative shipping *schedule* is created for the package. The transportation provider must maintain such schedules for millions of packages. Such schedules specify which flight (or truck) the shipment is scheduled to leave on, when the shipment will reach a way point, and so on. However, there is uncertainty about how long a particular part of the schedule will actually take. For example, Federal Express may ship a package from Boston to Chicago via Albany, NY. They have reliable statistics on how long the Boston-Albany leg takes, and how long the Albany-Chicago leg takes. A user who wants to know when his shipment

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is likely to reach him usually gets an *uncertain* answer of the form "Either today (36%) or tomorrow (64%)." A database system used by such a transportation vendor must have the ability to handle temporal modes of uncertainty.

- **Weather Applications:** Consider a weather database that tracks the weather at a fixed location (e.g. Washington). Such a weather database contains not only information about the weather in Washington in the past, but also contains projections for the future. Needless to say, any prediction about the future is liable to be uncertain. How often have we heard a TV newscaster say "There is a 39% probability of rain this afternoon"?

- **Time-Series Stock Applications:** There are a wide variety of programs that analyze the behavior of stocks, and predict their rise and fall in the future. Most such programs associate with their predictions a level of uncertainty. Such programs may say "We expect, with 60-70% certainty, that IBM stock will fall by 30% sometime in the next 2 weeks." When the output of such programs is to be stored in a relational database system, we must have the ability to represent and manipulate such statements.

- **Video Extraction:** A completely different application arises in the case of feature extraction in video databases [49]. A video may be viewed as a sequence of frames (still images). A video feature extraction algorithm attempts to identify objects and activities occurring in these frames. However, most image processing algorithms are uncertain in their identifications. Thus, the statement "Darth Vader appears in frames 26 and 27" is an *uncertain statement* about uncertainty and time (frame numbers are correlated tightly with time as most video players in the market today playback video at a rate of 15 to 30 frames per second). Thus, the creation of a video database which automatically identifies features and/or gestures encompasses some temporal aspects as well as some uncertainty.

All the above applications require the ability to make statements of the following kind: *Data tuple d is in relation R at some point of time in the interval $[t_i, t_j]$ with probability between p and p'*. For example, in the Transportation Application above, we must be able to store statements of the form "Package p will arrive in Albany at some time between 9am and 5pm on Nov. 8 with probability 50–60%." Similarly, in the weather application, we must be able to store statements of the form "Rain is expected to begin sometime between 2pm and 12 midnight on Nov. 8 with probability 5–20%." In the case of the stock market application, we must be able to store statements of the form "IBM stock will reach $300 per share some time during the time interval Nov 1-10 with probability 90-100%."

The main contributions of this paper may now be summarized as follows.

• First, in Section 3 we introduce the concept of a *temporal-probabilistic tuple* or TP-tuple, for short. Intuitively, a TP-tuple allows us to augment classical relational database tuples with temporal-probabilistic data, as well as arbitrary probability distributions. For example, not only can we say "Data tuple d is in relation r at some point of time in the interval $[t_i, t_j]$ with probability between p and p'" but we can also say that the probability mass is distributed over $[t_i, t_j]$ according to an *arbitrary* probability distribution. Throughout this paper, we will introduce definitions which allow us to make such statements in a TP-relation and which allow us to manipulate such TP-relations algebraically.

• Then, in Section 4 we show how given any TP-tuple tp, we may "flatten" tp into a set of *annotated tuples*. In general, the set of annotated tuples associated with a single TP-tuple can be very large — hence, annotated tuples serve as a *purely theoretical device*.

• We then define a *Theoretical Annotated Temporal Algebra* (TATA) in Section 5 and show how the classical relational algebra operations can be extended to the case of annotated tuples. Intuitively, the Theoretical Annotated Temporal Algebra provides a theoretical specification of how the TP-Algebra operations must be defined.

• We proceed in Section 6 by defining a *Temporal-Probabilistic Algebra* (TPA) which directly manipulates TP-tuples *without converting them to annotated tuples*. This has a great advantage, as TP-tuples are very succinct objects. We show that for each operation α in the Theoretical Annotated Temporal Algebra, there is a corresponding operation α' in the Temporal-Probabilistic Algebra which precisely captures it. Thus, the Temporal-Probabilistic Algebra is a sound and complete way of implementing the declarative semantics for temporal probabilistic data prescribed by the Theoretical Annotated Temporal Algebra. The correctness results are formally proved for every operation.

• In Sections 5 and 6 we also show how each operator, whether in the TP-Algebra, or in the Theoretical Annotated Temporal Algebra, can be parameterized by the user's knowledge of the dependencies between events. This is important because, as shown in [30], the probability of a complex event like $(e_1 \vee e_2)$ depends upon our knowledge of the dependencies between e_1 and e_2.

• Finally, in Section 7 we present an implementation of the TP-Algebra on top of ODBC and provide a set of experimental results.

The idea of handling uncertainty in temporal databases was first addressed by Dyreson and Snodgrass [13]. A detailed comparison of our framework to theirs is included in Section 8. Our paper provides an extension of the relational algebra to handle temporal probabilistic data, while [13] presents an extension of SQL, thus neatly complementing our work. As we will see in Section 8, there are many other interesting and important contributions made in [13] that complement the work reported here, leading to the potential for a very powerful system obtained by combining the two frameworks.

The relationship between relational, temporal, and TP-databases may be briefly summed up as follows. In classical relational databases [51] a data-relation R over schema $(A_1, \ldots, A_n)$ contains a set of tuples, as shown in Figure 1(a). In contrast, a temporal database relation R may be thought of as *shorthand* for a set of snapshots $R(1), R(2), \ldots$ — intuitively, given a time instant i, $R(i)$ is a data-relation that specifies what tuples are true w.r.t. relation R at time i. This is shown in Figure 1(b).

In contrast, a TP-relation is much more complicated, as shown in Figure 1(c). For every given point i in time, we cannot specify $R(i)$ precisely as we are uncertain about what is in R at time i. Thus, for each time instant i, a TP-relation specifies a *set* of relations, $\{R(i,1), \ldots, R(i,k_i)\}$ for some $k_i > 0$, together with a probability assignment, $\wp$ on $\{R(i,1), \ldots, R(i,k_i)\}$. Intuitively, $\wp(R(i,j)) = 0.3$ means that there is a 30% probability that the content of relation R at time i is $R(i,j)$.

2 Preliminaries and Basic definitions

In this section, we provide some basic definitions that are used in the algebras we develop later in the paper. The work reported in Subsections 2.1, 2.2, 2.3 and 2.4 is not new work, but form the basic definitions needed to describe our algebras. Subsection 2.5 describes new work.

Section 2.1 contains the description of a notion of a *calendar*, borrowing from definitions in [29]. Calendars are needed because all TP-relations will assume that time is specified w.r.t. an arbitrary but fixed calendar.

Section 2.2 defines temporal constraints over an arbitrary but fixed calendars. As specified earlier in the paper, our algebras use constraints to describe sets of time points.

Section 2.3 describes *distribution functions*.

Section 2.4 we specifies a set of axioms that a function must satisfy for it to be considered a probabilistic conjunction or disjunction strategy. As it is known that the probability of conjunctive and disjunctive combinations of two or more events depends not only on the probabilities of the events themselves but also on the

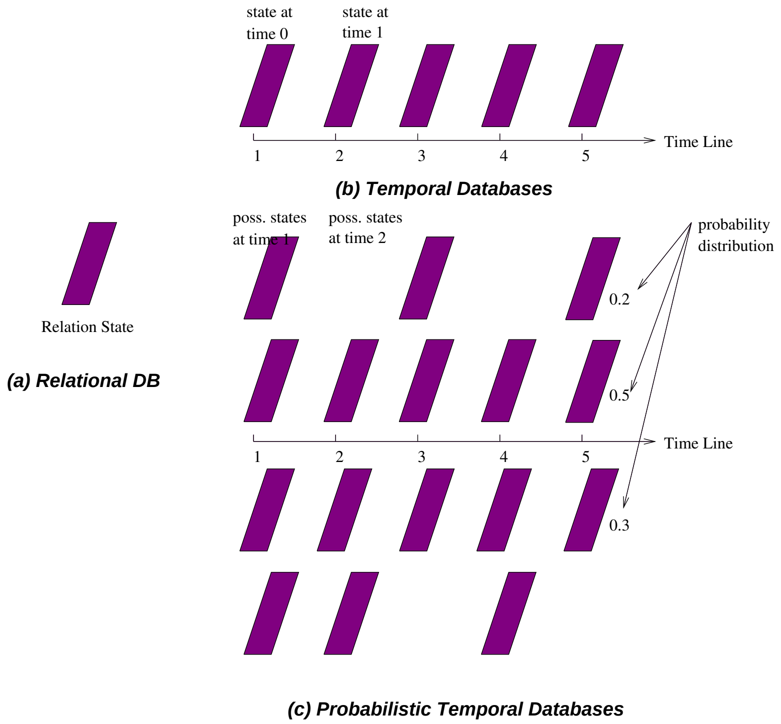


Figure 1: Relationship between Relations, Temporal Relations and TP-relations

dependencies between them, there is not a unique way of computing such probabilities. This section specifies what axioms a function must satisfy for it to be considered a possible way of computing the probabilities of compound events given the probabilities of simple events.

Section 2.5, finally, contains the description of the the means by which conflicting information about the probability of an event (which must be true at a certain time point) can be combined together. We introduce the concept of a *combination function* as a function that combines a set of probability intervals into one interval while satisfying a prerequisite set of axioms.

2.1 Calendars

In this section, we define the concept of a *calendar* that is used by a TP application. In our architecture, a TP application assumes the existence of an arbitrary but fixed calendar. The definitions in this section are not new, but taken from [29].

Definition 2.1 (time unit) A *time unit* consists of a *name* and a *time-value set*. The time-value set has a linear order, denoted $<_T$, where T is the name of the time unit. As usual, we let $\leq_T$ denote the reflexive closure of the $<_T$ relation. A time unit is either *finite* or *infinite*, depending on whether its time-value set is finite or infinite; an infinite time-value set is assumed to be countable. $\Box$

For instance, the time units named *day*, *month*, and *year* may have the time-value sets $\{1, \ldots, 31\}$, $\{1, \ldots, 12\}$, and $\{$all integers$\}$ respectively.

Definition 2.2 (linear hierarchy) A *linear hierarchy of time units*, denoted H, is a finite collection of distinct time units with a linear order $\sqsubseteq$ among those time units. The greatest time unit according to $\sqsubseteq$ may be either finite or infinite, while all other time units in the hierarchy must be finite. □

For instance, $H_1 = day \sqsubseteq month \sqsubseteq year$, $H_2 = minute \sqsubseteq hour \sqsubseteq day \sqsubseteq month \sqsubseteq year$, and $H_3 = hour \sqsubseteq day \sqsubseteq month$ are all linear hierarchies of time units.

Definition 2.3 (time point) Suppose $T_1 \sqsubseteq \cdots \sqsubseteq T_n$ is a linear hierarchy H of time units. A *time point* t in H is an n-tuple $(v_1, \ldots, v_n)$ such that for all $1 \leq i \leq n$, v_i is a time-value in the time-value set of T_i. Let $(v_1/.../v_n)$ be an abbreviation for time point t.

Time points are ordered according to the lexicographic ordering $<_H$ which is defined in the usual way. Thus, time point $t = (v_1, \ldots, v_n) <_H t' = (v'_1, \ldots, v'_n)$ iff there exists an i $(1 \leq i \leq n)$ such that $v_i <_{T_i} v'_i$ and $v_j = v'_j$ for all $j = i+1, \ldots, n$. Note that if $i = n$, then the $(v_j = v'_j)$ statement is vacuously true. When $t <_H t'$, we say that t *occurs before* t', and conversely, t' *occurs after* t. If $t = t'$, we say that t *occurs simultaneously with* t'. □

A time point in linear hierarchy H is simply an instantiation of each time unit in H (a specific point in time with respect to H). For instance, using hierarchy H_1 given above, "March 16, 1997" could be specified by the time point (16/3/1997). By using hierarchy H_2 given above, "3:45pm on March 12, 1997" could be specified by the time point (45, 15, 12, 3, 1997). For hierarchy H_1 given above, time point t *occurs before or simultaneously with* t', denoted $t = (v_{day}, v_{month}, v_{year}) \leq_{H_1} t' = (v'_{day}, v'_{month}, v'_{year})$, is true iff $((v_{year} < v'_{year}) \vee (v_{year} = v'_{year} \wedge v_{month} < v'_{month}) \vee (v_{year} = v'_{year} \wedge v_{month} = v'_{month} \wedge v_{day} \leq v'_{day}))$.

Definition 2.4 (calendar) A *calendar* τ consists of a linear hierarchy H of time units and a *validity predicate* denoted *valid*$_H$ (or simply *valid* if H is clear from context). A validity predicate specifies a non-empty set of valid time points; *valid*$_H(t)$ is true iff t is a valid time point. The *set of all time points over calendar* τ, denoted S_τ, is defined as $\{t \mid t$ is a time point in H and *valid*$_H(t)$ is true$\}$. □

For instance, if we are representing the Gregorian calendar τ by hierarchy H_1 given above, a suitable validity predicate states that valid(14/3/1996) = true but valid(29/2/1997) = false. (29/2/1997) is not a valid time point since February of 1997 only contains 28 days. Note that a calendar for the hierarchy *dayOfWeek* $\sqsubseteq$ *day* $\sqsubseteq$ *month* $\sqsubseteq$ *year* should have only one valid time point for each instantiation of (*day*, *month*, *year*) since these three time units uniquely determine the valid time-value for *dayOfWeek*.

Let $next_\tau(t)$ denote the next, consecutive time point after t. Thus, $next_\tau(t)$ denotes the time point $t' \in S_\tau$ where t' occurs after t and for all other $t'' \in S_\tau$ where t'' occurs after t, t'' also occurs after t'.

2.2 Constraints

When expressing a statement of the form "Data tuple d is in relation r at some time point in a set T of time points with probability in the interval $[p_1, p_2]$ and with the probability distributed according to distribution δ", we must be able to specify the set T of time points. Constraints are a natural way of specifying such sets. In this section, we recapitulate (from [29]) how *temporal constraints* can be used to specify sets of time points associated with a calendar.

Definition 2.5 (atomic temporal constraint) Suppose $T_1 \sqsubseteq \cdots \sqsubseteq T_n$ is a linear hierarchy H of time units over calendar τ. An *atomic temporal constraint over calendar* τ must take one of the following forms:
1. $(T_i$ *op* $v_i)$ where *op* is a member of the set $\{\leq, <, =, \neq, >, \geq\}$ and v_i is a time-value in the time-value set of time unit T_i. Here, $(T_i$ *op* $v_i)$ is called an *atomic time-value constraint*.

2. $(t_1 \sim t_2)$ where $t_1, t_2 \in S_\tau$ and $t_1 \leq_H t_2$. Here, $(t_1 \sim t_2)$ is called an *atomic time-interval constraint*. For convenience, let (t_1) be an abbreviation for $(t_1 \sim t_1)$. □

For example, (*day* $<$ 15), (*month* $\geq$ 8), and (12/3/1997 $\sim$ 10/4/1997) are all atomic temporal constraints, but (1996 = *year*) is not. Also, (*day* $<$ 45) is not an atomic temporal constraint since 45 is not in the time-value set of *day*. Similarly, (15/2/1997 $\sim$ 29/2/1997) is not an atomic temporal constraint since (29/2/1997) is not a valid time point in τ. Furthermore, (10/4/1997 $\sim$ 12/3/1997) is not an atomic temporal constraint since time point (10/4/1997) *occurs after* (12/3/1997).

Definition 2.6 (temporal constraint) A *temporal constraint* C *over calendar* τ is defined inductively in the following way:

- Any atomic temporal constraint over τ is a temporal constraint over τ.
- If C_1 and C_2 are temporal constraints over τ, then $(C_1 \wedge C_2)$, $(C_1 \vee C_2)$, and $(\neg C_1)$ are temporal constraints over τ.

If temporal constraint C is solely a boolean combination of *atomic time-value constraints*, then C is a *time-value constraint*. Similarly, if temporal constraint C is solely a boolean combination of *atomic time-interval constraints*, then C is a *time-interval constraint*. □

For instance, $((day > 5 \wedge day < 15) \wedge (month = 4 \vee month \geq 8) \wedge year = 1996)$ and ((12/3/1997 $\sim$ 10/4/1997) $\vee$ (10/7/1997 $\sim$ 10/7/1997)) are temporal constraints but $(day > 5 \neg \wedge day < 15)$ is not.

Definition 2.7 (solution set to an atomic temporal constraint) Suppose $T_1 \sqsubseteq \cdots \sqsubseteq T_n$ is a linear hierarchy H of time units over calendar τ. Then an atomic temporal constraint over τ is of the form $(T_i \; op \; v_i)$ or $(t_1 \sim t_2)$. The *solution set to an atomic temporal constraint over calendar* τ is the set S which is defined in the following way:

Case	S
$op = (\leq)$	$S = \{t \mid t \in S_\tau \wedge t.T_i \leq_{T_i} v_i\}$
$op = (<)$	$S = \{t \mid t \in S_\tau \wedge t.T_i <_{T_i} v_i\}$
$op = (=)$	$S = \{t \mid t \in S_\tau \wedge t.T_i = v_i\}$
$op = (\neq)$	$S = \{t \mid t \in S_\tau \wedge t.T_i \neq v_i\}$
$op = (>)$	$S = \{t \mid t \in S_\tau \wedge v_i <_{T_i} t.T_i\}$
$op = (\geq)$	$S = \{t \mid t \in S_\tau \wedge v_i \leq_{T_i} t.T_i\}$
$op = (\sim)$	$S = \{t \mid t \in S_\tau \wedge t_1 \leq_H t \leq_H t_2\}$

For instance, the solution set to (*day* $>$ 25) over the Gregorian calendar τ is the set of all time points (*day*, *month*, *year*) $\in \tau$ where *day* $>$ 25. Note that (29, 2, 1996) is in this set but (29, 2, 1997) is not since the latter is not a valid time point in S_τ. Also, the solution set to (1/1/1996 $\sim$ 31/12/1996) over the Gregorian calendar would contain 366 time points (one for each calendar day in 1996) while the solution set to (1/1/1997 $\sim$ 31/12/1997) over the same calendar would contain 365 time points.

Definition 2.8 (solution set to a temporal constraint) Let S_τ be the set of all valid time points over calendar τ. Then the *solution set to a temporal constraint* C *over calendar* τ, abbreviated sol(C), is the set S which is defined inductively in the following way:

Case	S
C is an atomic temporal constraint	$S = \text{sol}(C)$
$C = (C_1 \wedge C_2)$	$S = \text{sol}(C_1) \cap \text{sol}(C_2)$
$C = (C_1 \vee C_2)$	$S = \text{sol}(C_1) \cup \text{sol}(C_2)$
$C = \neg C_1)$	$S = S_\tau - sol(C_1)$

Each time point $t \in \text{sol}(C)$ is called *a solution to* C. □

For example, the solution set to ((5/8/1997 $\sim$ 10/8/1997) $\vee$ (7/8/1997 $\sim$ 12/8/1997)) would contain eight time points.

The following well known result states that any time-value constraint can be rewritten as an equivalent time-interval constraint (i.e., one which has an equal solution set) and vice-versa.

Proposition 1 (Folk theorem) *Time-value constraints and time-interval constraints have the same expressive power.*

Calendar τ is a *finite calendar* iff S_τ is finite. Note that when the greatest time unit of a calendar is finite, then the calendar is guaranteed to be finite. Furthermore, for all temporal constraints C over a finite calendar τ, sol(C) must be a finite subset of S_τ. Given a finite calendar τ, we use t_S^τ to denote the smallest time point of τ (w.r.t. the ordering $<_H$ associated with the calendar) and t_E^τ to denote the largest time point. When τ is clear from context, we will drop the superscript τ and just write t_S and t_E.

In the rest of this paper, all calendars are assumed to be finite unless we specifically state otherwise. Furthermore, all of our examples will use a finite version of the Gregorian calendar.

2.3 Distribution functions

Consider a simple statement saying that data tuple d is in relation r at some time point in the set $\{1, 2, 3, 4\}$ with probability 0.7. Suppose we are now asked "what the probability that d is in r at time 2?" There is no way to answer this question without assuming the existence of some probability distribution. In this paper, we wish to allow designers of TP-databases to specify probability distributions for each set of time points.

Definition 2.9 (probability distribution function) Let D be a temporal constraint over calendar τ such that $|\mathrm{sol}(D)| \geq 1$. Then a *probability distribution function (PDF) over calendar τ*, denoted pdf(D, t_j), is a function which takes D and a time point $t_j \in S_\tau$ as input, and returns as output a probability ρ_j which satisfies the following conditions:
1. For each $t_j \in S_\tau$, $0 \leq \rho_j \leq 1$.
2. For all $t_j \in S_\tau$ where $t_j \notin \mathrm{sol}(D)$, $\rho_j = 0$.
3. $\sum_{t_j \in S_\tau}(\rho_j) \leq 1$. This implies that $\sum_{t_j \in \mathrm{sol}(D)}(\mathrm{pdf}(D, t_j)) \leq 1$.

If the sum $\sum_{t_j \in S_\tau}(\rho_j)$ is strictly less than one, the function is called a *partial PDF*; if the sum is exactly equal to one, the function is called a *complete PDF*. A PDF is *determinate* if $\sum_{t_j \in S_\tau}(\rho_j)$ is computable in constant time. □

PDFs are both discrete and finite. Complete PDFs tell us what percentage of the total probability mass (i.e., 1.0) is associated with each $t_j \in \mathrm{sol}(D)$. Partial PDFs are useful when modeling infinite distributions; here, we are considering only a finite portion of the total probability mass. Determinate PDFs tell us up-front that a fixed percentage of the probability mass is unassigned. Thus, every complete PDF is determinate. In addition, a partial PDF which is known to allocate only a total of 0.9 to the values in S_τ is determinate.

To see how specific PDFs may be defined, let us examine some examples.

Example 2.1 (PDF; uniform) The *PDF for the uniform distribution over calendar τ*, denoted pdf$_u(D, t_j)$, is defined as $\rho_j = \frac{1}{|\mathrm{sol}(D)|}$ if $t_j \in \mathrm{sol}(D)$ or $\rho_j = 0$ otherwise. pdf$_u$ is a complete PDF.

Notice that for all $t_1, t_2 \in \mathrm{sol}(D)$, $\rho_1 = \rho_2$. In other words, we are equally dividing the probability mass among all of the relevant time points. Also, $\sum_{t_j \in S_\tau}(\rho_j) = 1$ is clearly true since there are $n = |\mathrm{sol}(D)|$

non-zero ρ_js, one for each $t_j \in \text{sol}(D)$, and $n \cdot \rho_j = |\text{sol}(D)| \cdot \frac{1}{|\text{sol}(D)|} = 1$. Furthermore, we will never divide by zero since by definition of PDFs, $|\text{sol}(D)| \geq 1$.

For the following PDF examples, let D be a temporal constraint and let $t_0, \ldots, t_n$ be a list of distinct time points in S_τ where sol(D) = $\{t_0, \ldots, t_n\}$ and t_i *occurs before* t_{i+1} for all $0 \leq i < n$, i.e. $sol(D)$ is enumerated in ascending order of time points. For instance if $D = (1/8/1997 \sim 3/8/1997)$ then $n = 2$, $t_0 = 1/8/1997, t_1 = 2/8/1997$, and $t_2 = 3/8/1997$.

Example 2.2 (PDF; geometric) Let p be a probability where $(0 < p < 1)$. Then the *PDF for the geometric distribution with parameter p over calendar* τ, denoted $\text{pdf}_{g,p}(D, t_j)$, is defined as $\rho_j = p \cdot (1 - p)^i$ if $t_j = t_i \in \text{sol}(D)$ or $\rho_j = 0$ otherwise. pdf_g is a partial PDF. Note that if $|\text{sol}(D_j)|$ is fixed (or constant time computable), then pdf_g is a determinate PDF.

If $p = \frac{1}{3}$, $\text{pdf}_{g,p}(D, t_0) = \frac{1}{3} \cdot (\frac{2}{3})^0$, $\text{pdf}_{g,p}(D, t_1) = \frac{1}{3} \cdot (\frac{2}{3})^1$, and $\text{pdf}_{g,p}(D, t_2) = \frac{1}{3} \cdot (\frac{2}{3})^2$. For all other time points $t_j \in S_\tau$, $\text{pdf}_{g,p}(D, t_j) = 0$. Notice that if $p = \frac{1}{2}$, then $\text{pdf}_{g,p}(D, t_0) = \frac{1}{2}$ and $\text{pdf}_{g,p}(D, t_i)$ will be half of $\text{pdf}_{g,p}(D, t_{i-1})$ for each $1 \leq i \leq n$.

Let $\text{pdf}_{gc,p}$ be defined in the same way as $\text{pdf}_{g,p}$ except $\text{pdf}_{gc,p}(D, t_n) = 1$ if $|\text{sol}(D)| = 1$ or $\text{pdf}_{gc,p}(D, t_n) = 1 - (\sum_{t_j \in \{t_0, \ldots, t_{n-1}\}} (\text{pdf}_{g,p}(D, t_j)))$ otherwise. We call $\text{pdf}_{gc,p}$ the *complete correlate* of $\text{pdf}_{g,p}$ since $\text{pdf}_{gc,p}(D, t_j) = \text{pdf}_{g,p}(D, t_j)$ for all $t_j \in S_\tau - \{t_n\}$ and since $\text{pdf}_{gc,p}$ is a complete PDF. In general, one can construct a complete correlate for any partial PDF in a similar way. Note that when $p = \frac{1}{2}$ and $|\text{sol}(D)| > 1$, $\text{pdf}_{gc,p}$ has the nice property that $\text{pdf}_{gc,p}(D, t_n) = \text{pdf}_{gc,p}(D, t_{n-1})$.

Example 2.3 (PDF; binomial) Let p be a probability where $(0 < p < 1)$. Then the *PDF for the binomial distribution with parameter p over calendar* τ, denoted $\text{pdf}_{b,p}(D, t_j)$, is defined as $\rho_j = \binom{n}{i} \cdot p^i \cdot (1 - p)^{n-i}$ if $t_j = t_i \in \text{sol}(D)$ or $\rho_j = 0$ otherwise. pdf_b is a complete PDF.

Example 2.4 (PDF; Poisson) Let $(\lambda > 0)$ be a rate and let e be the base of the natural logarithm (i.e., $e \simeq 2.71828$). Then the *PDF for the Poisson distribution with parameter* λ *over calendar* τ, denoted $\text{pdf}_{po,\lambda}(D, t_j)$, is defined as $\rho_j = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$ if $t_j = t_i \in \text{sol}(D)$ or $\rho_j = 0$ otherwise. pdf_{po} is a partial PDF. When $|\text{sol}(D)|$ is known, then pdf_{po} is a determinate PDF.

Techniques that specify how to associate and store probability distributions with events are provided by Dyreson and Snodgrass [13, p. 8] and by Dey and Sarkar [12]. Hence, we do not discuss this in further detail.

Throughout the rest of this paper, we will use (δ = "u"), (δ = "g, p"), (δ = "g_c, p"), (δ = "b, p"), and (δ = "po, λ") to represent the distribution functions for pdf_u, $\text{pdf}_{g,p}$, $\text{pdf}_{gc,p}$, $\text{pdf}_{b,p}$, and $\text{pdf}_{po,\lambda}$ respectively. Furthermore, unless we specifically state otherwise, assume that parameter $p = 0.5$. Thus, (δ = "g") represents the $\text{pdf}_{g,0.5}$ function.

2.4 Probabilistic strategies

Given the probabilities p_1 and p_2 of events e_1 and e_2, how do we compute the probability p of compound event $(e_1 \wedge e_2)$? As argued in [30], the answer depends on the relationship between e_1 and e_2. For instance if e_1 and e_2 are *mutually exclusive*, p should be zero; if e_1 and e_2 are independent of each other, p should be $(p_1 \cdot p_2)$. A similar situation arises when computing the probability of $(e_1 \vee e_2)$. We address these problems by consulting *probabilistic conjunction strategies* and *probabilistic disjunction strategies*. Both of these concepts were originally defined in [30] and are recapitulated below.

Before proceeding, recall that intervals obey the following definitions/properties:
1.$[L_1, U_1] \leq [L_2, U_2]$ iff $(L_1 \leq L_2 \wedge U_1 \leq U_2)$.
2.$[L_1, U_1] \geq [L_2, U_2]$ iff $(L_1 \geq L_2 \wedge U_1 \geq U_2)$.
3.$[L_1, U_1] \subseteq [L_2, U_2]$ iff $(L_1 \geq L_2 \wedge U_1 \leq U_2)$. 4.$[L, U] = ([L_1, U_1] \cap [L_2, U_2])$ **iff** $(L = \max(L_1, L_2) \wedge U = \min(U_1, U_2) \wedge L \leq U)$.

Definition 2.10 (probabilistic conjunction/disjunction strategy) *Let events e_1, e_2 be associated probabilistic intervals $[L_1, U_1]$ and $[L_2, U_2]$ respectively. Then a* probabilistic conjunction strategy *(*probabilistic disjunction strategy*) is a binary operation $\otimes$ ($\oplus$) which uses this information to compute the probabilistic interval $[L, U]$ for event "$e_1 \wedge e_2$" ("$e_1 \vee e_2$"). When the events involved are clear from context, we use $[L, U] = [L_1, U_1] \otimes [L_2, U_2]$ to denote $(e_1 \wedge e_2, [L, U]) = (e_1, [L_1, U_1]) \otimes (e_2, [L_2, U_2])$ and we use $[L, U] = [L_1, U_1] \oplus [L_2, U_2]$ to denote $(e_1 \vee e_2, [L, U]) = (e_1, [L_1, U_1]) \oplus (e_2, [L_2, U_2])$. Every conjunctive (disjunctive) strategy must conform to the following probabilistic postulates:*

Generic postulates *($* \in \{\otimes, \oplus\}$)*		
1. Commutativity	$([L_1, U_1] * [L_2, U_2]) = ([L_2, U_2] * [L_1, U_1])$	
2. Associativity	$(([L_1, U_1] * [L_2, U_2]) * [L_3, U_3]) = ([L_1, U_1] * ([L_2, U_2]) * [L_3, U_3]))$	
3. Monotonicity	$([L_1, U_1] * [L_2, U_2]) \leq ([L_1, U_1] * [L_3, U_3])$ *if* $[L_2, U_2] \leq [L_3, U_3]$	
Specific postulates		
4. Bottomline	$([L_1, U_1] \otimes [L_2, U_2]) \leq [\min(L_1, L_2), \min(U_1, U_2)]$	$([L_1, U_1] \oplus [L_2, U_2]) \geq [\max(L_1, L_2), \max(U_1, U_2)]$
5. Identity	$([L_1, U_1] \otimes [1, 1]) = [L_1, U_1]$	$([L_1, U_1] \oplus [0, 0]) = [L_1, U_1]$
6. Annihilator	$([L_1, U_1] \otimes [0, 0]) = [0, 0]$	$([L_1, U_1] \oplus [1, 1]) = [1, 1]$
7. Ignorance	$([L_1, U_1] \otimes [L_2, U_2]) \subseteq [\max(0, L_1 + L_2 - 1), \min(U_1, U_2)]$	$([L_1, U_1] \oplus [L_2, U_2]) \subseteq [\max(L_1, L_2), \min(1, U_1 + U_2)]$

Postulates 1–6 follow from the well-known properties of the probabilities of conjunctions and disjunctions. A brief explanation of axiom 7 is in order. Boole proved in 1854 [7] that if events e_1, e_2 are known to have probabilities in the intervals $[L_1, U_1], [L_2, U_2]$, *and we do not know anything about the relationship between these two events*, then the best that can be said about the probability for $(e_1 \wedge e_2)$ is that it lies in the interval shown above. Similarly, $[\max(L_1, L_2), \min(1, U_1 + U_2)]$ had been established by Boole as the probabilistic interval for the disjunction of e_1 and e_2. This forms the basis for numerous pieces of work in the AI and deductive database literature ([19, 34, 36] to name a few). This axiom merely says that if we know something about the dependency between e_1, e_2, then we must be able to infer a tighter probability interval than complete ignorance about dependencies would allow us to infer.

The following are some sample conjunctive and disjunctive strategies ([30]):

Conjunctive Strategies	
Ignorance	$([L_1, U_1] \otimes_{ig} [L_2, U_2]) = [\max(0, L_1 + L_2 - 1), \min(U_1, U_2)]$
Positive Correlation	$([L_1, U_1] \otimes_{pc} [L_2, U_2]) = [\min(L_1, L_2), \min(U_1, U_2)]$
Negative Correlation	$([L_1, U_1] \otimes_{nc} [L_2, U_2]) = [\max(0, L_1 + L_2 - 1), \max(0, U_1 + U_2 - 1)]$
Independence	$([L_1, U_1] \otimes_{in} [L_2, U_2]) = [L_1 \cdot L_2, U_1 \cdot U_2]$

Disjunctive Strategies	
Ignorance	$([L_1, U_1] \oplus_{ig} [L_2, U_2]) = [\max(L_1, L_2), \min(1, U_1 + U_2)]$
Positive Correlation	$([L_1, U_1] \oplus_{pc} [L_2, U_2]) = [\max(L_1, L_2), \max(U_1, U_2)]$
Negative Correlation	$([L_1, U_1] \oplus_{nc} [L_2, U_2]) = [\min(1, L_1 + L_2), \min(1, U_1 + U_2)]$
Independence	$([L_1, U_1] \oplus_{in} [L_2, U_2]) = [L_1 + L_2 - (L_1 \cdot L_2), U_1 + U_2 - (U_1 \cdot U_2)]$

Note that we use the more general notion of a probability interval $[L, U] \subseteq [0, 1]$ instead of a point probability $p \in [0, 1]$; intervals allow us to reason about the probabilities of compound events (through operators such as $\otimes_{ig}$) without making traditional assumptions like independence [30].

Probabilistic conjunctions will be useful when describing TATA and TPA semantics for *cartesian products*

(§5.6, §6.7). As conjunctive and disjunctive probabilistic strategies are commutative and associative, we can extend the definition of either strategy to apply to more than two arguments. We adopt the notations $([L_1, U_1] \otimes \ldots \otimes [L_k, U_k])$ and $([L_1, U_1] \oplus \ldots \oplus [L_k, U_k])$ to represent this generalization.

2.5 Combination functions

Suppose that we are trying to determine the probability that a single event e is true at time point t. Occasionally, we may have multiple sources of information where each source provides a different probability interval for e at time t. Here, combination functions can be used as a generic mechanism for combining these intervals into a single $[L, U]$ result.

Definition 2.11 (combination function) Let $S = \{[L_1, U_1], \ldots, [L_k, U_k]\}$ be a non-empty multiset of probabilistic intervals. Then a *combination function* χ is a function which takes S as input, and returns as output a probabilistic interval $[L, U]$ which satisfies the following axioms:
1. Identity: If $[L_1, U_1] = \ldots = [L_k, U_k]$, then $\chi(S) = [L_1, U_1]$. In other words, when all input intervals are equal, the output interval is also equal to all of the input intervals.
2. Bottomline: $L \leq \max\{L_i \mid [L_i, U_i] \in S\}$. In other words, the lower bound of the result cannot exceed the largest lower bound of the intervals in S. □

Combination functions will be useful when describing TATA and TPA semantics for *intersection* (§5.2, §6.3) and *union* (§5.3, §6.4). For instance, after a union merges all tuples from two relations, the resulting relation may contain more than one tuple for a single event. Here, we could *compact* (merge) these tuples into a single tuple by applying a combination function.

Definition 2.12 (conflict) A multiset S of probability intervals *conflict* iff $\bigcap_{[L,U] \in S}[L, U] = \emptyset$. □

Note that all combination functions must find a way to remove conflicts. A class of combination functions called *equity combination functions* prescribe to the view that if $S = \{[L_1, U_1], [L_2, U_2]\}$ does not conflict, then $\chi(S)$ should equal $[L_1, U_1] \cap [L_2, U_2]$. However, if these intervals conflicted, then different equity combination functions may resolve the conflict in different ways.

Definition 2.13 (equity combination function) An equity combination function χ_e is a combination function where $(\bigcap_{[L,U] \in S}[L, U] \neq \emptyset) \Rightarrow (\chi_e(S) = \bigcap_{[L,U] \in S}[L, U])$. □

Example 2.5 (example equity combination functions) :

Name	Interval Returned when $\bigcap_{[L,U] \in S}[L, U] = \emptyset$
Optimistic Equity	$\chi_{eq}(S) = [\max(\{L_i \mid [L_i, U_i] \in S\}), \max(\{U_i \mid [L_i, U_i] \in S\})]$
Enclosing Equity	$\chi_{ec}(S) = [\min(\{L_i \mid [L_i, U_i] \in S\}), \max(\{U_i \mid [L_i, U_i] \in S\})]$
Pessimistic Equity	$\chi_{ep}(S) = [\min(\{L_i \mid [L_i, U_i] \in S\}), \min(\{U_i \mid [L_i, U_i] \in S\})]$
Rejecting Equity	$\chi_{er}(S) = [0, 0]$
Skeptical Equity	$\chi_{esk}(S) = [0, 1]$
Quasi-independence Equity	$\chi_{eqi}(S) = [\Pi_{[L_i,U_i] \in S} L_i, \Pi_{[L_i,U_i] \in S} U_i]$

Note that when $\bigcap_{[L,U] \in S}[L, U] \neq \emptyset$, all of the functions above return $\bigcap_{[L,U] \in S}[L, U]$.

Proposition 2 *Every function listed in Example 2.5 is an equity combination function.*

3 TP-relations

In this section, we define the syntax and semantics of a Temporal-Probabilistic relation. Intuitively, a TP-relation is a multiset of TP-tuples. Each TP-tuple consists of a "data" part and a "probabilistic-temporal" part. This latter part is called a TP-case statement and it intuitively specifies the probability with which the "data" part of the tuple is in the relation at different instances of time. Once TP-cases are defined in Section 3.1 below, we will provide a formal definition of TP-tuples and TP-relations in Sections 3.2 and 3.3.

We intend for TP-tuples defined in this section to represent *events*. An event is *instantaneous* if it can only occur at a single point in time. For example, consider the event "Toss $toss\_id$ of coin C comes up heads." This is an instantaneous event since it can only be true at a single point in time — the same coin cannot be tossed twice at the same time and two different tosses of the same coin represent two distinct events. Our TP-tuples will represent such *instantaneous events*. It is important to note that a real world event e (which has a continuous duration) may be modeled in our framework through two instantaneous events — the event $st(e)$ denoting the start of e and the event $end(e)$ denoting the end of e. Thus here, without loss of generality, we only consider events that are instantaneous. A similar assumption is made by Dyreson and Snodgrass[13].

3.1 TP-case statements

Definition 3.1 (TP-case statement over calendar τ) A *TP-case statement over calendar* τ, denoted γ, is an expression of the form $\boxed{\{\langle C_1, D_1, L_1, U_1, \delta_1\rangle, \ldots, \langle C_n, D_n, L_n, U_n, \delta_n\rangle\}}$ where $n \geq 1$, C_i and D_i are a temporal constraints over τ, L_i and U_i are probabilities, δ_i is a distribution function over τ, and the following conditions are satisfied for all $1 \leq i \leq n$:
1. $(0 \leq L_i \leq U_i \leq 1)$.
2. $\mathrm{sol}(C_i) \subseteq \mathrm{sol}(D_i)$. This ensures that $\delta_i(D_i, t)$ is defined for each time point $t \in \mathrm{sol}(C_i)$.
3. $|\mathrm{sol}(C_i)| \geq 1$. In other words, C_i and D_i each have at least one solution in S_τ.
4. For all $1 \leq j \leq n$, $i \neq j \Rightarrow \mathrm{sol}(C_i) \cap \mathrm{sol}(C_j) = \emptyset$. In other words, $(C_i \wedge C_j)$ is always inconsistent. This ensures that each TP-case statement specifies at most one probability interval for each $t \in S_\tau$. Note that we do not have a similar requirement for $(D_i \wedge D_j)$.

For each $1 \leq i \leq n$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i\rangle$ is called *a TP-case* of γ. On occasion, we may want to assign probabilities to every time point in S_τ. Here, sol(C_n) = sol($\neg C_1 \wedge \neg C_2 \wedge \ldots \wedge \neg C_{n-1}$) and γ_n is called the *catch-all case*. For brevity, when γ_n is a catch-all case, we may use "$(*)$" to represent C_n.

Note: If sol(C_i) = sol(D_i), we let "$(\#)$" be an abbreviation for C_i. □

One may wonder why *two* constraints (C_i and D_i) occur in a TP-case $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i\rangle$. Intuitively, sol($C_i$) is the set of time points which γ_i is "interested in" while sol(D_i) is the set of time points used when distributing the probability interval $[L_i, U_i]$ according to δ_i. When this TP-case is associated with a data-tuple d, it says that d is in some relation at some time point $t \in \mathrm{sol}(C_i)$. The probability that d is true in the relation at such a t is $\delta_i(D_i, t)$. In other words, D_i is used to specify the set of time points used when distributing the probability interval $[L_i, U_i]$ according to δ_i. **This is an important distinction which is critically necessary.** Why ? Suppose that originally, $\mathrm{sol}(C_i) = \mathrm{sol}(D_i) = S = \{1, 2, 3, 4\}$ and $\delta_i =$ "$b, 0.5$". Thus, the probabilities associated with time points 1,2,3,4 are 0.125, 0.375, 0.375, 0.125. Now suppose we perform a *selection* operation (§5.4) which only asks for time points in the set $S' = \{2, 3\} \subset S$. If we had no D_i field in our TP-cases, then we would merely carry over the fact that S' has the binomial distribution on it. But applying this distribution to S' yields a probability of 0.5 to both 2 and 3 which is incorrect because selections should not change the probabilities assigned to time points $t \in S'$. Thus, some mechanism is needed to correctly compute the probabilities of relations resulting from algebraic operations executed.

Thus, in order to accurately compute probabilities, we must do one of two things: *either* carry with us the original set of values over which a probability distribution was defined, *or* determine how to refine accurately an *arbitrary* distribution to apply to a subset of the set to which the distribution was originally applicable. The latter option requires a complex algebraic theory of distributions and its implementation is likely to be extremely expensive. For this reason, we have chosen the first option above.

For another (simpler) example, consider a TP-case statement with one TP-case γ_1: $\{\langle(1/8/1997 \sim 5/8/1997), (1/8/1997 \sim 10/8/1997), 0.4, 0.8, u\rangle\}$ Intuitively, γ_1 says that some event occurred during the first five days of August 1997 (in other words, it occurred during one of the time points in $\mathrm{sol}(C_1)$). Since $\delta_1 =$ "u", the probability that it occurred on any of these days is the same. Specifically, this probability is $[\frac{1}{10} \cdot 0.4, \frac{1}{10} \cdot 0.8] = [0.04, 0.08]$ since we are uniformly distributing the probability mass $[0.4, 0.8]$ between all of the (10) time points in $\mathrm{sol}(D_1)$. In general, the probability interval for some time point $t \in \mathrm{sol}(C_i)$ is $[L_i \cdot \delta_i(D_i, t), U_i \cdot \delta_i(D_i, t)]$. Here, we see that a TP-case $\langle C_i, D_i, L_i, U_i, \delta_i\rangle$ is simply an extension to the P-tuple $\langle D_i, L_i, U_i, \delta_i\rangle$.

Note: Even though TP-cases contain two distinct constraint fields, viz. C and D, this distinction can be hidden from the user, especially in base relations where C and D are equal.

The expression on the left below is a TP-case statement. However, the expression on the right is not a TP-case statement as the solution set to C_1 (and D_1) is empty.

$\{\langle(\#), (month < 6 \wedge year = 1997), 0.4, 0.8, g\rangle,$	$\{\langle(\#), (year = 1996 \wedge year = 1997), 0.4, 0.8, g\rangle,$
$\langle(\#), (month \geq 6 \wedge year = 1997), 0.6, 0.6, u\rangle\}$	$\langle(\#), (year = 1998), 0.0, 0.0, u\rangle\}$

Furthermore, $\{\langle(\#), (month < 6 \wedge year = 1997), 0.4, 0.8, g\rangle, \langle(\#), (month \geq 3 \wedge year = 1997), 0.6, 0.6, u\rangle\}$ is not a TP-case statement as $(C_1 \wedge C_2)$ is not inconsistent (i.e., the probabilities for the overlapping time points are over-specified).

We reiterate that each temporal constraint in a TP-case statement must have a finite number of solutions (since S_τ is finite). We restrict ourselves to finite calendars and solution sets to avoid the complications which arise when trying to determine whether a constraint using negations is infinite or not.

3.2 TP-tuples

In this section, we will define the important concept of a Temporal Probabilistic tuple (TP-tuple for short). We will require that all TP-tuples contain a special field called a "hidden field." Intuitively, two TP-tuples with equal "data" parts may represent two different events. The hidden field (similar in function, but easier to implement than the concept of a *path* introduced in [30]) keeps track of how these TP-tuples were derived in order to determine whether or not two TP-tuples refer to the same event.

Definition 3.2 (hidden field) A *hidden field* holds a lexicographically sorted *hidden list* of *field-value pairs* (i.e., "<field$_1$>:<value$_1$>, . . ., <field$_n$>:<value$_n$>"). If there are no pairs to store, the hidden list will be EMPTY. □

In base relations, the contents of the hidden field will be EMPTY (since no fields have been projected out). For intermediate relations, the hidden field holds values of the form "<field>:<value>" for fields which have been projected out. Although these values should be hidden from the user, we shall see that they are important in determining whether two TP-tuples refer to the same event or not.

Definition 3.3 (TP-tuple) Let $T_1 \sqsubseteq \cdots \sqsubseteq T_m$ be the linear hierarchy of time units over calendar τ and suppose $A = (A_1, \ldots, A_k)$ is a relational schema where for all $1 \leq i < k$, $A_i \notin \{$"C", "D", "L", "U", "δ", "L_t", "U_t", "H"$\}$, and for all $1 \leq j \leq m$, $A_i \neq T_j$. Furthermore, let A_k be the *hidden field* "H", let

$d = (d_1, \ldots, d_k)$ be a (data) tuple over A, and let γ be a TP-case statement over τ. Then $tp = (d, \gamma)$ is a *TP-tuple over relational schema A and calendar τ*. Intuitively, γ gives the probability for each $t \in S_\tau$ that d occurs at time t.

For instance, suppose our relational schema is A = (Item, Origin, Dest, H). Then

Item	Origin	Dest	H	C	D	L	U	δ
I1	Rome	Vienna		$(\#)$	$day < 15 \wedge month = 11 \wedge year = 1996$	0.5	0.6	u
				$(\#)$	$day \geq 15 \wedge month = 11 \wedge year = 1996$	0.4	0.4	u

is a TP-tuple which indicates that item "I1" left from "Rome" and will arrive in "Vienna" in November 1996 at some time before the 15th (with $50 - 60\%$ probability) or on/after the 15th (with 40% probability). This TP-tuple is not concerned with I1's arrival before or after November 1996 since no probabilities are assigned to this time range.

If we were sure that I1 did not arrive in Vienna before or after November 1996, we could add the TP-case $\langle (\#), (*), 0, 0, u \rangle$ to the TP-tuple above. If we had no information regarding I1's arrival before or after November 1996 but we were assuming that the distribution function for this time was "u", we could add the TP-case $\langle (\#), (*), 0, 1, u \rangle$ to the TP-tuple above.

Finally, if we had no information whatsoever regarding I1's arrival before or after November 1996, we would not change the TP-tuple above. Here, we are implicitly assigning a probability interval of $[0, 1]$ to each time point t which lies outside of November 1996 since for all $1 \leq i \leq n$, $t \notin \mathrm{sol}(C_i)$.

In the following sections we employ the following operators which manipulate hidden lists.

Definition 3.4 (manifest projection) Let $A = (A_1, \ldots, A_k)$ be a relational schema where A_k is the *hidden field* ("H") and let $d = (d_1, \ldots, d_k)$ be a (data) tuple over A. Then the *manifest projection of data tuple d*, denoted $\mathcal{P}(d)$, is defined as $(d_1, \ldots, d_{k-1})$. In other words, the tuple $\mathcal{P}(d)$ contains every value in d except the *hidden list* (d.H). Here, A_1 to A_{k-1} are known as *manifest data fields*. □

Intuitively, the manifest projection of a TP-relation simply eliminates the hidden field of the TP-relation. The following definition specifies how hidden lists are concatenated.

Definition 3.5 (hidden list concatenation) The *concatenation of hidden lists d.H and d'.H*, denoted (d.H $\|$ d'.H), is a hidden list h'' which can be constructed by lexicographically merging every field-value pair in d.H and d'.H. For instance if d.H = "Fld3:Val3, Fld6:Val6" and d'.H = "Fld4:Val4, Fld8:Val8, Fld9:Val9", then h'' = "Fld3:Val3, Fld4:Val4, Fld6:Val6, Fld8:Val8, Fld9:Val9". □

Intuitively, the concatenation of two hidden lists can be obtained by taking the union of the two hidden lists, and then sorting them in lexicographic order.

3.3 TP-relations

Definition 3.6 (TP-relation) A *TP-relation over relational schema A and calendar τ*, denoted r, is a multiset of TP-tuples over relational schema A and calendar τ. Intuitively, a *base TP-relation* is a TP-relation which did not result from a query. If r is a base TP-relation, then for each TP-tuple $tp = (d, \gamma) \in r$, (i) d.H = EMPTY and (ii) for each TP-case $\langle C_i, D_i, L_i, U_i, \delta_i \rangle \in \gamma$, $C_i = D_i$.

We associate with each TP-relation a *primary key*. This key will be used when we describe the TPA's semantics for projection (§6.8). □

Recall that a *primary key* is a minimal set of fields which, taken collectively, allow us to uniquely identify a tuple in a relation [25]. In the worst case, a primary key may need to contain every manifest data field in a relation. In practice, well designed databases use tuple ids, transaction ids, SSNs, timestamps, etc. to help keep the primary keys small.

Definition 3.7 (TP-database) A *TP-database over calendar* τ is a pair $(Base, MView)$ where $Base$ is a set of base TP-relations over τ and $MView$ is a set of non-base TP-relations over τ. $\square$

For simplicity, we require all TP-relations in a TP-database to use the same calendar. Note that this requirement does not force us to lose any expressive power. Throughout this paper, we assume that all TP-relations are in the same TP-database unless we specifically state otherwise.

3.4 Semantics and consistency of TP-relations

We are now ready to define the formal semantics of TP-relations. In order to provide such a semantics, we will extend classical logic [44] to the case of TP-relations, by extending the concept of an interpretation in classical logic [44] to handle TP-relations. Before doing this, a preliminary definition is needed.

Definition 3.8 (data-identical TP-tuples) TP-tuples $tp = (d, \gamma)$ and $tp' = (d', \gamma')$ are *data-identical* iff $(d = d')$. Note that tp and tp' may come from different TP-relations as long as both TP-relations have the same schema. Also, note that $(d = d')$ only if ($d.\mathrm{H}$ = $d'.\mathrm{H}$). $\square$

Recall that without loss of generality, we interpret TP-relations under the assumption that all data-identical TP-tuples refer to the same, unique event. If tp and tp' are data-identical, we assume that they provide complementary information for the same event. If tp and tp' are not data-identical, we assume that they refer to different events. Let $tp \in r$. Then $r[tp]$ denotes the multiset of all TP-tuples in r which are data-identical to tp. Since "data-identical" is a reflexive, symmetric, transitive relation on TP-tuples, it is also an equivalence relation on r where each $r[tp]$ corresponds to an equivalence class in this relation.

A TP-relation r is *compact* iff for each data tuple d and each time point t there is at most one TP-tuple $tp = (d, \gamma) \in r$ where $t \in \mathrm{sol}(C_1 \vee \ldots \vee C_n)$. Otherwise, since r contains at least two TP-tuples which refer to the same event at the same time, r is an *uncompact* TP-relation. Later, we will describe a variety of *compaction* operators which convert uncompact TP-relations into compact TP-relations by consolidating probabilistic information for each $r[tp] \subseteq r$ (e.g., $\S 6.3$).

Intuitively, a TP-tuple $tp = (d, \gamma)$ is *consistent* if there exists a satisfying assignment of probabilities for each TP-case $\gamma_i \in \gamma$. This is given formal "teeth" through the following definition.

Definition 3.9 (TP-interpretation) Let $A = (A_1, \ldots, A_k)$ be a relational schema, let τ be a calendar, and let $dom(A) = dom(A_1) \times \cdots \times dom(A_k)$ be the domain of A. Then a *TP-interpretation* over the pair A, τ is a function $I_{A,\tau} : dom(A) \times S_\tau \mapsto [0, 1]$ such that $(\forall d \in dom(A))(\sum_{t \in S_\tau} I_{A,\tau}(d, t) \leq 1)$. $\square$

Let e be the event represented by data tuple d. Then $I_{A,\tau}(d, t) = p$ says that according to TP-interpretation $I_{A,\tau}$, the probability that e is true at time point t is p. Let D be a temporal constraint over τ. Then the *probability assigned by* $I_{A,\tau}$ *to* D, denoted $I_{A,\tau}(d, D)$, is equal to $\sum_{t \in \mathrm{sol}(D)} I_{A,\tau}(d, t)$. This intuition may be used to explain what it means for a TP-interpretation to satisfy a TP-tuple.

Definition 3.10 (satisfaction) Let d be a tuple in relational schema A and let $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$ be a TP-case. Then $I_{A,\tau}$ *satisfies* $\langle d, \gamma_i \rangle$, denoted $I_{A,\tau} \models \langle d, \gamma_i \rangle$, iff the following conditions hold:
1.$L_i \leq I_{A,\tau}(d, D_i) \leq U_i$, i.e. the probability that $I_{A,\tau}$ assigns to D_i lies in the interval $[L_i, U_i]$.
2.$(\forall t \in \text{sol}(C_i))(I_{A,\tau}(d, D_i) \cdot \delta_i(D_i, t) = I_{A,\tau}(d, t))$, i.e. $I_{A,\tau}$ distributes probabilities for each $t \in \text{sol}(C_i)$ according to δ_i.

TP-interpretation $I_{A,\tau}$ *satisfies* TP-tuple $tp = (d, \gamma)$ ($I_{A,\tau} \models tp$), iff $I_{A,\tau} \models \langle d, \gamma_i \rangle$ for all $\gamma_i \in \gamma$. □

For example, let us reconsider the following TP-tuple.

Item	Origin	Dest	H	C	D	L	U	δ
I1	Rome	Vienna		$(\#)$	$day < 15 \wedge month = 11 \wedge year = 1996$	0.5	0.6	u
				$(\#)$	$day \geq 15 \wedge month = 11 \wedge year = 1996$	0.4	0.4	u

Consider the TP-interpretation defined as follows:

$I(\langle$I1,Rome,Vienna$\rangle, (d, 11, 1996)) = 0.04$ when $d < 15$.
$I(\langle$I1,Rome,Vienna$\rangle, (d, 11, 1996)) = \frac{0.4}{16}$ when $d \geq 15$.
$I(\langle$item,origin,dest, $(d, m, y)) = 0$ otherwise.

This TP-interpretation satisfies the TP-tuple above because
$I(\langle$I1,Rome,Vienna$\rangle, \{t \mid t.day < 15 \wedge t.month = 11 \wedge t.year = 1996\}) = 0.04 \times 14 = 0.56$
which lies between 0.5 and 0.6.

Definition 3.11 (consistency and mutual consistency) A TP-tuple tp is *consistent* iff there exists a TP-interpretation $I_{A,\tau}$ where $I_{A,\tau} \models tp$. A TP-relation r is *consistent* iff $(\exists I_{A,\tau})(\forall tp \in r)(I_{A,\tau} \models tp)$. TP-relations r and r' are *mutually consistent* iff $(\exists I_{A,\tau})((\forall tp \in r)(I_{A,\tau} \models tp) \wedge (\forall tp' \in r')(I_{A,\tau} \models tp'))$. Note that if consistent TP-relations r, r' have different schemas, then r and r' must be mutually consistent. □

Later in this paper, we will provide algorithms to convert any TP-relation into a *compact* TP-relation. When a TP-relation is compact, there are no two TP-tuples that are data-identical, and hence, we can check consistency of a TP-relation by individually checking consistency of each TP-tuple. Suppose a TP-tuple $tp = (d, \gamma)$ has $\gamma = \{\langle C_1, D_1, L_1, U_1, \delta_1 \rangle, \ldots, \langle C_n, D_n, L_n, U_n, \delta_n \rangle\}$ as its TP-case statement. Then it suffices to check that $(L_1 + \cdots + L_n) \leq \min(1, \Sigma_{i=1}^{n} L_i \cdot \Sigma_{t \in S_\tau} \delta_i(D_i, t))$.

If this condition holds, then the TP-tuple is consistent. This forms the basis for the following claim. When the distribution function used is determinate, then the quantity on the right side of the above inequality can be computed in linear time because the determinacy condition guarantees constant time computation of the sum $\Sigma_{t \in S_\tau} \delta_i(D_i, t))$.

Proposition 3 *Checking consistency of a compact TP-relation which uses determinate PDFs is linear in the size of the TP-relation.*

4 Annotated relations

Annotated relations are the flat, relational equivalents of TP-relations. Let TP-case statement $\gamma = \{\langle C_1, D_1, L_1, U_1, \delta_1 \rangle, \ldots, \langle C_n, D_n, L_n, U_n, \delta_n \rangle\}$. Then we can *"flatten"* TP-tuple $tp = (d, \gamma)$ by creating an *annotated tuple* for each time point $t \in \text{sol}(C_1) \cup \ldots \cup \text{sol}(C_n)$.

Each annotated tuple (at) provides probabilistic information ($[L_t, U_t]$) for one data tuple (d) at one point in time (t). According to the definition for a TP-case statement, $t \in \text{sol}(C_i)$ for at most one $\gamma_i =$

$\langle C_i, D_i, L_i, U_i, \delta_i \rangle \in \gamma$ since $(i \neq j) \Rightarrow (\mathrm{sol}(C_i) \cap \mathrm{sol}(C_j) = \emptyset)$. If $t \notin \bigcup_{\gamma_i \in \gamma} \mathrm{sol}(C_i)$, then we do not create an annotated tuple for t. We denote the set of all annotated tuples derived from tp by ANN(tp). In the worst case, ANN(tp) may contain $|S_\tau|$ tuples.

Let r be a TP-relation containing n TP-tuples and let $\uplus$ denote the *multiset union* operator (in other words, a union operation without duplicate elimination). Then the *annotated relation* for r, denoted ANN(r), is defined as $\biguplus_{tp \in r} \mathrm{ANN}(tp)$. This means that ANN($r$) may contain up to $n \cdot |S_\tau|$ tuples! In general, annotated relations will always be finite since they are derived from a finite number of TP-cases, and each TP-case pertains to a finite number of time points (since τ is finite). Nonetheless, we can clearly see that ANN(tp) and ANN(r) will often be large and impractical. This is why we **only use annotation for theoretical purposes such as illustrating a process or proving equivalences between query expressions. In our implementation, we never create annotated relations.**

The table below contains an annotated relation ANN(r) for a TP-relation r consisting of one TP-tuple $tp = (d, \gamma)$, where d = ("D1", EMPTY) and $\gamma = \{\gamma_1\}$.

r

Data	H	C	D	L	U	δ
D1		$(\#)$	$day \leq 4 \wedge$ $month = 11 \wedge$ $year = 1996$	0.4	0.8	u

ANN(r)

Data	H	Day	Month	Year	L_t	U_t
D1		1	11	1996	0.1	0.2
D1		2	11	1996	0.1	0.2
D1		3	11	1996	0.1	0.2
D1		4	11	1996	0.1	0.2

However if γ_1's C_1 field was "$day \leq 3 \wedge month = 11 \wedge year = 1996$", then ANN($r$) would no longer contain the last tuple shown above. In general, note that changing C_i only affects the number of annotated tuples in ANN(r), not the probabilities for the remaining time points.

Notice the "0.1" and "0.2" values in the probabilistic fields above. These values were determined by uniformly distributing the available probability $[0.4, 0.8]$ among the four annotated tuples in ANN(r). We were only justified in making this uniformity assumption since δ_1 = "u". In general, TP-relations will only give us probability intervals for a range of time points, and determining (tight) probability intervals for each time point within that range requires us to apply a distribution function δ_i.

Definition 4.1 (annotated relation for a TP-tuple) Let $tp = (d, \gamma)$ be a TP-tuple over relational schema $(A_1, \ldots, A_k)$ and calendar τ where $d = (d_1, \ldots, d_k)$. Suppose γ contains n TP-cases of the form $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$ $(1 \leq i \leq n)$ and suppose τ consists of a linear hierarchy H containing m time units $T_1 \sqsubseteq \cdots \sqsubseteq T_m$. Here, each $t \in S_\tau$ will be of the form $t = (v_1, \ldots, v_m)$.

Then the *annotated relation for TP-tuple* tp *over calendar* τ, denoted ANN(tp), is defined as
$\{(d, t, L_t, U_t) \mid t \in \mathrm{sol}(C_i)$ for some $\gamma_i \in \gamma$ and $[L_t, U_t] = [L_i \cdot x, U_i \cdot x]$ where $x = \delta_i(D_i, t)\}$. □

Intuitively, in the definition above, x represents the percentage of $\mathrm{sol}(D_i)$'s probability which is associated with time point t according to δ_i. Note that when we explicitly show all fields of an annotated tuple, $at = (d_1, \ldots, d_k, v_1, \ldots, v_m, L_t, U_t)$ is over the schema $(A_1, \ldots, A_k, T_1, \ldots, T_m, L_t, U_t)$. Here, A_1 to A_{k-1} are *manifest data fields*, A_k is the *hidden field*, T_1 to T_m are *temporal fields*, and L_t, U_t are *probabilistic fields*.

Definition 4.2 (annotated relation for a TP-relation) Let r be a TP-relation over τ containing n TP-tuples $tp_1 \ldots tp_n$. Then the *annotated relation for TP-relation* r *over calendar* τ, denoted ANN(r), is defined as the multiset $(\mathrm{ANN}(tp_1) \uplus \ldots \uplus \mathrm{ANN}(tp_n))$ over τ.

We associate with each ANN(r) the *primary key* which is associated with r. This key will be used when we describe the TATA's projection operation (§5.7). □

4.1 Semantics and consistency of annotated relations

Our semantics for annotated relations closely parallels our semantics for TP-relations (§3.4). Annotated tuples $at = (d, t, L_t, U_t)$ and $at' = (d', t', L'_t, U'_t)$ are called *data-identical* iff $(d = d')$. We interpret annotated relations under the assumption that all data-identical annotated tuples refer to the same event. If at and at' are not data-identical, we assume that they refer to different events. Let d be a data tuple and let t be a time point. Then $\mathrm{ANN}(r)[d, t]$ denotes the equivalence class of the pair (d, t), i.e., the multiset of all $at \in \mathrm{ANN}(r)$ where $(at.d = d \wedge at.t = t)$.

Suppose $at = (d, t, L_t, U_t) \in \mathrm{ANN}(r)$, $at' = (d', t', L'_t, U'_t) \in \mathrm{ANN}(r)$, and $(d = d' \wedge t = t')$. Here, since $at, at' \in \mathrm{ANN}(r)$ refer to the same event at the same point in time, ANN(r) is an *uncompact* annotated relation. If there are no pairs of annotated tuples $at, at' \in \mathrm{ANN}(r)$ where $(d = d' \wedge t = t')$, then ANN(r) is a *compact* annotated relation. Later, we will describe a variety of *compaction* operators which convert uncompact annotated relations into compact annotated relations (e.g., §5.1). The following theorem states that the concept of "compact relation" for TP-relations and annotated relations coincide.

Theorem 1 *A TP-relation r is compact iff its annotated counterpart,* $\mathrm{ANN}(r)$*, is compact.*

Definition 4.3 (satisfaction of annotated tuples) Let d be a tuple in relational schema A, let t be a time point in S_τ, and let $[L, U]$ be a probability interval. Then a TP-interpretation $I_{A,\tau}$ *satisfies* annotated tuple $at = (d, t, L_t, U_t)$, denoted $I_{A,\tau} \models at$, iff $L_t \leq I_{A,\tau}(d, t) \leq U_t$. □

Definition 4.4 (consistency of annotated relations) An annotated tuple at is *consistent* iff $(\exists I_{A,\tau})(I_{A,\tau} \models at)$. An annotated relation ANN(r) is *consistent* iff $(\exists I_{A,\tau})(\forall at \in \mathrm{ANN}(r))(I_{A,\tau} \models at)$. Annotated relations ANN(r) and ANN(r') are *mutually consistent* iff $(\exists I_{A,\tau})((\forall at \in \mathrm{ANN}(r))(I_{A,\tau} \models at) \wedge (\forall at' \in \mathrm{ANN}(r'))(I_{A,\tau} \models at'))$. □

The following theorem tells us that if r is a consistent TP-relation, then ANN(r) is also consistent.

Theorem 2 *If $I_{A,\tau} \models r$, then $I_{A,\tau} \models \mathrm{ANN}(r)$. Hence if a TP-relation r is consistent, so is* $\mathrm{ANN}(r)$.

The converse of this theorem is not true, i.e., it may be the case that a TP-interpretation satisfies ANN(r), but does not satisfy r. This is shown in the following example.

Example 4.1 (satisfaction) Let r consist of one TP-tuple (d, γ) where $\gamma = \{\langle(\#), (1 \sim 2), 0.4, 0.8, u\rangle\}$. Then $\mathrm{ANN}(r) = \{at_1, at_2\}$ where $at_1 = (d, 1, 0.2, 0.4)$ and $at_2 = (d, 2, 0.2, 0.4)$.

Now consider the TP-interpretation $I_{A,\tau}$ such that $I_{A,\tau}(d, 1) = 0.3$ and $I_{A,\tau}(d, 2) = 0.4$. Clearly, $I_{A,\tau}$ satisfies $\mathrm{ANN}(r)$, but $I_{A,\tau} \not\models r$ because every TP-interpretation $J_{A,\tau}$ that satisfies r must have $J_{A,\tau}(d, 1) = J_{A,\tau}(d, 2)$.

This occurs since the details of the distribution get lost when annotating a relation — this is not surprising as annotated relations have no fields for including information about distributions. Instead, we can show that if r is compact, if $(d, \gamma) \in r$, and if $(d, t, L_t, U_t) \in \mathrm{ANN}(r)$, then there must be a TP-interpretation $I_{A,\tau}$ of r such that $I_{A,\tau}(d, t) = L_t$. A similar statement applies to U_t. This means that the bounds contained in $\mathrm{ANN}(r)$ are tight, and hence, $\mathrm{ANN}(r)$ correctly captures the implied probability intervals for data tuple d at time t.

Theorem 3 *Let r be a compact TP-relation containing a TP-tuple (d, γ), and suppose $(d, t, L_t, U_t) \in \mathrm{ANN}(r)$. Then there is a TP-interpretation $I_{A,\tau}$ satisfying r such that $I_{A,\tau}(d, t) = L_t$.*

Theorems 2 and 3 jointly tell us that as far as lower bounds are concerned, r and $\mathrm{ANN}(r)$ are equivalent when r is known to be compact. Later, we will describe mechanisms to make compact a TP-relation r.

Note that the above result does not hold for upper bounds — the reason for this is that in a TP-tuple, the upper bounds may often be loose (i.e., not tight). For instance, consider the following TP-tuple:

Data	H	C	D	L	U	δ
D1		$(\#)$	$(5/1/1998)$	0.6	1	u
		$(\#)$	$(6/1/1998)$	0.4	1	u

It is easy to see that the upper bounds of the TP-cases above can be tightened to 0.6 and 0.4 respectively. Hence, these upper bounds are "loose" and need to be tightened if a theorem similar to Theorem 3 is to hold.

Definition 4.5 (tightening) Let $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a TP-case statement where each TP-case $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$. A *tightening* of γ returns a TP-case statement $\gamma'' = \{\gamma''_1, \ldots, \gamma''_n\}$ where each $\gamma''_i = \langle C''_i, D''_i, L''_i, U''_i, \delta''_i \rangle$ and each $U''_i \leq U_i$ for all $1 \leq i \leq n$.

A TP-tuple (d, γ) is said to be *tight* iff there is no other TP-tuple (d, γ') such that: (i) γ' is a tightening of γ and (ii) for all TP-interpretations $I_{A,\tau}$, $I_{A,\tau} \models (d, \gamma)$ iff $I_{A,\tau} \models (d, \gamma')$.

A TP-relation is *tight* iff every TP-tuple in it is tight. □

Now we can show that for tight and compact TP-relations, Theorem 3 holds for upper bounds as well.

Theorem 4 *Let r be a compact, tight, TP-relation containing a TP-tuple (d, γ), and suppose $(d, t, L_t, U_t) \in \mathrm{ANN}(r)$. Then there is a TP-interpretation $I_{A,\tau}$ satisfying r such that $I_{A,\tau}(d, t) = U_t$.*

Theorems 3 and 4 jointly tell us that the conversion of a TP-relation r to annotated form preserves bounds when r is tight and compact. Later, in Section 6.10, we will describe a procedure for tightening TP-relations.

4.2 Sample annotated relations

Let r consist of one TP-tuple which contains two TP-cases as shown below.

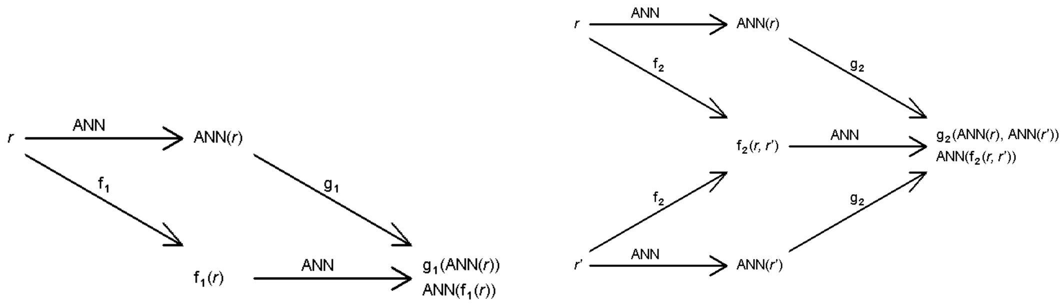
Item	Origin	Dest	H	C	D	L	U	δ
I1	Rome	Paris		$(\#)$	$day \leq 2 \wedge month = 8 \wedge year = 1997$	0.5	0.7	Φ
				$(\#)$	$day \geq 5 \wedge day \leq 7 \wedge month = 8 \wedge year = 1997$	0.3	0.6	Φ

Note that the variable Φ must be instantiated. If $\Phi =$ "u", ANN(r) will be

Item	Origin	Dest	H	Day	Month	Year	L_t	U_t
I1	Rome	Paris		1	8	1997	0.25	0.35
I1	Rome	Paris		2	8	1997	0.25	0.35
I1	Rome	Paris		5	8	1997	0.10	0.20
I1	Rome	Paris		6	8	1997	0.10	0.20
I1	Rome	Paris		7	8	1997	0.10	0.20

where $[L_t, U_t] = \frac{1}{2} \cdot [0.5, 0.7]$ for the first two tuples and $[L_t, U_t] = \frac{1}{3} \cdot [0.3, 0.6]$ for the remaining tuples in $\mathrm{ANN}(r)$. However if $\Phi =$ "g", ANN(r) will be

Item	Origin	Dest	H	Day	Month	Year	L_t	U_t
I1	Rome	Paris		1	8	1997	0.25	0.35
I1	Rome	Paris		2	8	1997	0.125	0.175
I1	Rome	Paris		5	8	1997	0.15	0.30
I1	Rome	Paris		6	8	1997	0.075	0.15
I1	Rome	Paris		7	8	1997	0.0375	0.075



(a) Unary Operators (b) Binary Operators

Figure 2: Commutativity between operators of the TATA and TPA algebras

where $[L_t, U_t] = \frac{1}{2} \cdot [0.5, 0.7], \frac{1}{4} \cdot [0.5, 0.7], \frac{1}{2} \cdot [0.3, 0.6], \frac{1}{4} \cdot [0.3, 0.6]$, and $\frac{1}{8} \cdot [0.3, 0.6]$ for the first through fifth tuples of ANN(r) respectively. Notice that modifying Φ (i.e., the distribution function δ) only affects the L_t and U_t fields of ANN(r).

5 Theoretical Annotated Temporal Algebra

In this section, we define the Theoretical Annotated Temporal Algebra and provide definitions for *compaction*, *intersection*, *union*, *selection*, *difference*, *cartesian product*, *projection*, and *join* on annotated relations.

Figure 2 shows what we hope to accomplish through this section. We know that every TP-relation can be converted into a (potentially very large) annotated relation. As annotated relations are *explicit* representations of TP-relations, the definition of the above operations on annotated relations can be explicitly defined and justified — this is what we will do in this section. Then, in Section 6, we will show how these operations can be implemented in the TP-Algebra in such a way that the TP-Algebra operations implement the annotated algebra operations on the *implicit* (smaller) TP-relations, rather than their larger annotated counterparts.

The definitions in this section will produce a new annotated relation ANN(r'') based on input from consistent annotated relations $\mathrm{ANN}(r), \mathrm{ANN}(r')$. Oftentimes, these definitions will refer to annotated tuples at, at' which are assumed to be of the form $at = (d, t, L_t, U_t)$ and $at' = (d', t', L'_t, U'_t)$.

Note: Our examples illustrating the Theoretical Annotated Temporal Algebra and the TP-Algebra will be based on the relations shown in Figure 3.

5.1 Compaction of an annotated relation

The first operation we define will be *compaction* as this operator is needed to define other operators. Compaction is the TP analog of duplicate elimination in the relational algebra.

Definition 5.1 (Compaction of an annotated relation) A function κ from annotated relations to annotated relations is called a *compaction operation* if it satisfies the following axioms:

- Compactness : $\kappa(\mathrm{ANN}(r))$ is *compact* for all annotated relations $\mathrm{ANN}(r)$.
- No Fooling Around (NFA) : If $\mathrm{ANN}(r)$ is compact, then $\kappa(\mathrm{ANN}(r)) = \mathrm{ANN}(r)$.
- Conservativeness : If $at = (d, t, L_t, U_t) \in \kappa(\mathrm{ANN}(r))$, then $\exists at' = (d, t, L'_t, U'_t) \in \mathrm{ANN}(r)$. □

r_1

Data	H	C	D	L	U	δ
D1		$(\#)$	$(1/8/1997 \sim 3/8/1997)$	0.64	0.88	g
		$(\#)$	$(5/8/1997 \sim 8/8/1997)$	0.40	0.80	u

ANN(r_1)

Data	H	Day	Month	Year	L_t	U_t
D1		1	8	1997	0.32	0.44
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11
D1		5	8	1997	0.10	0.20
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20

r_2

Data	H	C	D	L	U	δ
D1		$(\#)$	$(2/8/1997 \sim 3/8/1997)$	0.20	0.50	g
		$(\#)$	$(6/8/1997 \sim 9/8/1997)$	0.40	0.80	u

ANN(r_2)

Data	H	Day	Month	Year	L_t	U_t
D1		2	8	1997	0.10	0.25
D1		3	8	1997	0.05	0.125
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20
D1		9	8	1997	0.10	0.20

r_3

Data1	Data2	H	C	D	L	U	δ
D1	D2		$(\#)$	$(2/8/1997 \sim 2/8/1997)$	0.20	0.40	u
D1	D3		$(\#)$	$(2/8/1997 \sim 3/8/1997)$	0.60	0.80	g

ANN(r_3)

Data1	Data2	H	Day	Month	Year	L_t	U_t
D1	D2		2	8	1997	0.20	0.40
D1	D3		2	8	1997	0.30	0.40
D1	D3		3	8	1997	0.15	0.20

r_4

Data1	Data2	H	C	D	L	U	δ
D1	D2		$(\#)$	$(2/8/1997 \sim 3/8/1997)$	0.20	1.00	g
D1	D3		$(\#)$	$(3/8/1997 \sim 3/8/1997)$	0.50	0.50	u
D4	D5		$(\#)$	$(1/8/1997 \sim 1/8/1997)$	0.70	0.80	u

ANN(r_4)

Data1	Data2	H	Day	Month	Year	L_t	U_t
D1	D2		2	8	1997	0.10	0.50
D1	D2		3	8	1997	0.05	0.25
D1	D3		3	8	1997	0.50	0.50
D4	D5		1	8	1997	0.70	0.80

Figure 3: Example Base TP and Annotated Relations

The Compactness axiom assures us that the result of a compaction operation will be a compact relation. The NFA axiom states that applying compaction operation to a compact relation should not change the relation. The Compactness and NFA axiom jointly guarantee that compaction operations are idempotent, i.e. $\kappa(\kappa(\mathrm{ANN}(r))) = \kappa(\mathrm{ANN}(r))$. The Conservativeness axiom says that any information which appears in the result of a compaction has to originate from information in the initial relation; no information about "new" events, or events at "new" time points gets added during compaction.

It should be clear that there are many possible ways to compact a relation. One possible class of compaction strategies involves the use of a combination function (as defined in Section 2.5).

Definition 5.2 (χ-compaction of an annotated relation) Let χ be a combination function. Then the *χ-compaction of annotated relation* $\mathrm{ANN}(r)$, denoted $\kappa_\chi(\mathrm{ANN}(r))$, is defined as $\kappa_\chi(\mathrm{ANN}(r)) = \{at = (d, t, L_t, U_t) \mid [L_t, U_t] = \chi(\{[L_1^{(d,t)}, U_1^{(d,t)}], \ldots, [L_k^{(d,t)}, U_k^{(d,t)}]\})\}$ where $\mathrm{ANN}(r)[d, t] = \{at_1^{(d,t)}, \ldots at_k^{(d,t)}\}$ and $at_i^{(d,t)} = (d, t, L_i^{(d,t)}, U_i^{(d,t)})$. □

Intuitively, in combination function based compactions, χ is applied to the multiset of all $[L_i, U_i]$s associated with (d, t). The resulting $[L, U]$ then becomes the only probability interval associated with (d, t). The following proposition states that an operation defined in this manner is indeed a *compaction operation*.

Proposition 4 *Let χ be any combination function. Then $\kappa_\chi(\mathrm{ANN}(r))$ is a compaction operation.*

Theorem 5 *If $at' = (d, t, L'_t, U'_t) \in \mathrm{ANN}(r)$, then $\exists at = (d, t, L_t, U_t) \in \kappa_\chi(\mathrm{ANN}(r))$.*

Theorem 5 indicates that $\kappa_\chi(\mathrm{ANN}(r))$ operations also possess another important property: Reasonableness. Intuitively, this property is similar to the converse of Conservativeness — every tuple in ANN(r) leads to a corresponding tuple in the result of the compaction.

Another possible class of compaction strategies uses *p-strategies* (i.e., probabilistic conjunction or disjunction strategies as defined in Section 2.4). These compactions, denoted κ_ρ, are defined in the same way as $\kappa_\chi(\mathrm{ANN}(r))$ except we let $[L_t, U_t] = ([L_1^{(d,t)}, U_1^{(d,t)}] \otimes_\rho \ldots \otimes_\rho [L_k^{(d,t)}, U_k^{(d,t)}])$ when ρ is a conjunctive p-strategy, and let $[L_t, U_t] = ([L_1^{(d,t)}, U_1^{(d,t)}] \oplus_\rho \ldots \oplus_\rho [L_k^{(d,t)}, U_k^{(d,t)}])$ when ρ is a disjunctive p-strategy.

Proposition 5 *Let ρ be any p-strategy. Then $\kappa_\rho(\mathrm{ANN}(r))$ is a compaction operation.*

5.2 Intersection of two annotated relations

The intersection of annotated relations ANN(r) and ANN(r') is viewed as the operation of extracting information which is common to both relations. In our algebra, we break intersection into two suboperations: First, a *multiset intersection* will extract all tuples from both ANN(r) and ANN(r') which contain "common information". Then, we will use one of our previously-defined compaction operators to compact the result of this multiset intersection. Finally, *intersection* will be defined as a combination of these suboperations. Note that intersection (and multiset intersection) is only defined when both relations have the same schema.

Definition 5.3 (multiset intersection of two annotated relations) The *multiset intersection of annotated relations ANN(r) and ANN(r')*, denoted $\mathrm{ANN}(r) \cap \mathrm{ANN}(r')$, is defined as ANN($r''$) = $\{at \in \mathrm{ANN}(r) \mid (\exists at' \in \mathrm{ANN}(r'))(d = d' \wedge t = t')\} \cup \{at' \in \mathrm{ANN}(r') \mid (\exists at \in \mathrm{ANN}(r))(d = d' \wedge t = t')\}$. □

Intuitively, ANN(r'') will contain all $at \in \mathrm{ANN}(r)$ and all $at' \in \mathrm{ANN}(r')$ where at and at' refer to the same event at the same point in time. Recall that "$(\exists at \in \mathrm{ANN}(r))$" and "$(\exists at' \in \mathrm{ANN}(r'))$" are shorthand for "$(\exists(d, t, L_t, U_t) \in \mathrm{ANN}(r))$" and "$(\exists(d', t', L'_t, U'_t) \in \mathrm{ANN}(r'))$" respectively. For greater clarity and conciseness, our definitions will make use of this implicit notation. For example, $\mathrm{ANN}(r'') = \mathrm{ANN}(r_1) \cap \mathrm{ANN}(r_2)$ will be

Data	H	Day	Month	Year	L_t	U_t
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20
D1		2	8	1997	0.10	0.25
D1		3	8	1997	0.05	0.125
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20

Clearly, ANN(r'') above is uncompacted. To obtain a compact annotated relation, we may use any κ_χ compaction operator. Using this operator when defining intersection makes sense because r, r' may both contain data tuple d at some time point t, but with different probabilities. In this case we are combining two different probabilities assigned to the same event by two different sources (relations r and r'). This is exactly what combination functions χ were designed to support.

Definition 5.4 (intersection of two annotated relations) The *intersection of annotated relations ANN(r) and ANN(r') under the χ combination function*, denoted $\mathrm{ANN}(r) \cap_\chi \mathrm{ANN}(r')$, is defined as $\kappa_\chi(\mathrm{ANN}(r) \cap \mathrm{ANN}(r'))$. □

For example, $\mathrm{ANN}(r_1) \cap_{eq} \mathrm{ANN}(r_2) = \kappa_{eq}(\mathrm{ANN}(r''))$ will be

Data	H	Day	Month	Year	L_t	U_t
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20

5.3 Union of two annotated relations

Just like intersection, the union of two annotated relations will be presented as a combination of two suboperations: multiset union, which combines the information from two relations together and compaction, which compacts the result. As always, union is only defined when both relations have the same schema.

Definition 5.5 (multiset union of two annotated relations) The *multiset union of annotated relations ANN(r) and ANN(r')*, denoted $\mathrm{ANN}(r) \cup \mathrm{ANN}(r')$, is defined as $\mathrm{ANN}(r'') = \mathrm{ANN}(r) \uplus \mathrm{ANN}(r')$. □

Intuitively, ANN(r'') will contain all $at \in \mathrm{ANN}(r)$ and all $at' \in \mathrm{ANN}(r')$.As in the case of intersection, ANN(r'') may over-specify probabilistic information. We can consolidate this information by using a κ_χ compaction operator. The reason for using the operator κ_χ instead of a conjunction strategy is exactly for the same reason that we used the κ_χ compaction operator when defining intersection (see discussion preceding Definition 5.4).

Definition 5.6 (union of two annotated relations) The *union of annotated relations ANN(r) and ANN(r') under the χ combination function*, denoted $\mathrm{ANN}(r) \cup_\chi \mathrm{ANN}(r')$, is defined as $\kappa_\chi(\mathrm{ANN}(r) \cup \mathrm{ANN}(r'))$. □

The tables below show the results of $\mathrm{ANN}(r'') = \mathrm{ANN}(r_1) \cup \mathrm{ANN}(r_2)$ and $\mathrm{ANN}(r_1) \cup_{eq} \mathrm{ANN}(r_2)$.

$\mathrm{ANN}(r_1) \cup \mathrm{ANN}(r_2)$

Data	H	Day	Month	Year	L_t	U_t
D1		1	8	1997	0.32	0.44
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11
D1		5	8	1997	0.10	0.20
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20
D1		2	8	1997	0.10	0.25
D1		3	8	1997	0.05	0.125
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20
D1		9	8	1997	0.10	0.20

$\mathrm{ANN}(r_1) \cup_{eq} \mathrm{ANN}(r_2) = \kappa_{eq}(\mathrm{ANN}(r''))$

Data	H	Day	Month	Year	L_t	U_t
D1		1	8	1997	0.32	0.44
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11
D1		5	8	1997	0.10	0.20
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20
D1		8	8	1997	0.10	0.20
D1		9	8	1997	0.10	0.20

Definition 5.7 (union of two annotated relations) The *union of annotated relations ANN(r) and ANN(r') under the χ combination function*, denoted $\mathrm{ANN}(r) \cup_\chi \mathrm{ANN}(r')$, is defined as $\kappa_\chi(\mathrm{ANN}(r) \cup \mathrm{ANN}(r'))$. □

Note that although ANN(r_1) and ANN(r_2) are both *consistent*, $\mathrm{ANN}(r_1) \cup_{eq} \mathrm{ANN}(r_2)$ above is an *inconsistent* annotated relation (since for some data tuple d, the sum of the L_t values exceeds 1.0). This occurs since ANN(r_1) and ANN(r_2) are not *mutually consistent* (§4.1). In general, if consistent annotated relations ANN(r) and ANN(r') are also mutually consistent, then $\mathrm{ANN}(r) \cup_{eq} \mathrm{ANN}(r')$ will always be consistent.

5.4 Selection on an annotated relation

We represent a *selection condition over calendar τ* by the symbol $\mathcal{C}$. If $\mathcal{C}$ is of the form $(F\ op\ v)$ or $(t_1 \sim t_2)$, then $\mathcal{C}$ is an *atomic condition over τ*. Let $\mathcal{C}$ be an atomic condition, let $T_1 \sqsubseteq \cdots \sqsubseteq T_m$ be a linear hierarchy H of time units over τ, and suppose TP-relation r is over relational schema $A = (A_1, \ldots, A_k)$. Then one of the following cases must hold:
• If $F = A_i$ for some $1 \leq i < k$, then $\mathcal{C}$ is a *data condition*.
• If $F = T_j$ for some time unit T_j in H or if $\mathcal{C}$ is of the form $(t_1 \sim t_2)$, then $\mathcal{C}$ is a *temporal condition*.
• If $F =$ "L" or $F =$ "U", then $\mathcal{C}$ is a *probabilistic condition*.
• Otherwise, $\mathcal{C}$ is an *inapplicable condition*. In this case, $\sigma_{\mathcal{C}}(r)$ and $\sigma_{\mathcal{C}}(\mathrm{ANN}(r))$ are not defined. Notice that selections on the *hidden field* (i.e., $F = A_k$) are not permitted. Throughout this paper, we will assume that $\mathcal{C}$ is not an inapplicable condition.

Definition 5.8 (selection on an annotated relation; atomic condition) The *selection of atomic condition $\mathcal{C}$ on annotated relation ANN(r)*, denoted $\sigma_{\mathcal{C}}(\mathrm{ANN}(r))$, is defined in the following way:
• If $\mathcal{C}$ is a data condition, $\mathrm{ANN}(r'') = \{at \in \mathrm{ANN}(r) \mid d \text{ satisfies } \mathcal{C}\}$. In this case, our selection is based on the classical relational algebra.
• If $\mathcal{C}$ is a temporal condition, $\mathrm{ANN}(r'') = \{at \in \mathrm{ANN}(r) \mid t \in \mathrm{sol}(\mathcal{C})\}$.
• If $\mathcal{C}$ is a probabilistic condition, $\mathrm{ANN}(r'') = \{at \in \mathrm{ANN}(r) \mid ([L, U] = [L_t, U_t]) \text{ satisfies } \mathcal{C}\}$. □

For example if $\mathcal{C} = (2/8/1997 \sim 7/8/1997)$, $\sigma_{\mathcal{C}}(\mathrm{ANN}(r_1))$ and $\sigma_{\mathcal{C}}(\mathrm{ANN}(r_2))$ will be

Data	H	Day	Month	Year	L_t	U_t
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11
D1		5	8	1997	0.10	0.20
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20

Data	H	Day	Month	Year	L_t	U_t
D1		2	8	1997	0.10	0.25
D1		3	8	1997	0.05	0.125
D1		6	8	1997	0.10	0.20
D1		7	8	1997	0.10	0.20

but if $\mathcal{C} = (L \neq 0.10)$, $\sigma_{\mathcal{C}}(\mathrm{ANN}(r_1))$ and $\sigma_{\mathcal{C}}(\mathrm{ANN}(r_2))$ will be

Data	H	Day	Month	Year	L_t	U_t
D1		1	8	1997	0.32	0.44
D1		2	8	1997	0.16	0.22
D1		3	8	1997	0.08	0.11

Data	H	Day	Month	Year	L_t	U_t
D1		3	8	1997	0.05	0.125

Later, we will describe how to perform selections with non-atomic selection conditions ($\S 6.5$).

5.5 Difference of two annotated relations

As in the classical relational algebra, difference is only defined when both relations have the same schema. There are many possible ways of defining difference, but we have chosen to base our definition on the intuition that if two relations r and r' represent the information that two different "agents" have about the same world, then $r - r'$ should represent the information about the world that r has and r' does not.

Definition 5.9 (difference of two annotated relations) The *difference of annotated relations ANN(r) and ANN(r')*, denoted $\mathrm{ANN}(r) - \mathrm{ANN}(r')$, is defined as $\mathrm{ANN}(r'') = \{at \in \mathrm{ANN}(r) \mid (\forall at' \in \mathrm{ANN}(r'))\ (d \neq d' \vee t \neq t')\}$. □

Thus, ANN(r'') will not include $at \in \mathrm{ANN}(r)$ if there exists an $at' \in \mathrm{ANN}(r')$ which refers to the same event at the same point in time. For example, $\mathrm{ANN}(r_1) - \mathrm{ANN}(r_2)$ and $\mathrm{ANN}(r_2) - \mathrm{ANN}(r_1)$ will be

Data	H	Day	Month	Year	L_t	U_t
D1		1	8	1997	0.32	0.44
D1		5	8	1997	0.10	0.20

Data	H	Day	Month	Year	L_t	U_t
D1		9	8	1997	0.10	0.20

Suppose $at_1 = (d, t, 0.2, 0.4) \in \mathrm{ANN}(r)$ and $at_2 = (d, t, 0, 1) \in \mathrm{ANN}(r')$. Then by definition, ANN(r'') will not contain at_1. Now suppose we removed from ANN(r') all annotated tuples where $[L'_t, U'_t] = [0, 1]$. Here, ANN(r'') will contain at_1. Apparently, we cannot simply throw out tuples where $[L'_t, U'_t] = [0, 1]$.

Intuitively, if we do not have an annotated tuple for data tuple d at time t, then "we do not know anything about (d, t)'s probability". In this case, (d, t) is implicitly assigned a probability interval of $[0, 1]$. On the other hand, at_2 indicates that "we know that we do not know anything about (d, t)'s probability". This distinction is subtle yet important; by keeping these two cases distinct, we allow both the *closed world assumption* (where (d, t) is implicitly assigned a probability interval of $[0, 0]$) and the *open world assumption*.

5.6 Cartesian product of two annotated relations

Each tuple in the result of a cartesian product reflects the conjunction of two events. Suppose that at time t, events e_1 and e_2 have probability intervals $[L_1, U_1]$ and $[L_2, U_2]$ respectively. In order to compute the probability interval $[L, U]$ for the event $(e_1 \wedge e_2)$ at time t, we must apply a *probabilistic conjunction strategy* α, i.e., $[L, U] = [L_1, U_1] \otimes_\alpha [L_2, U_2]$ ($\S 2.4$). This allows users to ask queries such as "Compute the cartesian product of annotated relations $\mathrm{ANN}(r)$ and $\mathrm{ANN}(r')$ under the assumption that there is no information about dependencies between events in these relations."

Definition 5.10 (cartesian product of two annotated relations) The *cartesian product of annotated relations ANN(r) and ANN(r′) under the α probabilistic conjunction strategy*, denoted $\mathrm{ANN}(r) \times_\alpha \mathrm{ANN}(r')$, is defined as $\mathrm{ANN}(r'') = \{(d'', t, L''_t, U''_t) \mid (\exists at \in \mathrm{ANN}(r)) \wedge (\exists at' \in \mathrm{ANN}(r')) \wedge$ $(d'' = (\mathcal{P}(d), \mathcal{P}(d'), h'')) \wedge (h'' = (d.\mathrm{H} \parallel d'.\mathrm{H})) \wedge (t = t') \wedge ([L''_t, U''_t] = [L_t, U_t] \otimes_\alpha [L'_t, U'_t])\}$. □

Note that cartesian products only combine annotated tuples which refer to the same time point. It computes the combined data tuple d'' by merging (i) manifest data fields from ANN(r) (i.e., $\mathcal{P}(d)$), (ii) manifest data fields from ANN(r') (i.e., $\mathcal{P}(d')$), and (iii) $h'' = d.\mathrm{H} \parallel d'.\mathrm{H}$ (i.e., the hidden list concatenation of $d.\mathrm{H}$ and $d'.\mathrm{H}$). It then computes the combined probability interval by applying user selected conjunction strategy α. This is highly appropriate because when computing Cartesian Products, we are looking at the probability that the concatenation of the two data tuples is in the (ordinary set theoretic) cartesian product of the two relations at a given instant of time. This is therefore a *conjunctive* event, and hence, the use of a conjunctive p-strategy when performing cartesian products.

For example,

$\mathrm{ANN}(r_1) \times_{ig} \mathrm{ANN}(r_2)$

r_1.D	r_2.D	H	Day	Month	Year	L_t	U_t
D1	D1		2	8	1997	0.00	0.22
D1	D1		3	8	1997	0.00	0.11
D1	D1		6	8	1997	0.00	0.20
D1	D1		7	8	1997	0.00	0.20
D1	D1		8	8	1997	0.00	0.20

$\mathrm{ANN}(r_1) \times_{pc} \mathrm{ANN}(r_2)$

r_1.D	r_2.D	H	Day	Month	Year	L_t	U_t
D1	D1		2	8	1997	0.10	0.22
D1	D1		3	8	1997	0.05	0.11
D1	D1		6	8	1997	0.10	0.20
D1	D1		7	8	1997	0.10	0.20
D1	D1		8	8	1997	0.10	0.20

5.7 Projection on an annotated relation

A list $\mathcal{F}$ of fields is said to be *projectable* w.r.t. TP-relation r if (i) every field in $\mathcal{F}$ is a manifest data field of r, and (ii) $\mathcal{F}$ is non-empty. $\mathcal{F}$ is *projectable* w.r.t. annotated relation ANN(r) iff $\mathcal{F}$ is *projectable* w.r.t. r. It is important to note that hidden fields cannot be projected out.

Definition 5.11 (projection on an annotated relation) Let $\mathcal{F}$ be a list of fields which are *projectable* w.r.t. $\mathrm{ANN}(r)$ and let "$A_1, \ldots, A_n$" be the (possibly empty) list of all manifest data fields which appear in the primary key of ANN(r) but do not appear in $\mathcal{F}$. Then the *projection of field list $\mathcal{F}$ on annotated relation ANN(r)*, denoted $\pi_{\mathcal{F}}(\mathrm{ANN}(r))$, is defined as $\mathrm{ANN}(r'') = \{(d'', t, L_t, U_t) \mid (\exists at \in \mathrm{ANN}(r)) \wedge (d'' = (\pi_{\mathcal{F}}(\mathcal{P}(d)), h'')) \wedge (h'' = (d.\mathrm{H} \parallel$ "A_1:d.A$_1$, ..., A_n:d.A$_n$"$))\}$. □

Here, $\pi_{\mathcal{F}}(\mathcal{P}(d))$ works in the same way as projection in the classical relational algebra except it does not remove duplicates and it gracefully ignores fields in $\mathcal{F}$ which do not appear in $\mathcal{P}(d)$'s schema.

For example if $\mathcal{F}$ = "Data1" and if our primary key for ANN(r_3) was "Data1,Data2", then $\mathrm{ANN}(r'') = \pi_{\mathcal{F}}(\mathrm{ANN}(r_3))$ will be

Data1	H	Day	Month	Year	L_t	U_t
D1	Data2:D2	2	8	1997	0.20	0.40
D1	Data2:D3	2	8	1997	0.30	0.40
D1	Data2:D3	3	8	1997	0.15	0.20

Notice that if we did not have the hidden field h'', then we would not be able to tell whether (D1) refers to event (D1,D2) or to event (D1,D3). In other words, the hidden field helps us to prevent a loss of information.

Now suppose that after a projection, we wanted (D1) to refer to all events where Data1 = D1. For the example above, this would mean that event (D1) should refer to the compound event ((D1,D2) $\vee$ (D1,D3)). This interpretation is not directly supported by our algebras since our framework only allows *instantaneous* events (§3). However, our algebra is rich enough to express it indirectly by (i) setting each h'' in ANN(r'') to EMPTY and then (ii) invoking a disjunctive p-strategy based compaction (§5.1) on the result of step (i). The resulting annotated relation may be inconsistent. All tuples in the result of this operation denote instantaneous events.

To help reduce the size of the hidden field, projection only retains field-value pairs for fields which appear in the relation's primary key. Thus when the primary key is small, h'' will also be small.

5.8 Join of two annotated relations

For simplicity, this paper will only consider the "natural join" operation.

Definition 5.12 (join of two annotated relations) Let selection condition $\mathcal{C}$ be defined as $((\mathrm{ANN}(r).\mathcal{L}_1 = \mathrm{ANN}(r').\mathcal{L}_1) \wedge \ldots \wedge (\mathrm{ANN}(r).\mathcal{L}_n = \mathrm{ANN}(r').\mathcal{L}_n))$ where "$\mathcal{L}_1 \ldots \mathcal{L}_n$" is the list of all manifest data fields which occur in the schema for both ANN(r) and ANN(r'). Then the *join of annotated relations ANN(r) and ANN(r') under the α probabilistic conjunction strategy*, denoted $\mathrm{ANN}(r) \bowtie_\alpha \mathrm{ANN}(r')$, is defined as $\pi_{\mathcal{F}}(\sigma_{\mathcal{C}}(\mathrm{ANN}(r) \times_\alpha \mathrm{ANN}(r')))$ where $\mathcal{F}$ is the list of all manifest data fields which occur in the schema for either ANN(r) or ANN(r') after removing duplicate field names. $\Box$

For example, $\mathrm{ANN}(r'') = \mathrm{ANN}(r_3) \bowtie_{pc} \mathrm{ANN}(r_4)$ will be

Data1	Data2	H	Day	Month	Year	L_t	U_t
D1	D2		2	8	1997	0.10	0.40
D1	D3		3	8	1997	0.15	0.20

Notice that all of the hidden fields in ANN(r'') above are EMPTY. This occurs since in our example, $\mathcal{F}$ = "Data1,Data2" so when we perform a projection, the list of manifest data fields not appearing in $\mathcal{F}$ (i.e., the "$A_1, \ldots, A_n$" list in Definition 5.11) is empty.

Although our definition of join in this section only corresponds to a natural join, it can easily be extended to handle other types of join. For instance, an implementation which uses an SQL-like interface may allow users to explicitly specify appropriate values for $\mathcal{C}$ and $\mathcal{F}$.

6 TP-Algebra

This is the most important section of the paper. As we have mentioned several times before, for every data-tuple d and every time-point t, the TATA algebra *explicitly* represents the probability that a data-tuple d is in a given relation at time t. As pointed out by Dyreson and Snodgrass[13], this leads to a completely unacceptable explosion in the size of annotated relations and leads to major scalability problems. **In this section, we will show that TP-relations, which *implicitly* and *compactly* represent temporal probabilistic data, can be very efficiently manipulated by algebraic operations that correctly implement (as defined below) all the operations on the TP-algebra. In other words, we can use the TP-representation to efficiently implement operations analogous to the TATA algebra operations.**

In this section, we provide definitions for *TP-compression*, *compaction*, *intersection*, *union*, *selection*, *difference*, *cartesian product*, *projection*, *join*, and *tightening* of TP-relations. With the exception of TP-compression and tightening (which have no analogs in TATA), we will show that each of these operations *correctly implement* the corresponding operations in the TATA. The advantage is immediate: as TP-relations are relatively small when compared to their annotated counterparts, a huge savings, both in space (of storing TP- vs. annotated relations) and time (in terms of the time to process these operations) will result. The correctness theorems are stated in this section and the proofs are given in Appendix C.

Definition 6.1 (correctly implements) Unary TPA operator op^T *correctly implements* the semantics for unary TATA operator op^A iff $\mathrm{ANN}(op^T(r)) = op^A(\mathrm{ANN}(r))$ for every TP-relation r.

Furthermore, binary TPA operator op^T *correctly implements* the semantics for binary TATA operator op^A iff $\mathrm{ANN}(r\ op^T\ r') = \mathrm{ANN}(r)\ op^A\ \mathrm{ANN}(r')$ for every pair of TP-relations r, r'. $\square$

Note that as usual, intersection, union, and difference are only defined when both TP-relations have the same schema, selections are only defined when $\mathcal{C}$ is not an *inapplicable condition*, and projections are only defined when field list $\mathcal{F}$ is *projectable*.

The definitions in this section will produce a new TP-relation r'' based on input from consistent TP-relations r, r'. Oftentimes, these definitions will refer to TP-tuples tp, tp' which are assumed to be of the form $tp = (d, \gamma)$ and $tp' = (d', \gamma')$. Here, let γ contain n TP-cases of the form $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle \in \gamma$ and let γ' contain n' TP-cases of the form $\gamma'_j = \langle C'_j, D'_j, L'_j, U'_j, \delta'_j \rangle \in \gamma'$.

6.1 TP-compression of a TP-relation

The basic idea behind TP-compression is to allow the data in a TP-relation (either base or derived) to be compressed. Let $N(r)$ and $N(tp)$ denote the number of TP-cases in TP-relation r and TP-tuple tp respectively. When we apply a *TP-compression function* to r, we may be able to reduce the size of $N(r)$.

Definition 6.2 (TP-compression function) A *TP-compression function* $\Xi(r)$ is a function which takes TP-relation r as input, and returns as output a TP-relation r'' where (i) $N(r'') \leq N(r)$ and (ii) there exists a bijection between ANN(r) and ANN(r'') which maps each $(d, t, L_t, U_t) \in \mathrm{ANN}(r)$ to a $(d, t, L_t, U''_t) \in \mathrm{ANN}(r'')$ such that $L_t \leq U''_t \leq U_t$. $\square$

In other words, ANN(r'') and ANN(r) must have the same data tuples, time points, and lower bounds...but we allow TP-compressions to tighten upper bounds. Note that there are many functions which satisfy the definition of a TP-compression function given above. For instance, the following TP-compression function combines TP-cases which share the same distribution.

Definition 6.3 (TP-compression of a TP-relation; same-distribution) The *same-distribution TP-compression of TP-relation* r, denoted $\Xi^{sd}(r)$, is equal to the multiset S which can be constructed in the following way: Initially, let $S = r''$. Then for each $(d, \gamma'') \in r''$ and for each pair of TP-cases $\gamma_i = \langle C_i, D_i, L, U, \delta \rangle, \gamma_j = \langle C_j, D_j, L, U, \delta \rangle \in \gamma''$ where $\mathrm{sol}(D_i) = \mathrm{sol}(D_j)$, remove γ_i, γ_j from γ'' and add TP-case $\langle (C_i \vee C_j), D_i, L, U, \delta \rangle$ to γ''. $\square$

Another possible TP-compression function takes advantage of the uniform distribution's regularity.

Definition 6.4 (TP-compression of a TP-relation; u-based) The *u-based TP-compression of TP-relation* r, denoted $\Xi^u(r)$, is equal to the multiset S which can be constructed in the following way: Initially, let $S = r''$. Then for each $(d, \gamma'') \in r''$ and for each pair of TP-cases $\gamma_i = \langle C_i, D_i, L_i, U_i, u \rangle, \gamma_j =$

$\langle C_j, D_j, L_j, U_j, u\rangle \in \gamma''$ where $n_i = |\mathrm{sol}(D_i)|$, $n_j = |\mathrm{sol}(D_j)|$, and $([L_t, U_t] = \frac{1}{n_i}\cdot[L_i, U_i] = \frac{1}{n_j}\cdot[L_j, U_j])$, remove γ_i, γ_j from γ'' and add TP-case $\langle (C_i \vee C_j), (D_i \vee D_j), L_{ij}, \min(1, U_{ij}), u\rangle$ to γ'' where $n_{ij} = |\mathrm{sol}(D_i \vee D_j)|$ and $[L_{ij}, U_{ij}] = n_{ij} \cdot [L_t, U_t]$. □

Note that when $U_{ij} > 1$, the upper bounds in ANN(r'') will be tighter than the ones in ANN(r).

Definition 6.5 (TP-compression of a TP-relation; hybrid) The *hybrid TP-compression of TP-relation* r, denoted $\Xi^{hy}(r)$, is defined as $\Xi^{sd}(\Xi^{u}(r))$. □

The following theorem indicates that the functions above satisfy our definition for a TP-compression.

Theorem 6 *$\Xi^{sd}(r)$, $\Xi^{u}(r)$ and $\Xi^{hy}(r)$ are all TP-compression functions.*

More sophisticated TP-compression operators are also possible. For instance, let $p \in (0,1)$ be a probability and let $t_1, \ldots, t_n$ be a list of (consecutive) time points in S_τ where for each $1 \leq i < n$, $t_{i+1} = next_\tau(t_i)$. Then if γ'' contains n TP-cases of the form $\langle (\#), (t_i), L''_{t_i}, U''_{t_i}, u\rangle$ where ($[L''_{t_{i+1}}, U''_{t_{i+1}}] = p \cdot [L''_{t_i}, U''_{t_i}]$ for all $1 \leq i < n$) and ($U''_{t_1} \div p \leq 1$), apply an operator which replaces these n TP-cases with the TP-case $\langle (\#), (t_1 \sim t_n), L'', U'', \delta''\rangle$ where $[L'', U''] = [L''_{t_1} \div p, U''_{t_1} \div p]$ and $\delta'' =$ "g, p".

The aforementioned operator performs a *g-based TP-compression*. Note that for any distribution function δ, one can define a corresponding δ-based TP-compression operator. Also note that one can obtain optimal TP-compression (i.e., the smallest possible value for $N(r'')$) by allowing a TP-compression operator to dynamically create new distribution functions which fit the resulting data.

6.2 Compaction of a TP-relation

As in the case of the TATA, the *compaction* operation in the TPA will be used to define the operations of *intersection*, *union* and *projection*.

Definition 6.6 (Compaction of a TP-relation) A function κ from TP-relations to TP-relations is called a *compaction operation* if it satisfies the following axioms:
- Compactness : $\kappa(r)$ is *compact* for all TP-relations r.
- No Fooling Around (NFA) : If r is compact then $\mathrm{ANN}(\kappa(r)) = \mathrm{ANN}(r)$.
- Conservativeness : If $at = (d, t, L_t, U_t) \in \mathrm{ANN}(\kappa(r))$, then $\exists at' = (d, t, L'_t, U'_t) \in \mathrm{ANN}(r)$. □

The Compactness, NFA, and Conservativeness axioms for TP-relations are similar in spirit and intuition to the same concepts defined earlier for annotated relations.

As in the case of TATA, there are many different compaction operations on TP-relations. Below, we present the TP analogs of χ-compactions and p-strategy based compactions.

Definition 6.7 (χ-compaction of a TP-relation) Let χ be a combination function. Then the *χ-compaction of TP-relation* r, denoted $\kappa_\chi(r)$, is defined as $\mathrm{ANN}(\kappa_\chi(r)) =$
$\{at = (d, t, L_t, U_t) \mid [L_t, U_t] = \chi(\{[L_1^{(d,t)}, U_1^{(d,t)}], \ldots, [L_k^{(d,t)}, U_k^{(d,t)}]\})\}$ where
$\mathrm{ANN}(r)[d,t] = \{at_1^{(d,t)}, \ldots at_k^{(d,t)}\}$ and $at_i^{(d,t)} = (d, t, L_i^{(d,t)}, U_i^{(d,t)})$. □

The following lemma states that an operation defined in this manner is indeed a *compaction* operation.

Lemma 1 *Let χ be a combination function. Then $\kappa_\chi(r)$ is a compaction operation.*

Note that some parts of Definitions 6.6 and 6.7 were presented in a *declarative manner* — via the contents of the annotation of the result. Algorithm **Compute-Compaction** shown below provides a mechanism to efficiently compute compactions without resorting to annotation. This algorithm can perform compactions using either a combination function χ or a p-strategy ρ. The boxed line in this algorithm shows exactly where a combination function or p-strategy is applied to compact data-identical tuples. Note that when f is a p-strategy, the application of f to a set X of intervals merely represents the iterative application of f to pairs of intervals in X — as p-strategies are associative and commutative, this is well defined.

Algorithm Compute-Compaction(r,f):
Input: TP-relation r and combination function or p-strategy f
Output: TP-relation $r'' = \kappa_f(r)$
$r'' := \emptyset$; `// Initialize the resulting relation`
$r' := r$; `// Obtain a working copy of the initial relation`
`// For each (maximal) multiset S of data-identical TP-tuples in r`
while $(r' \neq \emptyset)$ **do** {
    Select a TP-tuple $tp \in r'$;
    $S := r'[tp]$; `// Extract the next equivalence class from r'`
    $r' := r' - S$; $\gamma'' := \emptyset$;
    $\Gamma_S := \biguplus_{(d,\gamma) \in S} \gamma$; $\Gamma_I := \Gamma_S$;
    foreach $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle \in \Gamma_I$ {
        Remove γ_i from Γ_I and Γ_S; $C_S := \bigvee_{\langle C,D,L,U,\delta \rangle \in \Gamma_S} C$;
        if $\mathrm{sol}(C_i \wedge \neg C_s) \neq \emptyset$ **then** Add $\langle (C_i \wedge \neg C_s), D_i, L_i, U_i, \delta_i \rangle$ to γ'';
        if $\mathrm{sol}(C_i \wedge C_s) \neq \emptyset$ **then** Add $\langle (C_i \wedge C_s), D_i, L_i, U_i, \delta_i \rangle$ to Γ_S; }
    $C_S := \bigvee_{\langle C,D,L,U,\delta \rangle \in \Gamma_S} C$;
    `// Note: Each` $t \in \mathrm{sol}(C_S)$ `will refer to more than one TP-case`
    foreach $t \in \mathrm{sol}(C_S)$ {
        $X := \emptyset$; `//` X `will contain the probability intervals to be combined`
        $\Gamma_t := \{\langle C, D, L, U, \delta \rangle \in \Gamma_S \mid t \in \mathrm{sol}(C)\}$;
        foreach $\langle C, D, L, U, \delta \rangle \in \Gamma_t$ {
            $x_t := \delta(D, t)$; $[L_t, U_t] := [L \cdot x_t, U \cdot x_t]$;
            $X := X \cup \{[L_t, U_t]\}$; }
        $\boxed{[L''_t, U''_t] := f(X);}$
        Add TP-case $\langle (\#), (t), L''_t, U''_t, u \rangle$ to γ''; }
    Add TP-tuple (d, γ'') to r''; }
return r'';
End-Algorithm

Theorem 7 *Let χ be a combination function. Then algorithm* **Compute-Compaction(**r, χ**)** *correctly computes the $\kappa_\chi(r)$ compaction operation.*

We define p-strategy based compactions of TP-relations in the same way as $\kappa_\chi(r)$ except we let $[L_t, U_t] = ([L_1^{(d,t)}, U_1^{(d,t)}] \otimes \ldots \otimes [L_k^{(d,t)}, U_k^{(d,t)}])$ and let $[L_t, U_t] = ([L_1^{(d,t)}, U_1^{(d,t)}] \oplus \ldots \oplus [L_k^{(d,t)}, U_k^{(d,t)}])$ when defining $\kappa_\otimes(r)$ and $\kappa_\oplus(r)$ respectively.

Lemma 2 *Let ρ be a p-strategy. Then $\kappa_\rho(r)$ is a compaction operation.*

As p-strategy based compaction of TP-relations is defined declaratively, we need an explicit algorithm (mentioned above) to compute it. The following result states the correctness of this algorithm.

Theorem 8 *Let ρ be a (conjunctive or disjunctive) p-strategy. Then algorithm* **Compute-Compaction(r, ρ)** *correctly computes the $\kappa_\rho(r)$ compaction operation.*

Thus far, we have separately defined compaction operators on annotated relations and on TP-relations. The following definition specifies when a compaction operator on the annotated side corresponds to a compaction operator on the TP-side.

Definition 6.8 (compatible pair of compactions) A pair $\langle \kappa^A(\mathrm{ANN}(r)), \kappa^T(r)\rangle$ of compaction operators is a *compatible pair* iff for every TP-relation r, $\kappa^A(\mathrm{ANN}(r)) = \mathrm{ANN}(\kappa^T(r))$. □

The following two theorems say that for any arbitrary combination function χ and for any arbitrary p-strategy ρ, $\langle \kappa_\chi(\mathrm{ANN}(r)), \kappa_\chi(r)\rangle$ and $\langle \kappa_\rho(\mathrm{ANN}(r)), \kappa_\rho(r)\rangle$ are compatible pairs.

Theorem 9 *Let χ be any combination function. Then $\langle \kappa_\chi(\mathrm{ANN}(r)), \kappa_\chi(r)\rangle$ is a compatible pair.*

Theorem 10 *Let ρ be any p-strategy. Then $\langle \kappa_\rho(\mathrm{ANN}(r)), \kappa_\rho(r)\rangle$ is a compatible pair.*

6.3 Intersection of two TP-relations

In this section, we show how we can *correctly implement* the intersection of two TP-relations. Intersection consists of of two suboperations — *multiset intersection* and *combination function based compaction.*

Definition 6.9 (multiset intersection of two TP-relations) The *multiset intersection of TP-relations r and r'*, denoted $r \cap r'$, can be constructed in the following way: Initially, let $r'' = \emptyset$. Then for each $tp = (d, \gamma) \in r$ and each $tp' = (d', \gamma') \in r'$ where $(d = d')$,
1. Let $\Gamma = \Gamma' = \emptyset$.
2. For each $\gamma_i \in \gamma$ and each $\gamma'_j \in \gamma'$ where $|\mathrm{sol}(C_i \wedge C'_j)| \geq 1$, add TP-case $\langle (C_i \wedge C'_j), D_i, L_i, U_i, \delta_i\rangle$ to Γ and add TP-case $\langle (C_i \wedge C'_j), D'_j, L'_j, U'_j, \delta'_j\rangle$ to Γ'. Note that $(C_i \wedge C'_j)$ is shared by both TP-cases.
3. If $\Gamma \neq \emptyset$, add TP-tuples (d, Γ) and (d, Γ') to r''. Note that $\Gamma \neq \emptyset \Rightarrow \Gamma' \neq \emptyset$. Γ will be empty if there are no overlapping time points. □

Definition 6.10 (intersection of two TP-relations) The *intersection of TP-relations r and r' under the χ combination function*, denoted $r \cap_\chi r'$, is defined as $\kappa_\chi(r \cap r')$. □

As in the case of the TATA, we apply a κ_χ compaction operator to the result of a multiset intersection. Also, to keep the size of $r \cap_\chi r'$ manageable, we usually perform a TP-compression on the result of a compaction.

For example, $r'' = r_1 \cap r_2$ will be

Data	H	C	D	L	U	δ
D1		$(2/8/1997 \sim 3/8/1997)$ $(6/8/1997 \sim 8/8/1997)$	$(1/8/1997 \sim 3/8/1997)$ $(5/8/1997 \sim 8/8/1997)$	0.64 0.40	0.88 0.80	g u
D1		$(2/8/1997 \sim 3/8/1997)$ $(6/8/1997 \sim 8/8/1997)$	$(2/8/1997 \sim 3/8/1997)$ $(6/8/1997 \sim 9/8/1997)$	0.20 0.40	0.50 0.80	g u

and, $\Xi^{hy}(r \cap_{eq} r') = \Xi^{hy}(\kappa_{eq}(r''))$ will be

Data	H	C	D	L	U	δ
D1		$(\#)$	$(2/8/1997 \sim 2/8/1997)$	0.16	0.22	u
		$(\#)$	$(3/8/1997 \sim 3/8/1997)$	0.08	0.11	u
		$(\#)$	$(6/8/1997 \sim 8/8/1997)$	0.30	0.60	u

The following shows that our definition of intersection correctly implements the TATA semantics. This lets us completely avoid the construction of the (huge) annotated expansion while preserving the same semantics.

Theorem 11 (Correctness of intersection) $\mathrm{ANN}(r \cap_\chi r') = \mathrm{ANN}(r) \cap_\chi \mathrm{ANN}(r')$.

6.4 Union of two TP-relations

In this section, we show how we can *correctly implement* the union of two TP-relations. Union consists of two suboperations – *multiset union* and *combination function based compaction.*

Definition 6.11 (multiset union of two TP-relations) The *multiset union of TP-relations r and r'*, denoted $r \cup r'$, is defined as $r'' = r \uplus r'$. □

Intuitively, r'' will contain all $tp \in r$ and all $tp' \in r'$. For example, $r'' = r_1 \cup r_2$ will be

Data	H	C	D	L	U	δ
D1		$(\#)$	$(1/8/1997 \sim 3/8/1997)$	0.64	0.88	g
		$(\#)$	$(5/8/1997 \sim 8/8/1997)$	0.40	0.80	u
D1		$(\#)$	$(2/8/1997 \sim 3/8/1997)$	0.20	0.50	g
		$(\#)$	$(6/8/1997 \sim 9/8/1997)$	0.40	0.80	u

As in the case of the TATA, we apply a κ_χ compaction operator to the result of a multiset union.

Definition 6.12 (union of two TP-relations) The *union of TP-relations r and r' under the χ combination function*, denoted $r \cup_\chi r'$, is defined as $\kappa_\chi(r \cup r')$. □

For example, $\Xi^{hy}(r \cup_{eq} r') = \Xi^{hy}(\kappa_{eq}(r''))$ will be

Data	H	C	D	L	U	δ
D1		$(1/8/1997)$	$(1/8/1997 \sim 3/8/1997)$	0.64	0.88	g
		$(\#)$	$(2/8/1997 \sim 2/8/1997)$	0.16	0.22	u
		$(\#)$	$(3/8/1997 \sim 3/8/1997)$	0.08	0.11	u
		$(\#)$	$(5/8/1997 \sim 9/8/1997)$	0.50	1.00	u

The following shows that our definition of union correctly implements the TATA semantics.

Theorem 12 (Correctness of union) $\mathrm{ANN}(r \cup_\chi r') = \mathrm{ANN}(r) \cup_\chi \mathrm{ANN}(r')$.

6.5 Selection on a TP-relation

In this section, we show how we can *correctly implement* selection on a TP-relation. The *TP-filter* operator defined below will help us handle selections of *probabilistic conditions* (§5.4) on TP-relations.

Definition 6.13 (TP-filter) Let $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$ be a TP-case, let $\mathcal{C}$ = $(F \; op \; v)$ be a probabilistic condition, and let $x = L_i$ if F = "L" or let $x = U_i$ otherwise. Then a *TP-filter* is a function which takes γ_i and $\mathcal{C}$ as input, and returns as output a temporal constraint C''_i where
1. $\text{sol}(C''_i) \subseteq \text{sol}(C_i)$
2. For each time point $t \in \text{sol}(C''_i)$, $(x_t \; op \; v)$ must be true when $x_t = \delta_i(D_i, t) \cdot x$
3. There is no temporal constraint C'_i where $(\text{sol}(C'_i) \supset \text{sol}(C''_i))$ and C'_i satisfies the previous cases.

Intuitively, a TP-filter returns a temporal constraint whose solution set consists of all time points $t \in \text{sol}(C_i)$ where $[L_t, U_t] = [L_i \cdot \delta_i(D_i, t), U_i \cdot \delta_i(D_i, t)]$ satisfies $\mathcal{C}$. If no $t \in \text{sol}(C_i)$ satisfies this condition, TP-filter$(\gamma_i, \mathcal{C})$ returns an inconsistent temporal constraint. □

For example if $\gamma_i = \langle (\#), (5/8/1997 \sim 8/8/1997), 0.4, 0.8, g \rangle$ and $\mathcal{C} = (U > 0.15)$, TP-filter$(\gamma_i, \mathcal{C})$ will be $C''_i = (5/8/1997 \sim 6/8/1997)$ since $(0.4 > 0.15)$ for $5/8/1997$ and $(0.2 > 0.15)$ for $6/8/1997$.

In general, $n = |\text{sol}(C_i)|$ may be a large number. With arbitrary distribution functions, this can be problematic since the TP-filter function may have to test all n time points. Fortunately, this problem can be alleviated by exploiting regularities in our distribution functions. For instance if δ_i = "u", then we only need to test one time point $t \in \text{sol}(C_i)$; if t should be in sol(C''_i), then $C''_i = C_i$ or $C''_i = \emptyset$ otherwise. This "all or none" behavior occurs since each $t \in \text{sol}(C_i)$ will have the same probability value after distributing uniformly.

Implementations of TP-filters can also exploit regularities in the geometric PDF by searching sol(C_i) in chronological (or reverse chronological) order and then ending the search after finding the first t which should not be in sol(C''_i). The exact search method to use will, of course, depend on which *op* is present in $\mathcal{C}$. For instance if *op* = $(\neq)$, it may be cheaper to let C'_i = TP-filter$(\gamma_i, \neg\mathcal{C})$ and then return $C''_i = (C_i \wedge \neg C'_i)$.

Definition 6.14 (Selection on a TP-tuple; atomic condition) The *selection of atomic condition $\mathcal{C}$ on TP-tuple* $tp = (d, \gamma)$, denoted $\sigma_{\mathcal{C}}(tp)$, can be constructed in the following way: Initially, let $\gamma'' = \emptyset$.
- If $\mathcal{C}$ is a data condition, let $\gamma'' = \gamma$ if d satisfies $\mathcal{C}$.
- If $\mathcal{C}$ is a temporal condition, then for each $\gamma_i \in \gamma$ where $C''_i = (C_i \wedge \mathcal{C})$ is consistent, add TP-case $\langle C''_i, D_i, L_i, U_i, \delta_i \rangle$ to γ''.
- If $\mathcal{C}$ is a probabilistic condition, then for each $\gamma_i \in \gamma$ where C''_i = TP-filter$(\gamma_i, \mathcal{C})$ is consistent, add TP-case $\langle C''_i, D_i, L_i, U_i, \delta_i \rangle$ to γ''.

If $\gamma'' = \emptyset$ then $\sigma_{\mathcal{C}}(tp) = \emptyset$. Otherwise, $\sigma_{\mathcal{C}}(tp) = (d, \gamma'')$. □

Definition 6.15 (Selection on a TP-relation; atomic condition) The *selection of atomic condition $\mathcal{C}$ on TP-relation* $r = \{tp_1, \ldots, tp_n\}$, denoted $\sigma_{\mathcal{C}}(r)$, is defined as $(\sigma_{\mathcal{C}}(tp_1) \uplus \ldots \uplus \sigma_{\mathcal{C}}(tp_n))$. □

Note that for all $tp_i, tp_j \in r$, $\sigma_{\mathcal{C}}(tp_i)$ does not affect the results of $\sigma_{\mathcal{C}}(tp_j)$ when computing $\sigma_{\mathcal{C}}(r)$.

For example if $\mathcal{C} = (2/8/1997 \sim 7/8/1997)$, $\sigma_{\mathcal{C}}(r_1)$ will be

Data	H	C	D	L	U	δ
D1		$(2/8/1997 \sim 3/8/1997)$ $(5/8/1997 \sim 7/8/1997)$	$(1/8/1997 \sim 3/8/1997)$ $(5/8/1997 \sim 8/8/1997)$	0.64 0.40	0.88 0.80	g u

and $\sigma_{\mathcal{C}}(r_2)$ will be

Data	H	C	D	L	U	δ
D1		$(2/8/1997 \sim 3/8/1997)$ $(6/8/1997 \sim 7/8/1997)$	$(2/8/1997 \sim 3/8/1997)$ $(6/8/1997 \sim 9/8/1997)$	0.20 0.40	0.50 0.80	g u

but if $\mathcal{C} = (L \neq 0.10)$, $\sigma_{\mathcal{C}}(r_1)$ will be

Data	H	C	D	L	U	δ
D1		$(\#)$	$(1/8/1997 \sim 3/8/1997)$	0.64	0.88	g

and $\sigma_{\mathcal{C}}(r_2)$ will be

Data	H	C	D	L	U	δ
D1		$(3/8/1997)$	$(2/8/1997 \sim 3/8/1997)$	0.20	0.50	g

We can extend selection to handle non-atomic selection conditions by using the following definition.

Definition 6.16 (Selection on a TP-relation) The *selection of condition $\mathcal{C}$ on TP-relation r*, denoted $\sigma_{\mathcal{C}}(r)$, is defined inductively in the following way:
• If $\mathcal{C}$ is an atomic condition, then $r'' = \sigma_{\mathcal{C}}(r)$ by way of our previous definition.
• If $\mathcal{C}$ is of the form $(\mathcal{C}_1 \wedge \mathcal{C}_2)$, then $r'' = \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$.
• If $\mathcal{C}$ is of the form $(\mathcal{C}_1 \vee \mathcal{C}_2)$, then $r'' = \sigma_{\mathcal{C}_1}(r) \cup_{eq} \sigma_{\mathcal{C}_2}(r)$. (Note that as long as r is compact, it follows by the Identity axiom that irrespective of which combination function is used, we obtain the same results, i.e. "eq" in the above definition can be replaced with any other combination without the result being changed.)
• If $\mathcal{C}$ is of the form $(\neg\mathcal{C}_1)$, then

⋆ If $\mathcal{C}_1$ is of the form $(\mathcal{C}_2 \wedge \mathcal{C}_3)$, $(\mathcal{C}_2 \vee \mathcal{C}_3)$, or $(\neg\mathcal{C}_2)$, then $r'' = \sigma_{\mathcal{C}_4}(r)$ where $\mathcal{C}_4 = (\neg\mathcal{C}_2 \vee \neg\mathcal{C}_3)$, $\mathcal{C}_4 = (\neg\mathcal{C}_2 \wedge \neg\mathcal{C}_3)$, or $\mathcal{C}_4 = (\mathcal{C}_2)$ respectively.

⋆ If $\mathcal{C}_1$ is a *data*, *temporal*, or *probabilistic* condition, then $r'' = \sigma_{\mathcal{C}_4}(r)$ where $\mathcal{C}_4$ is the atomic, logical negation of $\mathcal{C}_1$.

⋆ Otherwise, $\mathcal{C}_1$ is an *inapplicable* condition and so r'' is not defined.

To perform selections with non-atomic conditions on annotated relations, use the definition above except replace all instances of r and r'' with ANN(r) and ANN(r'') respectively. □

For example if $\mathcal{C}_1 = (2/8/1997 \sim 7/8/1997), \mathcal{C}_2 = (L \neq 0.10)$, and $\mathcal{C} = (\mathcal{C}_1 \wedge \mathcal{C}_2)$, then $\sigma_{\mathcal{C}}(r_1)$ will be

Data	H	C	D	L	U	δ
D1		$(2/8/1997 \sim 3/8/1997)$	$(1/8/1997 \sim 3/8/1997)$	0.64	0.88	g

and $\sigma_{\mathcal{C}}(r_2)$ will be

Data	H	C	D	L	U	δ
D1		$(3/8/1997)$	$(2/8/1997 \sim 3/8/1997)$	0.20	0.50	g

Theorem 13 $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) = \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$.

Description	Specification	Query	Conditions
e before e_q	$end(e) \leq st(e_q)$	$\sigma_{\mathcal{C}}(r)$	$\mathcal{C} = ((t_S \sim t_1) \wedge (\text{Kind} = E))$
e equal e_q	$st(e) = st(e_q) \wedge$ $end(e) = end(e_q)$	$\sigma_{\mathcal{C}_1}(r) \cap_{eq} \sigma_{\mathcal{C}_2}(r)$	$\mathcal{C}_1 = ((t_1 \sim t_1) \wedge (\text{Kind} = S))$ $\mathcal{C}_2 = ((t_2 \sim t_2) \wedge (\text{Kind} = E))$
e meets e_q	$end(e) = st(e_q)$	$\sigma_{\mathcal{C}}(r)$	$\mathcal{C} = ((t_1 \sim t_1) \wedge (\text{Kind} = E))$
e overlaps e_q	$st(e) \leq st(e_q) \wedge$ $st(e_q) \leq end(e)$	$\sigma_{\mathcal{C}_1}(r) \cap_{eq} \sigma_{\mathcal{C}_2}(r)$	$\mathcal{C}_1 = ((t_S \sim t_1) \wedge (\text{Kind} = S))$ $\mathcal{C}_2 = ((t_1 \sim t_E) \wedge (\text{Kind} = E))$
e during e_q	$st(e_q) \leq st(e) \wedge$ $end(e) \leq end(e_q)$	$\sigma_{\mathcal{C}_1}(r) \cap_{eq} \sigma_{\mathcal{C}_2}(r)$	$\mathcal{C}_1 = ((t_1 \sim t_E) \wedge (\text{Kind} = S))$ $\mathcal{C}_2 = ((t_S \sim t_2) \wedge (\text{Kind} = E))$
e starts e_q	$st(e) = st(e_q) \wedge$ $end(e) \leq end(e_q)$	$\sigma_{\mathcal{C}_1}(r) \cap_{eq} \sigma_{\mathcal{C}_2}(r)$	$\mathcal{C}_1 = ((t_1 \sim t_1) \wedge (\text{Kind} = S))$ $\mathcal{C}_2 = ((t_S \sim t_2) \wedge (\text{Kind} = E))$
e finishes e_q	$st(e_q) \leq st(e)$ $end(e) = end(e_q)$	$\sigma_{\mathcal{C}_1}(r) \cap_{eq} \sigma_{\mathcal{C}_2}(r)$	$\mathcal{C}_1 = ((t_1 \sim t_E) \wedge (\text{Kind} = S))$ $\mathcal{C}_2 = ((t_2 \sim t_2) \wedge (\text{Kind} = E))$

The preceding table shows how one may generate queries on TP-relation r which correspond to seven of J. F. Allen's thirteen possible temporal relationships [1]. The six remaining possibilities correspond to the inverses of these original seven (the inverse of "equal" is identical to "equal" so it is not counted). Here, we assume that r uses two TP-tuples for each event e; one for $st(e)$ (where the value of the "Kind" field is "S"), and one for $end(e)$ (where the value of the "Kind" field is "E"). Assume that for for each event e, if $st(e)$ or $end(e)$ satisfies the selection condition, then the TP-tuples for both $st(e)$ and $end(e)$ should be included in the result. Our queries will return every event e which satisfies some relationship w.r.t. event e_q where $st(e_q) = t_1$ and $end(e_q) = t_2$.

Theorem 14 (Correctness of selection) $\mathrm{ANN}(\sigma_{\mathcal{C}}(r)) = \sigma_{\mathcal{C}}(\mathrm{ANN}(r))$.

6.6 Difference of two TP-relations

In this section, we show how we can *correctly implement* the difference of two TP-relations.

Definition 6.17 (difference of two TP-relations) The *difference of TP-relations r and r'*, denoted $r - r'$, can be constructed in the following way: Initially, let $r'' = r$. Then for each $tp = (d, \gamma) \in r''$ and each $tp' = (d', \gamma') \in r'$ where $(d = d')$,
1. Let $\gamma'' = \emptyset$ and let $C' = (C'_1 \vee \ldots \vee C'_{n'})$. Recall that tp' contains exactly n' TP-cases.
2. For each $\gamma_i \in \gamma$ where $C''_i = (C_i \wedge \neg C')$ is consistent, add TP-case $\langle C''_i, D_i, L_i, U_i, \delta_i \rangle$ to γ''.
3.Remove tp from r''. Then if $\gamma'' \neq \emptyset$, add TP-tuple (d, γ'') to r''. □

For example $r_1 - r_2$ and $r_2 - r_1$ will be

Data	H	C	D	L	U	δ
D1		$(1/8/1997)$ $(5/8/1997)$	$(1/8/1997 \sim 3/8/1997)$ $(5/8/1997 \sim 8/8/1997)$	0.64 0.40	0.88 0.80	g u

and

Data	H	C	D	L	U	δ
D1		$(9/8/1997)$	$(6/8/1997 \sim 9/8/1997)$	0.40	0.80	u

Theorem 15 (Correctness of difference) $\mathrm{ANN}(r - r') = \mathrm{ANN}(r) - \mathrm{ANN}(r')$.

6.7 Cartesian product of two TP-relations

In this section, we show how we can *correctly implement* the cartesian product of two TP-relations. Recall, as in the annotated case, that when taking the cartesian product of two relations, we must know the relationship, if any, between the events denoted by the tuples in the two relations because the probability of a concatenated tuple being present in the result of the Cartesian Product is the probability of a conjunctive event. Thus, conjunction strategies *parameterize* the Cartesian Product operation.

Definition 6.18 (cartesian product of two TP-relations) The *cartesian product of TP-relations r and r' under the α probabilistic conjunction strategy*, denoted $r \times_\alpha r'$, can be constructed in the following way: Initially, let $r'' = \emptyset$. Then for each $tp = (d, \gamma) \in r$ and each $tp' = (d', \gamma') \in r'$,
1. Let $\gamma'' = \emptyset$.
2. For each time point t where $t \in \mathrm{sol}(C_i)$ for some $\gamma_i \in \gamma$ and $t \in \mathrm{sol}(C'_j)$ for some $\gamma'_j \in \gamma'$,
   (a) Let $[L_t, U_t] = [L_i \cdot x_t, U_i \cdot x_t]$ where $x_t = \delta_i(D_i, t)$.
   (b) Let $[L'_t, U'_t] = [L'_j \cdot x'_t, U'_j \cdot x'_t]$ where $x'_t = \delta'_j(D'_j, t)$.
   (c) Let $[L''_t, U''_t] = ([L_t, U_t] \otimes_\alpha [L'_t, U'_t])$.
   (d) Add TP-case $\langle (\#), (t), L''_t, U''_t, u \rangle$ to γ''.
3. If $\gamma'' \neq \emptyset$, add TP-tuple (d'', γ'') to r'' where $d'' = (\mathcal{P}(d), \mathcal{P}(d'), h'')$ and $h'' = (d.\mathrm{H} \parallel d'.\mathrm{H})$. □

For example, $\Xi^{hy}(r_1 \times_{ig} r_2)$ will be

r_1.Data	r_2.Data	H	C	D	L	U	δ
D1	D1		$(\#)$	$(2/8/1997 \sim 2/8/1997)$	0.00	0.22	u
			$(\#)$	$(3/8/1997 \sim 3/8/1997)$	0.00	0.11	u
			$(\#)$	$(6/8/1997 \sim 8/8/1997)$	0.00	0.60	u

but $\Xi^{hy}(r_1 \times_{pc} r_2)$ will be

r_1.Data	r_2.Data	H	C	D	L	U	δ
D1	D1		$(\#)$	$(2/8/1997 \sim 2/8/1997)$	0.10	0.22	u
			$(\#)$	$(3/8/1997 \sim 3/8/1997)$	0.05	0.11	u
			$(\#)$	$(6/8/1997 \sim 8/8/1997)$	0.30	0.60	u

The use of a TP-compression operation when executing a Cartesian product operation is important because sometimes, Cartesian product can produce a large number of TP-cases when an existing tp-case gets broken into "pieces." TP-compressions prevent this from happening. The following shows that our definition of cartesian product correctly implements the TATA semantics.

Theorem 16 (Correctness of cartesian product) $\mathrm{ANN}(r \times_\alpha r') = \mathrm{ANN}(r) \times_\alpha \mathrm{ANN}(r')$.

6.8 Projection on a TP-relation

In this section, we show how we can *correctly implement* projection on a TP-relation.

Definition 6.19 (projection on a TP-relation) Let $\mathcal{F}$ be a list of fields which are *projectable* w.r.t. r and let "$A_1, \ldots, A_n$" be the (possibly empty) list of all manifest data fields which appear in the primary key of r but do not appear in $\mathcal{F}$. Then the *projection of field list $\mathcal{F}$ on TP-relation* r, denoted $\pi_{\mathcal{F}}(r)$, can be constructed in the following way: Initially, let $r'' = \emptyset$. Then for each $(d, \gamma) \in r$, add TP-tuple (d'', γ) to r'' where $d'' = (\pi_{\mathcal{F}}(\mathcal{P}(d)), h'')$ and $h'' = (d.\mathrm{H} \mid\mid$ "A_1:$d.\mathrm{A}_1$, ..., A_n:$d.\mathrm{A_n}$"). (Recall that the $\pi_{\mathcal{F}}(\mathcal{P}(d))$ operator was defined in section 5.7.) □

For example if $\mathcal{F}$ = "Data1" and our primary key for r_3 was "Data1,Data2", $r'' = \pi_{\mathcal{F}}(r_3)$ will be

Data1	H	C	D	L	U	δ
D1	Data2:D2	$(\#)$	$(2/8/1997 \sim 2/8/1997)$	0.20	0.40	u
D1	Data2:D3	$(\#)$	$(2/8/1997 \sim 3/8/1997)$	0.60	0.80	g

Theorem 17 (Correctness of projection) $\mathrm{ANN}(\pi_{\mathcal{F}}(r)) = \pi_{\mathcal{F}}(\mathrm{ANN}(r))$.

6.9 Join of two TP-relations

In this section, we show how we can *correctly implement* the join of two TP-relations.

Definition 6.20 (join of two TP-relations) Let selection condition $\mathcal{C}$ be $((r.\mathcal{L}_1 = r'.\mathcal{L}_1) \wedge \ldots \wedge (r.\mathcal{L}_n = r'.\mathcal{L}_n))$ where "$\mathcal{L}_1 \ldots \mathcal{L}_n$" is the list of all manifest data fields which occur in the schema for both r and r'. Then the *join of TP-relations r and r' under the α probabilistic conjunction strategy*, denoted $r \bowtie_\alpha r'$, is defined as $\pi_{\mathcal{F}}(\sigma_{\mathcal{C}}(r \times_\alpha r'))$ where $\mathcal{F}$ is the list of all manifest data fields which occur in the schema for either r or r' after removing duplicate field names. □

For example, $r_3 \bowtie_{pc} r_4$ will be

Data1	Data2	H	C	D	L	U	δ
D1	D2		$(\#)$	$(2/8/1997)$	0.10	0.40	u
D1	D3		$(\#)$	$(3/8/1997)$	0.15	0.20	u

Theorem 18 (Correctness of join) $r \bowtie_\alpha r' = \mathrm{ANN}(r) \bowtie_\alpha \mathrm{ANN}(r')$.

6.10 Tightening of a TP-relation

Recall that TP-tuples can be "loose" in the sense that a TP-tuple may, for example, have two TP-cases, γ_1, γ_2 with $[L_1, U_1] = [0.3, 0.6]$ and $[L_2, U_2] = [0.5, 0.8]$. In this case, it is easy to see that we can tighten the upper bounds of these ranges, and reset them to $[0.3, 0.5]$ and $[0.5, 0.7]$ respectively. The reason is that the two lower bounds sum up to 0.8, thus allowing us an upper bound in each case that is no more than 0.2 more than the lower bound. This is a simple example of tightening. Recall that tightening played in establishing Theorem 4, which showed that the annotated expansion of a TP-relation faithfully represents the TP-relation as long as the TP-relation is compact and tight.

In this section, we describe a procedure for *tightening* a compact TP-relation r. Tight TP-relation r'' can be constructed from r in the following way: Initially, let $r'' = \emptyset$ and let $r = \kappa_\chi(r)$ for some combination function χ. Then for each $tp \in r$, add Tighten-TP-Tuple(tp) (see algorithm below) to r''.

Note that for each $tp_1, tp_2 \in r$, Tighten-TP-Tuple(tp_1) and Tighten-TP-Tuple(tp_2) do not affect each other since r is compact and hence tp_1 and tp_2 must refer to different events. Also note that Tighten-TP-Tuple works much faster when the distribution functions are uniform since here, all upper bounds for a single TP-case will be tightened by the same amount.

7 Implementation and Experiments

All of the TPA operators described in this paper have been implemented under Borland C++ version 5.01. Our code can run on any 32 bit Windows platform (i.e., Win95, Win98, and WinNT). This code communicates with standard, relational databases by using the Borland Database Engine's API (BDE version 3.0). Here, the same API can be used to interface with a variety of databases including Paradox, dBASE, Oracle, Microsoft SQL Server, InterBase, Sybase, and any ODBC (Open Database Connectivity) data source. Note however that the underlying, relational database should (i) be capable of storing 32 bit integers and (ii) be able to process basic SQL queries. A demonstration of this implementation can be accessed from the web by clicking on the "TP-Databases" link in the "`http://bester.cs.umd.edu`" page — our user interface is fully compatible with the Internet Explorer 4.0 browser.

In the implementation, a TP-database is a collection of TP-relations which have the same chronon. Each TP-relation r is actually stored as two tables; the *data-table* r_d stores r's data-items and hidden fields while the *case-table* r_c stores r's TP-cases. Both of these tables have an indexed "Id" field which stores an integer used to reconstruct r from the join of r_d and r_c. Basically, if two tuples have the same "Id", then we assume that both tuples refer to the same event. Note however that unless we just performed a compaction, two tuples which refer to different events may have different "Id"s.

Temporal constraints are represented by an array of integers. Basically, each temporal constraint C is broken up into a set of n disjoint, non-adjacent ranges of time points. Each range consists of two time points (the starting and ending time) and each time point consists of two integers (the first represents the number of milliseconds since the start of a day while the second represents the number of days since some fixed reference

Algorithm Tighten-TP-Tuple(tp):
Input: TP-tuple $tp = (d, \gamma)$ where $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ and for all $1 \leq i \leq n$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$
Output: Tight TP-tuple tp'' which is a tightening of tp
Note: In this algorithm, let $\delta(D, C)$ be a "shortcut" for the following expression: $\Sigma_{t \in \mathrm{sol}(C)} \delta(D, t)$

$L := 0$; $U := 0$; `// [L,U] will hold the sum of the lower and upper bounds`
for $i := 1$ **to** n **do** {
  $L'_i := \delta_i(D_i, C_i) \cdot L_i$; $L := L + L'_i$;
  $U'_i := \delta_i(D_i, C_i) \cdot U_i$; $U := U + U'_i$; }
if $U \leq 1.0$ **then** **return** $tp'' := tp$; `// If U ≤ 1.0, then tp was already tight`
$\gamma'' = \emptyset$;
for $i := 1$ **to** n **do** {
  if $\delta_i \neq u$ **then** {
    foreach $t \in \mathrm{sol}(C_i)$ {
      $L_t := \delta_i(D_i, t) \cdot L_i$; $U_t := \delta_i(D_i, t) \cdot U_i$; $U' := 1 - (L - L_t)$;
      if $U' < U_t$ **then** $U_t := U'$;
      Add TP-case $\langle (\#), (t), L_t, U_t, u \rangle$ to γ''; } }
  else {
    $m = |\mathrm{sol}(D_i)|$; $L_t = \frac{L_i}{m}$; $U_t = \frac{U_i}{m}$;
    $U' := 1 - (L - L_t)$; $U''_i := m \cdot \min(U_t, U')$;
    Add TP-case $\langle C_i, D_i, L_i, U''_i, \delta_i \rangle$ to γ''; } }
return $tp'' := (d, \gamma'')$;
End-Algorithm

date). Thus, the first element of C's array will hold n, the next four elements will hold C's first range, the next four elements will hold C's second range, etc. Here, ranges are stored in chronological order. Note that temporal constraints can be indexed by using an auxiliary data structure, e.g. a segment tree [40] or constraint indexing methods such as those in [5, 6, 22].

7.1 Experiments

We conducted two sets of experiments. The first set was intended to demonstrate the relative efficiency of TP-algebra operations when compared to TATA algebra operations. In addition, this set of experiments was designed to study how different distribution functions affected the efficiency of operations. The second set of experiments tested scalability of the TP-algebra operations. The TATA algebra was implemented for these experiments by forcing TP-tuples to have only one TP-case with $C = D =$, $|\mathrm{sol}(D)| == 1.$ and $\delta =$"u".

We should mention that all our experiments were conducted by executing queries "as is." Once a query optimizer for TP-databases is built (which we are currently working on [10]), the timings reported should improve substantially. With hand-optimized versions of some of the queries, we noticed significant improvements in running time. However, due to space reasons, we have chosen to defer the important topic of query optimization and probabilistic indexes to a future paper [10]. *In some of the charts shown in Appendix B reflecting the results of the experiments, readers may sometimes see only two lines instead of eight, because the four lines denoting the TP-computations and the four lines denoting the TATA-computations are almost identical.*

7.1.1 Comparing TATA vs. TP-Algebra

Our experiments were conducted as follows. We generated TP-relations containing $nTuples$ TP-tuples where $nTuples \in \{100, 500, 1000\}$. Each TP-tuple had one TP-case $\langle C_i, D_i, L_i, U_i, \delta_i\rangle$ where $C_i = D_i = (t_1 \sim t_2)$, $t_1 = \mathrm{random}(\{t \in \mathrm{sol}(1/1/1998 \sim 31/12/1998)\})$, t_2 is the time point which occurs *nTimePoints* days after t_1. Probabilities were assigned randomly. We allowed different probability distributions (independence, geometric, binomial,or a mix of these three) in TP-relations. Using these relations, we calculated the (median of 3) computation times for each of the following operations:

Intersection and Union Computations
$\Xi^{hy}(r \cap_{eq} r')$, $\mathrm{ANN}(r) \cap_{eq} \mathrm{ANN}(r')$, $\Xi^{hy}(r \cup_{eq} r')$, and $\mathrm{ANN}(r) \cup_{eq} \mathrm{ANN}(r')$.
Chart **(a)** in Appendix B shows that intersection takes time that is more or less linear in the number of tuples. Furthermore, as the number of TP-tuples increases, the savings rendered by using TP-tuples instead of annotated tuples increases significantly. Chart **(b)** in Appendix B shows that increasing the total number of time-points (i.e. increasing the effect of uncertainty) has no effect whatsoever on TP-tuples, but the effect on annotated tuples is very significant.

Charts **(a)** and **(b)** jointly show that as far as intersection is concerned, the distributions used have no significant impact on the efficiency of computing intersection.

Similar results hold for union as seen from Charts **(c)** and **(d)**.

Selection Computations
$\sigma_{\mathcal{C}}(r)$ and $\sigma_{\mathcal{C}}(\mathrm{ANN}(r))$ for each type of selection condition $\mathcal{C}$ (i.e., data, temporal, and probabilistic).
We ran three types of experiments with selections involving conditions on data attributes (Charts **(e)** and **(f)**), temporal attributes (Charts **(g)** and **(h)**), and probabilistic attributes (Charts **(i)** and **(j)**), respectively.

When we held the average number of time points per TP-case constant to 16, and increased the number of tuples, we notice that the TP-algebra significantly outperforms the TATA algebra. Furthermore, as the number of data tuples increases, there is very little increase in time on the TP-side, in contrast to the much larger increase on the TATA side. The same phenomenon may be noted when the number of tuples is held constant, but the amount of uncertainty is increased.

An important point to note is that Charts **(i)** and **(j)** indicate that performing probabilistic selections on TP-databases that use uniform distributions is faster than on identical TP-databases that use other distributions!

Difference and Projection Computations
$r - r'$, $\mathrm{ANN}(r) - \mathrm{ANN}(r')$, $\pi_{\mathcal{F}}(r)$, and $\pi_{\mathcal{F}}(\mathrm{ANN}(r))$.
Charts **(k)** and **(l)** show what happens with Difference, while Charts **(m)** and **(n)** show what happens with Projection. The results mirror those in the case of union and intersection.

Join Computations
$\Xi^{hy}(r \bowtie_{\alpha} r')$ and $\mathrm{ANN}(r) \bowtie_{\alpha} \mathrm{ANN}(r')$ for each conjunction strategy $\alpha \in \{ig, pc, nc, in\}$.
We first studied what happens with join under the positive correlation conjunction strategy (Charts **(o)** and **(p)**). Subsequently, we studied what happens with join when we vary the conjunction strategy used. In the first case, we noticed that the performance of TP-join is affected relatively little when we increase number of tuples and/or the the amount of uncertainty. However, as seen in charts Charts **(q)** and **(r)**, using negative correlation as the conjunction strategy is actually much more efficient than using the other strategies, both on the TP and the TATA side — an observation that we have not seen made before. (This is in interesting contrast to previous beliefs that using independence assumptions leads to greater efficiency).

7.1.2 Scalability of TP-Algebra Operations

We studied the performance of two operations in the TP-algebra — selection, and join, as these are two of the most widely used operations. Our interest was to see what happens to the performance of the TP-algebra operations when we execute queries with massive amounts of uncertainty.

Charts **(s)** and **(t)** show what happens when we use a mix of distribution functions, and use either 100 or 1000 TP-tuples per TP-relation, and vary the number of solutions to TP-cases over the set 4, 96, 5760 and $345,600$. *Due to the size of these numbers, the charts shown use a log-scale.* Chart **(s)** shows the results of performing both selects and joins when we are looking at the case of 100 TP-tuples.

As the reader can see, temporal selections are almost completely unaffected by the amount of uncertainty both in the case of 100 TP-tuples and 1000 TP-tuples (where the time taken stays constant). However, probabilistic selects are expensive to compute (almost as expensive as joins), because they require that the distribution function be applied to all time points in a TP-case. Notice that even when we have 345,600 time-points inside each of these 100 tuples (making up a "flat relation" of size 34,560,000), it takes only about 60 seconds to evaluate the probabilistic select. When we have 345,600 time-points inside each of the 1000 TP-tuples shown in Chart **(t)** (making a flat relation of size 3.5 billion approximately), we see that the time taken is about 125 seconds, reflecting a doubling in the time, though the data increased in size by a factor of 10. We feel this is quite efficient.

Our framework is also quite efficient for computing TP-joins. As can be seen from Chart **(s)**, when we compute a join of two relations consisting of 100 TP-tuples each and 345,600 time-points inside each of these 100 TP-tuples, the join takes about 75 seconds — a bit more expensive than a probabilistic select, but not too bad. When we use a 1000 TP-tuples (and the same 345,600 time-points inside each of these 1000 TP-tuples), the join takes about 580 seconds — a five fold increase when the data tuples in the two joined relations were both increased ten fold.

8 Related Work

There has been almost no work to date on the integration of probabilistic databases and temporal reasoning. A notable exception is the work of Dyreson and Snodgrass [13]. We therefore organize this section into three parts — the first part compares our work with that of Dyreson and Snodgrass, the second part compares our work with existing work on probabilistic databases, and the third part compares our work with relevant work on temporal indeterminacy.

8.1 Comparison with Dyreson and Snodgrass

Dyreson and Snodgrass [13] were one of the first to model temporal uncertainty using probabilities by proposing the concept of an *indeterminate instant*. Intuitively, an indeterminate instant is an interval of time-points with an associated probability distribution. They propose an extension of SQL that supports (i) specifying which temporal attributes are indeterminate, (ii) correlation credibility which allows a query to use uncertainty to modify temporal data — for example, by using an EXPECTED value correlation credibility, the query will return a *determinate* relation that retains the most probable time point for the event, (iii) ordering plausibility which is an integer between 1 and 100 where 1 denotes that any possible answer to the query is desired while 100 denotes that only a definite answer is desired, and (iv) specifying that certain temporal intervals are indeterminate. Dyreson and Snodgrass [13] develop a semantics for their version of SQL. In addition, they show how to compute probabilities of temporal relationships such as "event e_1 occurs before event e_2,” ‘event e_1 occurs at the same time as event e_2,” etc., and provide efficient data structures to represent probability mass functions.

Our framework may be viewed as in improvement over the the Dyreson-Snodgrass framework in the manner described below. In addition, there is an important philosophical difference between our work and theirs — we are adding time to “kosher” probabilistic databases, while they are adding probability to “kosher” temporal databases.

SQL vs. Algebra. First and foremost, Dyreson and Snodgrass present a version of SQL for temporally indeterminate databases. In contrast, we present an *algebra* and prove that all our algebraic operations are correct. Both are clearly needed for a database that supports probabilities over temporal attributes.

Base Relations. In the Dyreson and Snodgrass [13] framework, the base relations are temporal relations. In effect, base relations in [13] may be viewed as special cases of TP-relations where the C and D fields are *atomic time-interval constraints*. In contrast, our framework:

• Allows C and D to be arbitrary temporal constraints, thus generalizing their approach. As a consequence, TP-relations can be much more succinct than the base relations used in [13]. This is because a single TP-tuple can often express information about a union of disjoint time intervals.

• In [13], no explicit lower/upper bounds are considered; all probabilities used are point probabilities. This is a special case of our framework, and as we have already seen. Recall that in 1854 [7], Boole noticed that we must use probability intervals whenever we are ignorant of the relationship between events.

Conversely, there are some things that can be expressed in the Dyreson-Snodgrass framework [13] which we do not handle — for example, in the current paper, we have assumed tuples have only one indeterminate temporal attribute while [13] allows more than one. Furthermore, we have no analog of correlation credibility or ordering plausibility.

Distribution Functions. In [13], all PDFs are assumed to be complete. In contrast, in this paper, we allow both complete and incomplete PDFs. In fact, we noticed for the first time that determinate PDFs (all complete PDFs are determinate) guarantee linear time consistency checks for TP-databases.

Independence Assumptions. In [13], all indeterminate events are assumed to be independent. This assumption is valid for many applications, and invalid for others. For instance, a transportation plan that involves shipping a packet and then trucking it involves two dependent events — changes in the ship’s arrival time will change the time at which the packet is loaded onto the truck. The independence assumption allows for efficient computation of temporal relationships such as “event e_1 occurs before event e_2.” In contrast, in our paper, we allow *users* to specify in their query what the relationship between events is. Thus, independence can be used in our framework when appropriate, and other dependencies can be used when deemed appropriate by the user. Our framework supports computation of the probabilistic versions of all 13 operators postulated by Allen [1].

Operators developed. Our algebra supports a host of operations that do not appear to be supported in the Dyreson-Snodgrass framework [13]. For instance, we provide whole families of compaction methods, combination functions, and compression functions.

Semantics. We provide formal, model-theoretic descriptions of consistency in our paper for TP-relations, and provide very efficient means to check consistency (linear time) when determinate PDFs are used. When indeterminate PDFs are used, consistency checking is more complex.

Prototype Implementation. We have implemented our framework on top of the Borland Database Engine, and our experiments complement those of Dyreson and Snodgrass in the sense that we examine how different distributions fundamentally affect the efficiency of the algebraic operations. Note that distribution functions can be stored according to the methods described in [13].

8.2 Relationship with work in Probabilistic Databases

Dyreson and Snodgrass [13, p.46] stated that they "could not adopt the PDM approach or its successors to support temporal indeterminacy, since there might be several million elements in a set of possible chronons. Representing each alternative with an associated probability is impractical." This statement is certainly correct if *PDM* is taken to mean the TATA approach. In fact, our experiments on the TATA validate Dyreson and Snodgrass' concern — TATA should be used only for theoretical purposes and should not be implemented. However, as we have seen, the TP-Algebra is a much more succinct PDM representation of temporal probabilistic data which can efficiently deal with large sets of chronons.

Kiessling and his group [23, 50, 41] have developed a framework called DUCK for reasoning with uncertainty. They provide an elegant, logical, axiomatic theory for uncertain reasoning in the presence of rules. In the same spirit as Kiessling et al., Ng and Subrahmanian [34, 36] have provided a probabilistic semantics for deductive databases — they assume absolute ignorance, and furthermore, assume that rules are present in the system. In contrast, in our framework, rules are not present; rather, our interest is in extending the relational algebra to capture probabilistic and temporal information. Time is not handled by the DUCK approach.

In an important paper, Lakshmanan and Sadri [32] show how selected probabilistic strategies can be used to extend the previous probabilistic models. Lakshmanan and Shiri [33, 43] have shown how deductive databases may be parametrized through the use of conjunction and disjunction strategies, an approach also followed by Dekhtyar and Subrahmanian [11]. While the ideas behind these frameworks have been used here through the notions of conjunction and disjunction strategies, no notion of time is discussed.

Barbara et al. [3] develop a probabilistic data model and propose probabilistic operators. Their work is based on the assumption that probabilities of compound events can always be precisely determined, an assumption valid for few combination strategies. In contrast, we allow interval probabilities which permit margins of error in the probability data. In addition, when performing joins, they assume that Bayes' rule applies (and hence, as they admit up front, they make the assumption that all events are independent). Also, as they point out, unfortunately their definition leads to a "lossy" join. No temporal data is handled.

Cavallo and Pittarelli [9] propose a model for probabilistic relational databases. In their model, tuples in a probabilistic relation are interpreted using an exclusive or, meaning at most one of the data-tuples is assumed to be present in the underlying classical relation. This is a rather restrictive assumption, and we make no such assumptions. Furthermore, due to the above assumptions, Cavallo and Pittarelli [9] only propose probabilistic projection and join operations, but the other relational algebra operations are not specified.

An important paper on the topic is by Dey and Sarkar [12] who propose an elegant `1NF` approach to handling probabilistic databases. Their paper is a significant improvement to the work of Barbara et al. [3]. In particular, their framework (i) supports having uncertainty about some objects but certain information about others, (ii) uses first normal form which is easy to understand and use, (iii) introduces elegant new operations like conditionalization, and (iv) removes assumptions about deterministic keys prevalent in previous approaches. The `1NF` representation used by them is a special case of the annotated representation in this paper — as discussed in Section 8.2 and as pointed out by Dyreson and Snodgrass [13], this representation is *not* suitable for directly representing temporal indeterminacy. Dey and Sarkar's approach does not handle time explicitly. Many of our operators generalize theirs — for instance, their notion of *union* clusters together all data-identical tuples and takes their `max`, difference clusters together all data-identical tuples and subtracts probability values, and their notion of projection clusters together all data-identical tuples and takes the sum of the tuples' probabilities (or 1, whichever is smaller) to be the probability. These computations are probabilistically legitimate only under some assumptions on the dependencies between the events involved. Our notion of *combination functions* generalize these substantially. In addition, their notion of join only applies under an independence assumption, which we have been able to remove through the notion of p-strategies.

Similarly, our notion of compaction operations may be viewed as extensions of the two *coalesce* operations proposed by them — we propose whole families of coalesce operations in contrast, and our algebra uses such operations as parameters. Dey and Sarkar [12] propose some operations such as *conditionalization* and N^{th}-Moment that have no analogs in our paper, and deserve further study.

This paper builds on top of the ProbView system for probabilistic databases[30]. ProbView extends the classical relational algebra by allowing users to specify in their query, what probabilistic strategy (or strategies) should be used to parameterize the query. ProbView removed the independence assumption from previous works. However, ProbView has no notion of time, and it was noted by Snodgrass [46] that though ProbView scaled up well to massive numbers of tuples, it did not scale up well when massive amounts of uncertainty are present as is the case with temporal probabilistic databases, where saying that an event sometime between Jan 1-4 yields a total of $4 \times 24 \times 60 \times 60 = 345,600$ seconds. Thus, if our temporal database uses seconds as it lowest level of temporal granularity, this gives rise to $345,600$ cases to represent just one statement — something that would quickly overwhelm ProbView. As the reader can see and as our experiments indicate, TP-databases were specifically designed to eliminate this problem.

Several other authors have handled uncertainty in databases through the use of fuzzy sets [15, 24, 38, 39] — as the differences between probabilities and fuzzy sets are well known, we do not address these works extensively here. In addition, uncertainty has been extensively studied in the context of deductive databases and logic programming [11, 24, 28, 32, 31, 33, 34, 36, 37, 42, 52] as well as probabilistic logic [18, 19, 37].

8.3 Relationship with work in Temporal Databases

Snodgrass was one of the first to model indeterminate instances in his doctoral dissertation [45] — he proposed the use of a model based on three valued logic. Dutta [17] later proposed a fuzzy logic based approach to handle *generalized temporal events* — events that may occur multiple times. This approach is also used by Dubois and Prade [15].

Gadia [20] proposes an elegant model to handle incomplete temporal information as well. He models values that are completely known, values that are unknown but are not to have occurred, values that are known if they have occurred, and values that are unknown even if they occurred. Gadia [20] shows that his model is sound. However, he makes no use of probabilistic information.

An important body of work is that of Koubarakis [26, 27] who proposes the use of constraints for representing temporal data. In this sense, our work is directly related and builds upon Koubarakis' work. Like us, Koubarakis uses constraints to represent when an even occurs. Koubarakis' framework allows stating the facts that event e_1 occurred between 8 and 11 AM, and that event e_2 occurs after 12pm. From this, we may conclude that event e_2 occurs after e_1 — our framework can support this conclusion as well. However, inside our TP-tuples, we cannot state that event e_2 occurs after e_1 — something we can do in a query, but which Koubarakis [26, 27] can explicitly encode in his tuples.

Another important body of work is that of Brusoni *et al.*[8] who developed a system called LaTeR. LaTeR restricts constraints to conjunctions of linear inequalities, as does Koubarakis' work. LaTeR makes a compromise — when tuples are inserted, it builds a constraint network (which increases insertion time), but this pays off because at query time, queries can be efficiently processed. We can benefit from this strategy in our work — as constraint networks are main memory data structures, an adaptation to disk-based structures would greatly enhance scalability. We will report on such efforts in part II of this series of papers [10].

9 Conclusions and Future Work

There are a large variety of applications where there is uncertainty about when certain real-world events occurred, or are predicted to occur. Such applications range from shipping and transportation applications, where extensive statistical data is available about shipping times for packages from one location to another, to data mining and time series applications where predictions about when certain stock market activity may occur is inherently uncertain. The same is true of weather applications where predictions about the likelihood of rain in a certain time interval also has probabilistic attributes. A variety of other important applications involving uncertainty about when events occur have been identified by Dyreson and Snodgrass in [13].

The only previous work whose explicit goal was to incorporate uncertainty into temporal databases is due to Dyreson and Snodgrass [13]. In this paper, we choose a philosophically different approach to incorporating probabilistic temporal reasoning in relational databases — instead of adding probabilities to temporal databases, we instead add time to probabilistic databases. Our approach allows us to make the following important contributions over and above the important work of Dyreson and Snodgrass [13].

• We propose what is, to our knowledge, the first extension of the relational algebra that integrates both probabilities and time. This nicely complements the probabilistic temporal SQL language designed by Dyreson and Snodgrass [13].

• Second, our framework removes several assumptions made in previous work. First, our framework allows users to specify in their (algebraic) queries, what dependencies (if any) they assume between indeterminate instances. No conditional independence assumptions are required *unless desired* by the user. Instead, the user can parameterize his query with a variety of other probabilistic assumptions. Second, we allow the database to associate *partial distributions* with uncertain data. This is certainly very practical. Most statistical sampling methods do not provide total distributions, but distributions with associated margins of error. Third, by introducing the TP-Algebra, we show how the PDM model can be modified to support temporal indeterminacy, even if there might be several million elements in a set of possible chronons. This was an important open problem raised by Snodgrass in [53].

• We propose two algebras in this paper. The TATA-Algebra is intended for *purely theoretical purposes*. As the TATA-Algebra explicitly specifies the probability of an event occurring at any given time point, it leads to unacceptably large relations. However, the explicit specification allows us to easily specify *how* the relational operations should be defined, i.e. what their behavior should be so as to be "probabilistically and temporally kosher."

The TP-Algebra on the other hand is an *implementation oriented algebra*. First, TP-relations are very small compared to annotated relations. Second, for every operation *op* defined on the TATA-Algebra, we show how to define an analogous operation that directly manipulates the succinct TP-relations. We show that these TP-operations are all *correct* in the sense that they correctly implement the TATA-Algebra operations. **Thus, there is no need to implement the TATA-Algebra because the TP-Algebra can realize it in a sound, complete, and much more efficient manner.**

• We provide a host of new algebraic operations that have not been introduced before. These include a variety of *compaction operators*, *compression operators*, *combination operators*, and a *tightening* operator.

• We have conducted experiments on the feasibility of our approach by building a prototype TP-Algebra system on top of ODBC. Our experiments show that the distributions that are used definitely impact the performance of the system. TP-relations are shown to be far more scalable than their annotated counterparts.

This is the first in a long term research effort we have on probabilistic temporal databases. Our long term research program involves enhancing the framework of TP-programs in the following ways.

First, due to space reasons, we have not been able to study **query optimization** in this paper. We are working on establishing a set of rewrite rules, and a set of cost models Preliminary test runs with optimized versions of some queries show far superior performance than indicated in the experiments in this paper.

Second, we are working on the problem of **incremental view maintenance** in TP-databases. This is an important problem because in many TP applications (such as stock applications and transportation applications), updates occur all the time. Handling these updates efficiently is critical.

A *third* important point is that of **integrity constraints**. What does it mean for a database state to satisfy an integrity constraint, when in fact, the contents of the database is uncertain and reflects a set of possible states, together with an associated probability distribution.

A *fourth* major research topic is on **indexing** TP-databases. While extensive work has been carried out on indexing constraints, and indexing relational databases, the problem of indexing TP-cases introduces new twists when we wish to support efficient probabilistic queries.

A *fifth* major research topic is that of supporting **probabilistic aggregate operations**. For instance, this is needed to support queries of the form *Find the ten most probable events to occur at time* t.

Acknowledgments

Different parts of this work were supported by the Army Research Office under Grants DAAH-04-95-10174, DAAH-04-96-10297, DAAG-55-97-10047 and DAAH04-96-1-0398, by the Army Research Laboratory under contract number DAAL01-97-K0135, by an NSF Young Investigator award IRI-93-57756, and by a grant from Lockheed Martin Advanced Technology Laboratories.

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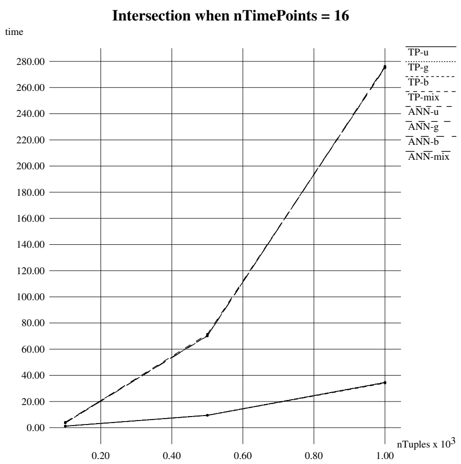
A Appendix: Experimental results



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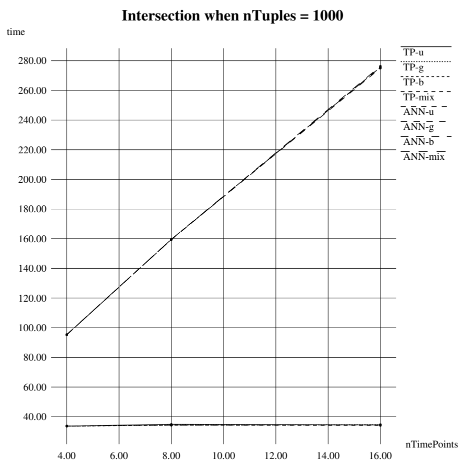
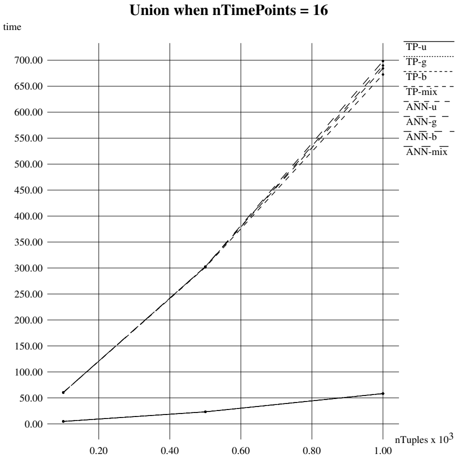
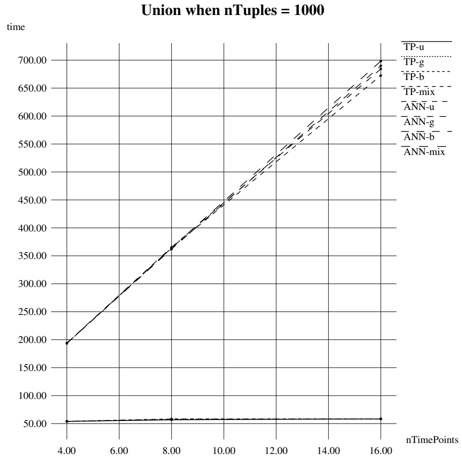


(b)



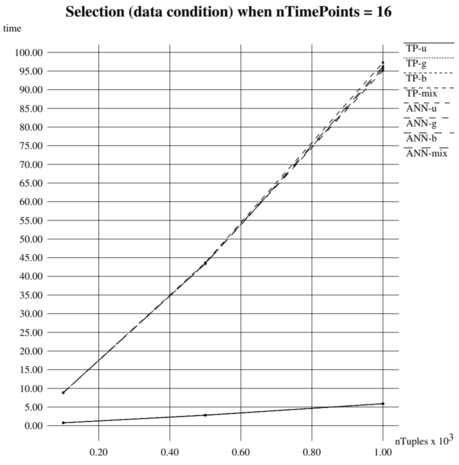
Union when nTimePoints = 16
time
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTuples x 10^3

(c)



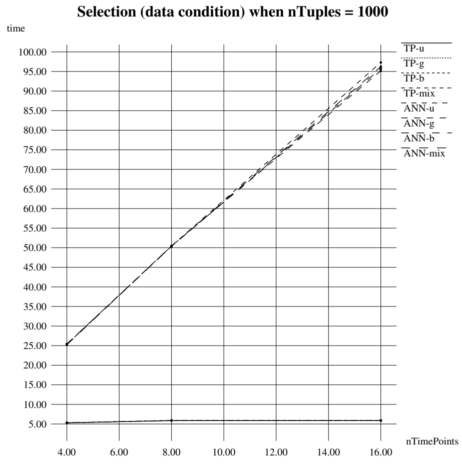
Union when nTuples = 1000
time
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTimePoints

(d)



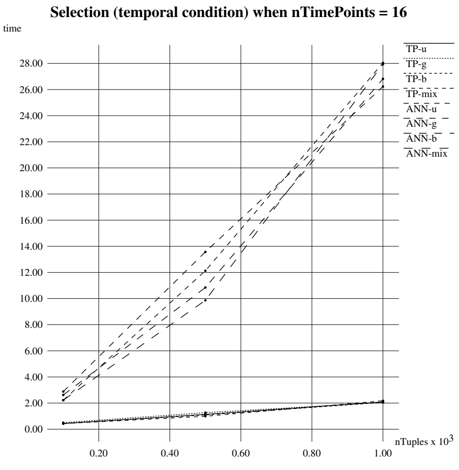
Selection (data condition) when nTimePoints = 16
time
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTuples x 10^3

(e)



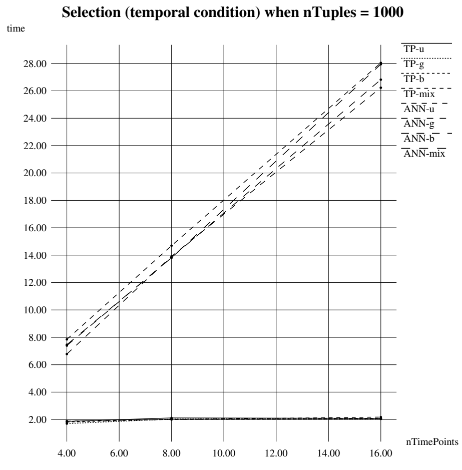
Selection (data condition) when nTuples = 1000
time
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTimePoints

(f)



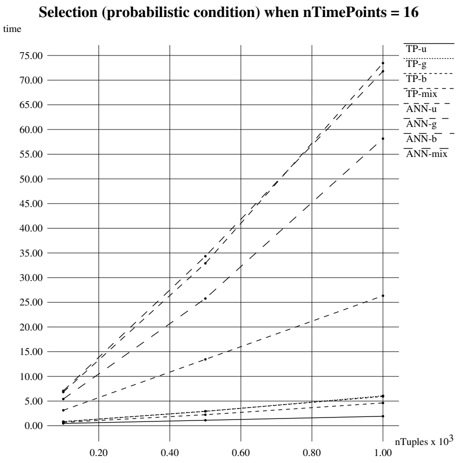
Selection (temporal condition) when nTimePoints = 16
time
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTuples x 10^3

(g)

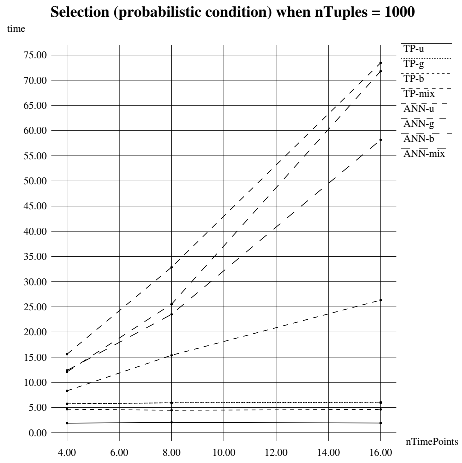


Selection (temporal condition) when nTuples = 1000
time
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTimePoints

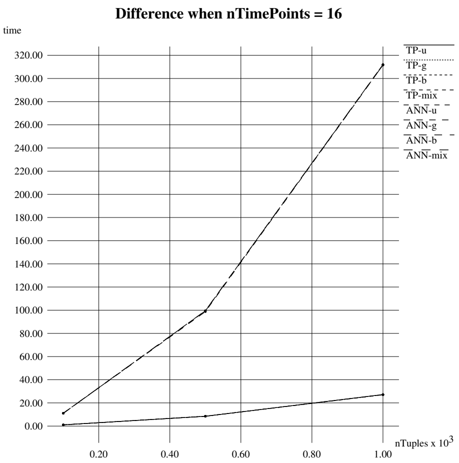
(h)


Selection (probabilistic condition) when nTimePoints = 16
time
75.00
70.00
65.00
60.00
55.00
50.00
45.00
40.00
35.00
30.00
25.00
20.00
15.00
10.00
5.00
0.00
0.20
0.40
0.60
0.80
1.00
nTuples x 10^3
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix

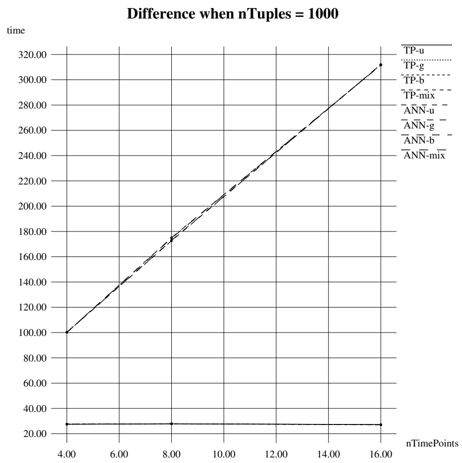
(i)


Selection (probabilistic condition) when nTuples = 1000
time
75.00
70.00
65.00
60.00
55.00
50.00
45.00
40.00
35.00
30.00
25.00
20.00
15.00
10.00
5.00
0.00
4.00
6.00
8.00
10.00
12.00
14.00
16.00
nTimePoints
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix

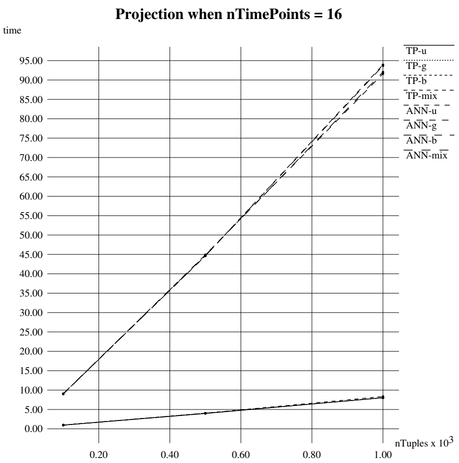
(j)


Difference when nTimePoints = 16
time
320.00
300.00
280.00
260.00
240.00
220.00
200.00
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.20
0.40
0.60
0.80
1.00
nTuples x 10^3
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix

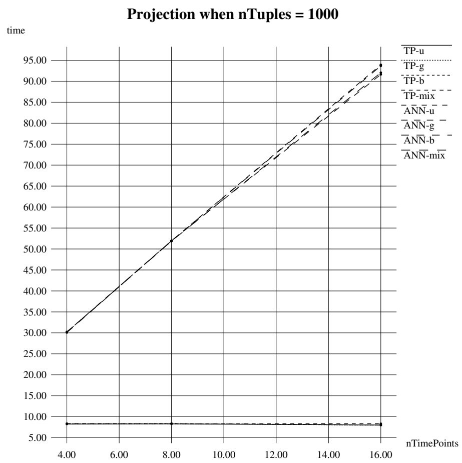
(k)


Difference when nTuples = 1000
time
320.00
300.00
280.00
260.00
240.00
220.00
200.00
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
4.00
6.00
8.00
10.00
12.00
14.00
16.00
nTimePoints
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix

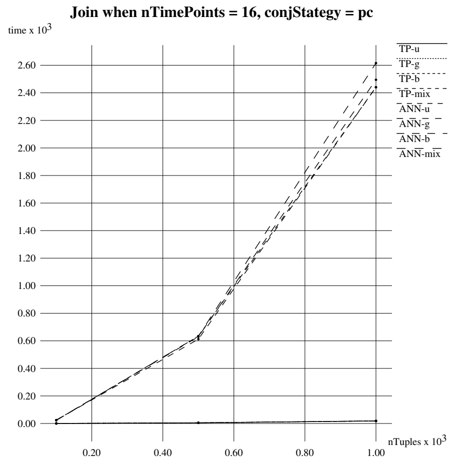
(l)


Projection when nTimePoints = 16
time
95.00
90.00
85.00
80.00
75.00
70.00
65.00
60.00
55.00
50.00
45.00
40.00
35.00
30.00
25.00
20.00
15.00
10.00
5.00
0.00
0.20
0.40
0.60
0.80
1.00
nTuples x 10^3
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix

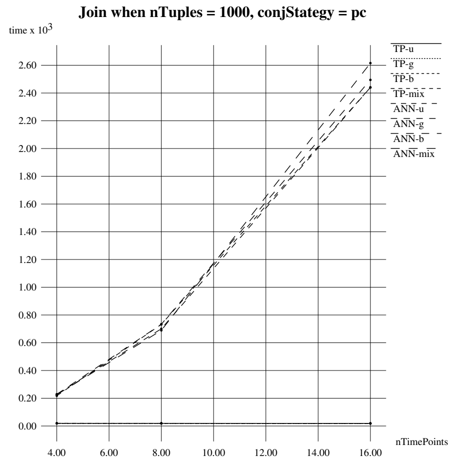
(m)


Projection when nTuples = 1000
time
95.00
90.00
85.00
80.00
75.00
70.00
65.00
60.00
55.00
50.00
45.00
40.00
35.00
30.00
25.00
20.00
15.00
10.00
5.00
4.00
6.00
8.00
10.00
12.00
14.00
16.00
nTimePoints
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix

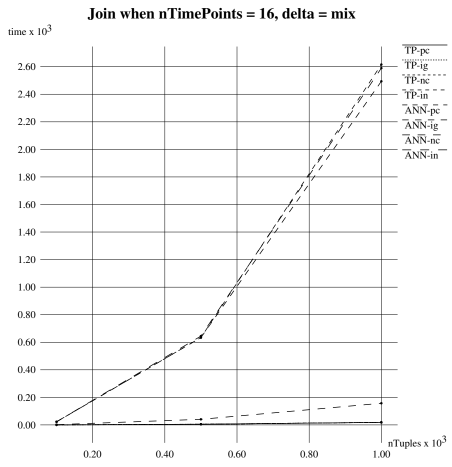
(n)


Join when nTimePoints = 16, conjStategy = pc
time x 10^3
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTuples x 10^3

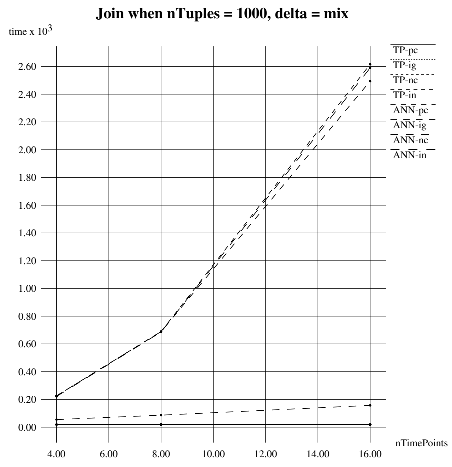
(o)


Join when nTuples = 1000, conjStategy = pc
time x 10^3
TP-u
TP-g
TP-b
TP-mix
ANN-u
ANN-g
ANN-b
ANN-mix
nTimePoints

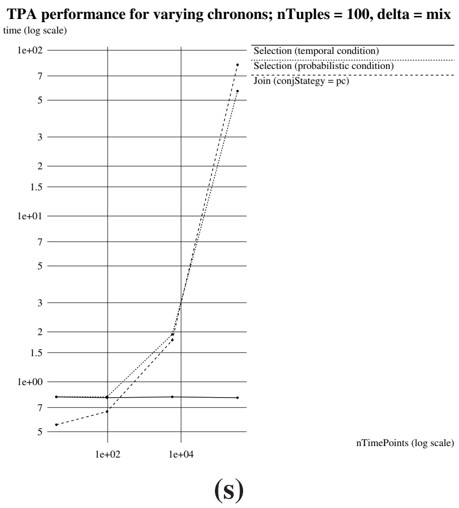
(p)


Join when nTimePoints = 16, delta = mix
time x 10^3
TP-pc
TP-ig
TP-nc
TP-in
ANN-pc
ANN-ig
ANN-nc
ANN-in
nTuples x 10^3

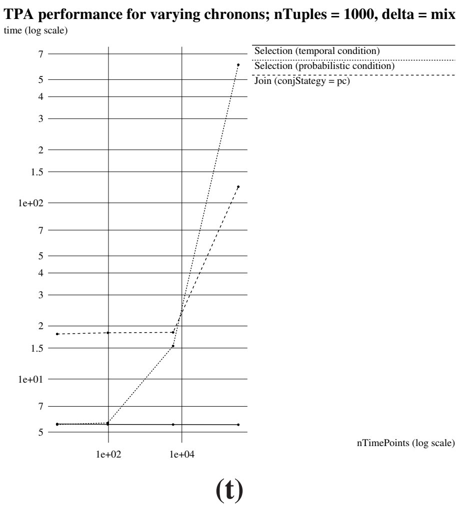
(q)


Join when nTuples = 1000, delta = mix
time x 10^3
TP-pc
TP-ig
TP-nc
TP-in
ANN-pc
ANN-ig
ANN-nc
ANN-in
nTimePoints

(r)


TPA performance for varying chronons; nTuples = 100, delta = mix
time (log scale)
Selection (temporal condition)
Selection (probabilistic condition)
Join (conjStategy = pc)
nTimePoints (log scale)

(s)


TPA performance for varying chronons; nTuples = 1000, delta = mix
time (log scale)
Selection (temporal condition)
Selection (probabilistic condition)
Join (conjStategy = pc)
nTimePoints (log scale)

(t)

The materials below are included for the benefit of the reviewers. They do not need to be included in the final version of the paper.

B Appendix: Notation tables

Symbol	Meaning	§
T	Time unit e.g. (*day*, $\{1,\ldots,31\}$)	2.1
$H = T_1 \sqsubseteq \cdots \sqsubseteq T_n$	Linear hierarchy of time units	2.1
$t = (v_1/\ldots/v_n)$	Time point e.g. (2/3/1996)	2.1
$t <_H t'$	Time point t occurs before t'	2.1
τ	Calendar e.g. Gregorian	2.1
$next_\tau(t)$	Next, consecutive time point after t	2.1
C, D	Temporal constraints e.g. $(t_1 \sim t_2)$	2.2
$S_\tau = \mathrm{sol}(t_S \sim t_E)$	Set of all valid time points over τ	2.2
$\mathrm{sol}(C) = S \subseteq S_\tau$	Solution set for temporal constraint C	2.2
pdf(D, t_j)	Determines probability ρ_j for $t_j \in \mathrm{sol}(D)$	2.3
δ	Distribution function (names a family of PDFs)	2.3
$\delta = u, g, g_c, b, po$	Uniform, geometric, complete g, binomial, Poisson	2.3
$\otimes_\alpha, \oplus_\beta$	Probabilistic conjunction/disjunction strategy	2.4
$\alpha/\beta = ig, pc, nc, in$	Ignorance, positive/negative correlation, independence	2.4
$\chi(S)$	Combination function; combines each $[L_i, U_i] \in S$	2.5
χ_{eq}	Optimistic equity combination function	2.5
$st(e), end(e)$	Instantaneous start/end events which bounding event e	3
$\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$	TP-case; C_i may be "$(\#)$" if sol(C_i) = sol(D_i)	3.1
$\gamma = \{\gamma_1, \ldots, \gamma_n\}$	TP-case statement; each TP-case $\gamma_i \in \gamma$	3.1
$d = (d_1, \ldots, d_k)$	Data tuple over relational schema $A = (A_1, \ldots, A_k)$	3.2
$tp = (d, \gamma) \in r$	TP-tuple in TP-relation r	3.2
$\mathcal{P}(d)$	Manifest projection of data tuple d; removes $d.\mathrm{H}$	3.2
$d.\mathrm{H} \parallel d'.\mathrm{H}$	Hidden list concatenation; merge of field-value pairs	3.2
$r[tp]$	Multiset of all TP-tuples in r which are data-identical to tp	3.4
$dom(A)$	Domain of relational schema A; $dom(A_1) \times \cdots \times dom(A_k)$	3.4
$I_{A,\tau}(d, t)$	TP-Interpretation; probability that d's event is true at time t	3.4
$I_{A,\tau}(d, D)$	Probability assigned by $I_{A,\tau}$ to temporal constraint D	3.4
$I_{A,\tau} \models tp$	TP-Interpretation $I_{A,\tau}$ satisfies TP-tuple tp	3.4
$\mathrm{ANN}(tp), \mathrm{ANN}(r)$	Annotated relations for tp and r	4
$at = (d, t, L_t, U_t)$	Annotated tuple for $tp = (d, \gamma)$ at time t	4
$\mathrm{ANN}(r)[d, t]$	Multiset of all $at \in \mathrm{ANN}(r)$ where $(at.d = d \wedge at.t = t)$	4.1

Symbol	Meaning	§
$\kappa_\chi(\mathrm{ANN}(r))$, $\kappa_\rho(\mathrm{ANN}(r))$	Combination function/p-strategy based compaction	5.1
$\mathrm{ANN}(r) \cap \mathrm{ANN}(r')$	Multiset intersection	5.2
$\mathrm{ANN}(r) \cap_\chi \mathrm{ANN}(r')$	Denotes $\kappa_\chi^\cap(\mathrm{ANN}(r) \cap \mathrm{ANN}(r'))$	5.2
$\mathrm{ANN}(r) \cup \mathrm{ANN}(r')$	Multiset union	5.3
$\mathrm{ANN}(r) \cup_\chi \mathrm{ANN}(r')$	Denotes $\kappa_\chi^\cup(\mathrm{ANN}(r) \cup \mathrm{ANN}(r'))$	5.3
$\sigma_\mathcal{C}(\mathrm{ANN}(r))$	Selection of condition $\mathcal{C}$	5.4
$\mathrm{ANN}(r) - \mathrm{ANN}(r')$	Difference	5.5
$\mathrm{ANN}(r) \times_\alpha \mathrm{ANN}(r')$	Cartesian product under α	5.6
$\pi_\mathcal{F}(\mathrm{ANN}(r))$	Multiset projection of projectable field list $\mathcal{F}$	5.7
$\mathrm{ANN}(r) \bowtie_\alpha \mathrm{ANN}(r')$	Join; $\pi_\mathcal{F}(\sigma_\mathcal{C}(\mathrm{ANN}(r) \times_\alpha \mathrm{ANN}(r')))$	5.8
$N(r)$, $N(tp)$	Number of TP-cases in r/tp	6.1
$\Xi^{sd}(r)$, $\Xi^{u}(r)$, $\Xi^{hy}(r)$	Same distribution/uniform/hybrid TP-compression	6.1
$\kappa_\chi(r)$, $\kappa_\rho(r)$	Combination function/p-strategy based compaction	6.2
$r \cap r'$	Multiset intersection	6.3
$r \cap_\chi r'$	Denotes $\kappa_\chi^\cap(r \cap r')$	6.3
$r \cup r'$	Multiset union	6.4
$r \cup_\chi r'$	Denotes $\kappa_\chi^\cup(r \cup r')$	6.4
C_i'' = TP-filter$(\gamma_i, \mathcal{C})$	TP-filter of γ_i w.r.t. probabilistic condition $\mathcal{C}$	6.5
$\sigma_\mathcal{C}(tp)$, $\sigma_\mathcal{C}(r)$	Selection of condition $\mathcal{C}$ on tp/r	6.5
$r - r'$	Difference	6.6
$r \times_\alpha r'$	Cartesian product under α	6.7
$\pi_\mathcal{F}(r)$	Multiset projection of field list $\mathcal{F}$	6.8
$r \bowtie_\alpha r'$	Join; $\pi_\mathcal{F}(\sigma_\mathcal{C}(r \times_\alpha r'))$	6.9

C Appendix: Proofs of results

Proof of Proposition 1.

Let τ be a calendar and $(name_1, \ldots, name_k)$ be the list of the names of its time units.

To prove this proposition it suffices to show that (i) every atomic *time-value* constraint can be represented as a *time-interval* constraint and (ii) every atomic *time-interval* constraint can be represented as a *time-value* constraint.

We do just that below:

1. For every atomic *time-interval* constraint C there exists a *time-value* constraint C' such that $sol(C) = sol(C')$.

   Let $C = (t_1 \sim t_2)$ be an atomic time-interval constraint. Let $t_1 = (v_1, \ldots, v_k)$ and $t_2 = (v_1', \ldots, v_k')$.

   Now, let $sol(C) = \{s_1, \ldots, s_m\}$.

   Let $s = (v_1^s, \ldots v_k^s)$ be a time point. We construct a time-value constraint C_s as follows:

$$C_s = (name_1 = v_1^s \wedge name_2 = v_2^s \wedge \ldots \wedge name_k = v_k^s)$$

By definition of solution set for a temporal constraint $sol(C_s) = \{s\}$. But then for a time-value constraint

$$C' = C_{s_1} \vee C_{s_2} \vee \ldots \vee C_{s_k}$$

the solution set is defined as the union of solution sets of C_{s_i} ($1 \leq i \leq k$), i.e.:

$$sol(C') = sol(C_{s_1}) \cup \ldots \cup sol(C_{s_k}) = \{s_1\} \cup \ldots \cup \{s_k\} = \{s_1, \ldots, s_k\} = sol(C).$$

Note: C' constructed as described above is going to be a very large constraint. In practice it is always possible to construct much more compact time-value constraints for representing a time period.

2. For every atomic *time-interval* constraint C there exists a *time-value* constraint C' such that $sol(C) = sol(C')$.

   Let $C = (name_i opv_i)$ be an atomic *time-value* constraint. We construct an equivalent *time-interval* constraint as follows.

   Let t be a timepoint and let C_t be a time-interval constraint defined as follows:

$$C_t = (t \sim t)$$

   By definition of the solution set of a temporal constraint, $sol(C_s) = \{s\}$.

   Now, let $sol(C) = \{t_1, \ldots, t_m\}$. The solution set of a time-interval constraint C' defined as

$$C' = C_{t_1} \vee C_{t_2} \vee \ldots \vee C_{t_m}$$

   will be equal to

$$sol(C') = sol(C_{t_1}) \cup \ldots \cup sol(C_{t_m}) = \{t_1\} \cup \ldots \cup \{t_m\} = \{t_1, \ldots, t_m\} = sol(C)$$

   Note: C' constructed as described above is going to be a very large constraint. In practice it is always possible to construct much more compact time-value constraints for representing a time period. $\square$

Proof of Proposition 2.
If S is such that $\cap_{[L,U] \in S}[L, U] \neq \emptyset$ then the values of all functions defined in the example 2.5 will coincide on S, as they are all defined to be equal to the intersection. Clearly,$\cap_{[L,U] \in S}[L, U]$ satisfies Identity as $[L, U] \cap [L, U] = [L, U]$. Also, since $\cap_{[L,U] \in S}[L, U] = [\max_{[L,U] \in S}(L), \min{[L, U] \in S(U)}]$ and $[\max_{[L,U] \in S}(L), \min{[L, U] \in S(U)}] \leq [\max_{[L,U] \in S}(L), 1]$ the Bottomline is satisfied. We also notice that when $S = \{[L, U]\}$, $\cap_{[L,U] \in S}[L, U] \neq \emptyset$, i.e. Identity need not be considered when $\cap_{[L,U] \in S}[L, U] = \emptyset$.

Let now $\cap_{[L,U] \in S}[L, U] = \emptyset$. We prove Bottomline for all six functions:

- *Optimistic Equity* $\chi_{eq}(S) = [\max_{[L,U] \in S}(L), \max{[L, U] \in S(U)}] \leq [\max_{[L,U] \in S}(L), 1]$ i.e. χ_{eq} satisfies Bottomline.
- *Enclosing Equity* $\chi_{ec}(S) = [\min_{[L,U] \in S}(L), \max{[L, U] \in S(U)}] \leq [\max_{[L,U] \in S}(L), 1]$ i.e. χ_{ec} satisfies Bottomline.
- *Pessimistic Equity* $\chi_{ep}(S) = [\min_{[L,U] \in S}(L), \min{[L, U] \in S(U)}] \leq [\max_{[L,U] \in S}(L), 1]$ i.e. χ_{ep} satisfies Bottomline.
- *Rejecting Equity* $\chi_{er}(S) = [0, 0] \leq [\max_{[L,U] \in S}(L), 1]$ i.e. χ_{er} satisfies Bottomline.

- <u>*Skeptical Equity*</u> $\chi_{er}(S) = [0, 1] \leq [\max_{[L,U]\in S}(L), 1]$.
- <u>*Quasi-Independence Equity*</u> $\chi_{eqi}(S) = [\Pi_{[L,U]\in S}L, \Pi_{[L,U]\in S}U] \leq [\max_{[L,U]\in S}(L), 1]$. □

Proof of Theorem 1.

- <u>r is compact $\Longrightarrow$ $ANN(r)$ is compact.</u>

  As r is compact, for each data tuple d and timepoint t there is at most one tp-tuple $tp = (d, \gamma) \in r$, $\gamma = \gamma_1, \dots, \gamma_k$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$, for $1 \leq i \leq k$ such that $t \in sol(C_1 \vee \dots \vee C_k$.

  Let d be some data tuple such that there exists a tuple $tp = (d, \gamma) \in r$. We show that $ANN(r[tp])$ is compact. Let $r[tp] = \{tp_1, tp_2, \dots tp_m\}$. Let C^j be the disjunction of all C-temporal constraints in tp_j. As r is compact, we know that $(\forall 1 \leq i \neq j \leq m)(sol(C^j) \cap sol(C^i) = \emptyset)$. Let t be a timepoint in $sol(C^1 \vee \dots \vee C^m)$. Then there exists a unique $1 \leq h \leq m$ such that $t \in sol(C^h)$. Consider tp-tuple $tp_h = (d, \gamma')$ where $\gamma' = \gamma'_1, \dots, \gamma'_k$ and $\gamma'_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$ for all $1 \leq i \leq k$. By the definition of a TP-tuple, $(\forall 1 \leq i \neq j \leq k) sol(C_i) \cap sol(C_j) = \emptyset$. This means that for each $tinsol(C_1 \vee \dots \vee C_k)$ there will be *only one* annotated tuple $(d, t, L_t, U_t,)$ added to $ANN(tp)$, i.e., $ANN(tp_h)$ has *at most one* tuple $(d, t, L_t, U_t,)$ for each timepoint t. The latter means that $ANN(r[tp])$ is compact, since for two tuples $tp', tp'' \in r[tp]$, and two tuples $at' = (d, t', L', U') \in ANN(tp')$ and $at'' = (d, t'', L'', U'') \in ANN(tp'')$ it will **never** be the case that $t' = t''$. Compactness of $ANN(r[tp])$ for any $tp \in r$ yields the compactness of $ANN(r)$, as for any two **not data-identical** tuples tp and tp' in r, no two tuples $at \in ANN(r[tp])$ and $at' \in ANN(r[tp'])$ will be data-identical.

- <u>$ANN(r)$ is compact $\Longrightarrow$ r is compact.</u>

  Let r be a tp-relation. Consider an arbitrary annotated tuple $at = (d, t, L_t, U_t) \in ANN(r)$. As $ANN(r)$ is compact, $ANN(r)[d, t] = \{ar\}$, i.e. no other annotated tuple in $ANN(r)$ is data-time identical to at. By definition of the operator $ANN(\cdot)$, there exists a tp-tuple $tp \in r$ such that, $tp = (d, \gamma)$, $\gamma = \gamma_1, \dots, \gamma_k$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$, for $1 \leq i \leq k$ and $at \in ANN(tp)$, i.e. $t \in sol(C_1 \vee \dots C_k)$.

  As $ANN(r)[d, t] = \{at\}$, **for any other** tp-tuple $tp' = (d, \gamma') \in r$ such that $\gamma' = \gamma'_1, \dots, \gamma'_l$, $\gamma'_i = \langle C'_i, D'_i, L'_i, U'_i, \delta'_i \rangle$ we will have $t \notin sol(C'_1 \vee \dots \vee C'_l)$. Therefore, by definition 6.6, r is a compact tp-relation. □

Proof of Theorem 2. We show a stronger result, viz. that if $I \models r$ then also $I \models ANN(r)$.Let $tp = (d, \gamma) \in r$ be some tp-tuple and $at = (d, t, L, U) \in ANN(r)$. Since $I \models r$, $I \models tp$. We need to show that $I \models at$.

Let $\gamma = \gamma_1, \dots, \gamma_k$, and $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$ for $1 \leq i \leq k$. Since $at = (d, t, L, U) \in ANN(r)$, there exists some $1 \leq j \leq k$ such that, $t \in sol(C_j)$ and $[L, U] = [\delta(D_j, t)L_i, \delta(D_j, t)U_i]$.

Since $I \models tp$, $I \models \langle d, D_j, [L_j, U_j], \delta_j \rangle$. By the definition of satisfaction, $I(d, D_j) \in [L_j, U_j]$ and $I(d, t) = \delta_j(D_j, t)I(d, D_j)$. But then $L = \delta_j(D_j, t)L_j \leq \delta_j(D_j, t)I(d, D_j) = I(d, t) \leq \delta_j(D_j, t)U_j = U$, i.e., $I(d, t) \in [L, U]$. From the latter it follows that $I \models (d, t, L, U) = at$. □

Proof of Theorem 3. Let $I_{A,\tau}(d, t)$ be computed as follows. As r is compact, all tuples of the form $(d, -, -, -) \in ANN(r)$ are derived from $ANN(tp)$ where $tp = (d, \gamma)$ where $\gamma = \langle C_1, D_1, L_1, U_1, \delta_1 \rangle, \dots, \langle C_m, D_m, L_m, U_m, \delta_m \rangle$. Let $ANN(tp)$ consist of $(d_1, t_1, L_1, U_1), \dots, (d_k, t_k, L_k, U_k)$. We know that $\Sigma_{i=1}^k L_i \leq 1$. Without loss of generality, suppose t is a solution of C_i for some $1 \leq i \leq m$. Clearly, exactly one such i exists. Let $\{t'_1, \dots, t'_r, t'_{r+1}, \dots, t'_{r+s}\}$ be all *other* solutions of D_i, and let sol$(C_i) = \{t, t'_1, \dots, t'_r\}$. For all $1 \leq i \leq r$, (d, t'_i, L'_i, U'_i) is in $ANN(r)$. We know that $L_i \leq L + L'_1 + \dots + L'_r \leq U_i$. We now construct an interpretation $I_{A,\tau}$ that satisfies r and which assigns L to $I_{A,\tau}(d, t)$ as follows. We know that $x \cdot I_{A,\tau}(d, y) = \delta_i(D_i, y)$ for some $x > 0$. By setting $I_{A,\tau}(d, t) = L$, we know that $x = \frac{\delta_i(D_i,t)}{L}$. Therefore, we may set

$$I_{A,\tau}(d, t'_i) \quad = \quad \delta_i(D_i, t'_i) \times \frac{L}{\delta_i(D_i, t)}.$$

The same procedure may be used to extend $I_{A,\tau}$ to other tuples that are not data-identical to $(d, -, -, -)$. □

Proof of Theorem 4. Similar to the proof of Theorem 3. □

Proof of Proposition 3.
Suppose r is compact. As r contains no distinct data-identical tuples, r is consistent iff all tp-tuples in r are individually consistent. Suppose a TP-tuple $tp = (d, \gamma)$ has $\gamma =$ "$\langle C_1, D_1, L_1, U_1, \delta_1\rangle, \ldots, \langle C_n, D_n, L_n, U_n, \delta_n\rangle$" as its TP-case statement. Then it suffices to check that

$$(L_1 + \cdots + L_n) \quad \leq \quad \min(1, \Sigma_{i=1}^{n} L_i \times \Sigma_{t \in S_\tau} \delta_i(D_i, t)).$$

If this condition holds, then the probabilistic interpretation $I_{A,\tau}$ defined as follows satisfies tp.

$$I_{A,\tau}(d, t) = \begin{cases} L_i \times \delta_i(D_i, t) & \text{if } t \in \text{sol}(C_i), 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

As all the distributions δ_i are determinate, it is easy to see that this check is linear in the size of the relation r. □

Proof of Proposition 4.
Let χ be a combination function and let $ANN(r)$ be an annotated relation. We show that κ_χ satisfies the properties of compaction operations.

1. Compactness. By definition 5.2 $\kappa_\chi(ANN(r))$ contains *only one* annotated tuple for each pair (d, t) where d is data tuple and t is a timepoint. Therefore, $\kappa_\chi(ANN(r))$ is compact.

2. No Fooling Around. Let $ANN(r)$ be compact. Then for any data tuple d and timepoint t, $ANN(r)[d, t]$ will contain *at most* one tuple. Let $ANN(r)[d, t] = \{(d, t, L_t, U_t)\}$. Then, by definition 5.2, $\kappa_c hi(ANN(r))$ will contain a tuple $(d, t, \chi(\{[L_t, U_t]\})$. As χ is a combination function, it satisfies the Identity property of combination functions, i.e. $\chi(\{d, t, L_t, U_t\}) = [L_t, U_t]$, i.e., $(d, t, L_t, U_t) \in \kappa_\chi(ANN(r))$ or $ANN(r) \subseteq \kappa_\chi(ANN(r))$.

   To prove that $\kappa_\chi(ANN(r)) \subseteq ANN(r)$ we observe that by definition 5.2, **the only** tuples in $\kappa_\chi(ANN(r))$ are those inserted there by applying χ to the sets $\{[L_t, U_t]\}$ of probabilistic intervals from $ANN(r)[d, t]$ equivalence classes. As we have shown above, **all** such tuples will be the tuples from the original relation $ANN(r)$.

3. Conservativeness. We notice that if there exists a tuple $at = (d, t, L_t, U_t) \in \kappa_\chi(ANN(r))$, then by definition 5.2 there had to be a *nonempty* set $ANN(r)[d, t] \subset ANN(r)$, which means that for any such tuple in $\kappa_\chi(ANN(r))$ there is always at least one tuple $at' = (d, t, L'_t, U'_t)$ in $ANN(r)$. □

Proof of Theorem 5. Let χ be a combination function and $ANN(r)$ be an annotated relation. Let $at' = (d, t, L', U') \in ANN(r)$. Then $ANN(r)[d, t] \neq \emptyset$. Let $ANN(r)[d, t] = \{at'_1, \ldots, at'_k\}, at'_i = (d, t, L'_i, U'_i)$ for $1 \leq i \leq k$. By definition of combination function-based compaction, $\kappa_\chi(ANN(r))$ will contain the tuple $at = (d, t, L, U)$ where $[L, U] = \chi(\{[L'_1, U'_1], \ldots, [L'_k, U'_k]\})$. □

Proof of Proposition 5. We need to show that $\chi_\otimes$ satisfies the three axioms defining compaction operators.

1. Compactness. By the definition of a p-strategy based compaction, for any given pair (d, t), κ_ρ computes $ANN(r)[d, t]$ consisting of all tuples in $ANN(r)$ of the form $(d, t, -, -)$. All tuples in $ANN(r)[d, t]$ are combined into one by the construction of the definition of a p-strategy based compaction.

2. No Fooling Around. If $ANN(r)$ is compact, then for all (d, t), $ANN(r)$ contains at most one tuple of the form $(d, t, -, -)$. In this case, the definition of κ_ρ returns that tuple unchanged.

3. Conservativeness. Suppose $\kappa_\rho(ANN(r))$ contains a tuple of the form $(d, t, -, -)$. Then it is easy to see that by the definition a of p-strategy based compaction there must exist at least one tuple in $ANN(r)$ of the form $(d, t, -, -)$. □

Proof of Theorem 6. We need to show that $\Xi^{sd}(r)$, $\Xi^{u}(r)$ and $\Xi^{hy}(r)$ are all TP-compression functions.

To see that $\Xi^{sd}(r)$ is a TP-compression function, we must demonstrate that $N(\Xi^{sd}(r) \leq N(r)$ and that there exists a bijection, ϕ from $ANN(r)$ to $ANN(sd_(r))$. In fact we will show that a stronger statement holds: $ANN(r) = ANN(sd_(r))$.

1. $\underline{N(\Xi^{sd}(r) \leq N(r)}$.

   Let $r = \{tp_1, \ldots, tp_s\}$. Clearly, $N(r) = N(tp_1) + \ldots + N(tp_s)$.

   By definition of Ξ^{sd}, for every tp-tuple $tp \in r$, *exactly one* tp-tuple tp' is added to $\Xi^{sd}(r)$. Therefore $|r| = |\Xi^{sd}(r)|$. Let now $\Xi^{sd}(r) = \{tp'_1, \ldots, tp'_s\}$ where $tp'_i = \Xi^{sd}(tp_i)$, $1 \leq i \leq s$. As $N(\Xi^{sd}(r)) = N(tp'_1) + \ldots + N(tp'_s)$, it suffices to show that for all $1 \leq i \leq s$ $N(tp'_i) \leq N(tp_i)$.

   The latter statement is clearly true as by definition of Ξ^{sd} either $tp_i = tp'_i$ (in the case when no tp case statements were deleted) or, for every two tp-case statements deleted from the tp-tuple tp_i during its conversion into tp'_i only one tp-case statement is added.

2. $\underline{ANN(r) = ANN(\Xi^{sd}(r))}$.

   Let $tp \in r$. We will show that $ANN(tp) = ANN(\Xi^{sd}(tp))$.

   Let $tp' = \Xi^{sd}(tp)$. Let $tp = (d, \gamma), \gamma = \gamma_1, \ldots, \gamma_n$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i, 1 \leq i \leq n$ and let $tp' = (d, \gamma'), \gamma' = \gamma'_1, \ldots, \gamma'_m$, $\gamma'_i = \langle C'_i, D'_i, L'_i, U'_i, \delta'_i, 1 \leq i \leq m$.

   Consider some tp-case γ'_j from tp'. By the definition of Ξ^{sd}, one of two cases is possible:

   (a) There exists some $1 \leq k \leq n$, $\gamma'_j = \gamma_k$ (i.e., during the application of Ξ^{sd} tp-case $gamma_k$ did not change). In this case clearly $ANN((d, \gamma_k)) = ANN((d, \gamma'_j))$.

   (b) γ'_j is not equal to any $\gamma_i \in \gamma$, but, there exist $1 \leq k_1, \ldots k_q \leq n$ such that $D_{k_1} = \ldots = D_{k_q} = D'_j$, $L_{k_1} = \ldots = L_{k_q} = L'_j$, $U_{k_1} = \ldots = U_{k_q} = U'_j$, $\delta_{k_1} = \ldots = \delta_{k_q} = \delta'_j$ and $C_{k_1} \vee \ldots \vee C_{k_q} C'_j$.

   Let now $at = (d, t, L, U) \in ANN(d, (\gamma_{k_1}, \ldots, \gamma_{k_2}))$. Clearly, there exists some $1 \leq h \leq q$, such that $t \in sol(C_{k_h})$ and $[L, U] = [\delta_{k_h}(D_{k_h}, t) \cdot L_{k_h}, \delta_{k_h}(D_{k_h}, t) \cdot U_{k_h}]$. Since $t \in sol(C_{k_h})$, $t \in sol(C_{k_1} \vee \ldots \vee C_{k_q}) = sol(C'_j)$. Therefore, $at' = (d, t, \delta'_j(D'_j, t) \cdot L'_j, \delta'_j(D'_j, t) \cdot U'_j) \in ANN(d, \gamma'_j)$. But since $D_{k_h} = D'_j$, $L_{k_h} = L'_j$, $U_{k_h} = U'_j$ and $\delta_{k_h} = \delta'_j$, $[L, U] = [\delta'_j(D'_j, t) \cdot L'_j, \delta'_j(D'_j, t) \cdot U'_j]$ and therefore $at = at'$. Hence, $ANN(d, (\gamma_{k_1}, \ldots, \gamma_{k_2})) \subseteq ANN(ANN(d, \gamma'_j)$.

   The proof that $ANN(d, (\gamma_{k_1}, \ldots, \gamma_{k_2})) \supseteq ANN(ANN(d, \gamma'_j)$ is analogous.

   This establishes that $ANN(d, (\gamma_{k_1}, \ldots, \gamma_{k_2})) = ANN(ANN(d, \gamma'_j)$.

   Combining the results from above together we conclude that $ANN(tp) \supseteq ANN(\Xi^{sd}(tp))$.

   To show that $ANN(tp) \subseteq ANN(\Xi^{sd}(tp))$ it suffices to notice that for any $1 \leq i \leq n$ and tp-case γ_i there exists such $1 \leq j \leq m$ and tp-case γ'_j that $sol(C_i) \subseteq sol(C'_j)$.

The proofs for$\Xi^u(r)$ and $\Xi^{hy}(r)$ are similar. □

Proof of Lemma 1.
Similar to the proof of Proposition 4. □

Proof of Theorem 7.

Let us look at the relation r'' which is returned by **Compute-Compaction**(r, χ). The main loop of the algorithm (lines 04.–23.) works in the following way:

- In lines 05.–08. of the algorithm the initial relation r is broken into the equivalence classes by the relation of *data-identicity*. Lines 05.–0.6 select the next unprocessed equivalence class, and lines 07.–08. prepare the data for the next step. In particular, Γ_I (and Initially Γ_S) are set to be equal to the set of all tp-cases found in the tp-tuples of the current equivalence class.
- The **foreach** loop in lines 09.–12. then separates all timepoints which are in solution of some tp-case $\gamma_i \in \Gamma_I$ into two categories: timepoints that are referred to **only** by γ and timepoints referred to by more than one tp-case. The timepoints referred to only by one tp-case are handled by line 11. in which a new tp-case that contains only such timepoints from γ_i and adds it to the final tp-tuple.
- Line 13. collects together in one temporal constraint C_S all the timepoints that are referred to by more than one original tp-case.
- In the **foreach** loop in lines 15.–22. each timepoint t from $sol(C_S)$ gets processed. Lines 17.–20. collect together the set X of all intervals $[L, U]$ which are the probabilities associated with the timepoint t by all original tp-cases which refer to t.
- In line 21. the combination function χ is applied to the set X to obtain the interval $[L''_t, U''_t]$, which will become the part of the tp-case $((t), (t), L''_t, U''_t, u)$ which is added to the final tp-tuple tp''.
- In line 23. the tp-tuple tp'' constructed for the current equivalence class is added to the resulting relation r''.

Now, let some annotated tuple $at = (d, t, L, U) \in ANN(r'')$. Then there exists a tp-tuple $tp \in r''$, such that $at \in ANN(tp)$. Let us examine the values of L and U.

Since $at \in ANN(tp)$, $tp = (d, \gamma)$, $\gamma = \gamma_1, \ldots \gamma_n$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \le i \le n$ and there exists $1 \le j \le n$, $t \in sol(C_j)$ and $[L, U] = [\delta_j(D_j, t) \cdot L_j, \delta_j(D_j, t) \cdot U_j]$.

Let us look at how tp-case γ_j could have been added to tp. ¿From the analysis of the algorithm above, it is clear that there are only two possiblilties:

1. γ_i had been added to tp in line 11. Let us look at the equivalence class of tp in r (although tp itself is not in r it **must** be data-identical to some tp-tuples of r and hence $r[tp]$ makes sense even if $tp \notin r$). After line 08. is executed while the algorithm is processing $r[tp]$, Γ_I will contain the (multi)set of all tp-cases from the tuples in $r[tp]$. Since γ_i had been added to tp in line 11., it means that there exists such a tp-case $\gamma' = \langle C', D', L', U' \rangle \in Gamma_I$[1] that $sol(C_j) \subseteq sol(C')$, $D_j = D'$, $L_j = L'$, $U_j = U'$ and $\delta_j = \delta'$. Also this means that no other tp-case in Γ_I refers to any timepoints in $sol(C_j)$, in particular, no other tp-case in Γ_I refers to timepoint t. Clearly then $ANN(r)[d, t] = ANN(r[tp])[d, t]$ will contain **one and only one** annotated tuple: $at' = (d, t, \delta(D', t) \cdot L', \delta(D', t) \cdot U')$. But from the equalities

[1] here and further when we mention Γ_I we refer to its initial value assigned in line 08.

established above, $at' = (d, t, \delta_j(D_j, t) \cdot L_j, \delta_j(D_j, t) \cdot U_j) = (d, t, L, U) = at$. Now we notice that $\chi(\{[L, U]\}) = [L, U]$ as it is required by the definition of χ-compaction, since χ as a combination function satisfies Identity postulate.

2. γ_i had been added to tp in line 22. In this case in the set Γ_I built in line 08. for the equivalence class of tp in r there will be more than one tp-case which refers to t. From the analysis of the algorithm above we conclude that in this case $t \in sol(C_S)$ after C_S had been computed in line 13. Let us consider the iteration of the **foreach** loop 15.–22. which processes t. In line 17. Γ_t gets assigned the value of the set of all tp-cases $\gamma_i = \langle C'_i, D'_i, L'_i, U'_i, \delta'_i \rangle$ that $t \in sol(C'_i)$. **foreach** loop in lines 18.–20 builds the set $X = \{[L^*_i, U^*_i]\}$ where $[L^*_i, U^*_i] = [\delta_i(D'_i, t) \cdot L'_i, \delta_i(D'_i, t) \cdot U'_i]$ are the probabilistic intervals assigned to d, t by each tp-case γ_i.

   In line 21. $[L, U] = \chi(X)$ is computed. Now, all we have to prove is that $X = \{[L_1^{(d,t)}, U_1^{(d,t)}], \ldots, [L_k^{(d,t)}, U_k^{(d,t)}]\}$ such that
   $\mathrm{ANN}(r)[d, t] = \{at_1^{(d,t)}, \ldots at_k^{(d,t)}\}$ and $at_i^{(d,t)} = (d, t, L_i^{(d,t)}, U_i^{(d,t)})$. Let $\Gamma_t = \{\gamma'_1, \ldots \gamma'_s\}$. Since Γ_t contains **all** tp-cases of $r[tp]$ (and hence, all tp-cases of r) which refer to timepoint t for the data d, $ANN(r)[d, t] = ANN(r[tp])[d, t]$ will be equal to the set: $\{at'_1, \ldots at'_s\}$, where $at'_i = (d, t, \delta_i(D_i, t) \cdot L_i, \delta_i(D_i, t) \cdot U_i) = (d, t, L^*_i, U^*_i), 1 \leq i \leq s$. But then we also know that $X = \{[L^*_1, U^*_1], \ldots [L^*_1, U^*_1]\}$. This proves the theorem. □

Proof of Lemma 2.
Similar to the proof of Proposition 5. □

Proof of Theorem 8.
Similar to the proof of Theorem 7. □

Proof of Theorem 9.
By definition 6.7 $ANN(\kappa_\chi(r)) = \{at = (d, t, L, U) | [L, U] = \chi(\{[L_1,^{(d,t)}, U_1,^{(d,t)}], \ldots, [L_k,^{(d,t)}, U_k,^{(d,t)}]\})$, where $(\forall 1 \leq i \leq k)(at_i^{(d,t)} = (d, t, L_i^{(d,t)}, U_i^{(d,t)})$ and $\{at_1^{(d,t)}, \ldots at_k^{(d,t)}\} = ANN(r)[d, t]\}$. But by definition 5.2 $\kappa_\chi(ANN(r))$ is equal to the same expression. Therefore, $ANN(\kappa_\chi(r)) = \kappa_\chi(ANN(r))$. □

Proof of Theorem 10.
By the definition of a p-strategy based compaction of TP-relations $ANN(\kappa_\rho(r)) = \{at = (d, t, L, U) | [L, U] = [L_1^{(d,t)}, U_1^{(d,t)}] *_\rho \ldots *_\rho [L_k^{(d,t)}, U_k^{(d,t)}]$,
where $(\forall 1 \leq i \leq k)(at_i^{(d,t)} = (d, t, L_i^{(d,t)}, U_i^{(d,t)}); \{at_1^{(d,t)}, \ldots at_k^{(d,t)}\} = ANN(r)[d, t]\}$ and $* = \oplus$ if ρ is a disjunctive p-strategy and $* = \otimes$ if ρ is a conjunctive p-strategy.

But by the definition of a p-strategy based compaction of annotated relations $\kappa_\chi(ANN(r))$ is equal to the same expression. Therefore, $ANN(\kappa_\rho(r)) = \kappa_\rho(ANN(r))$. □

Proof of Theorem 11.
By definition $r \cap_\chi r' = \kappa_\chi(r \cap r')$. Our proof consists of three parts. show that:

1. *Claim 1:* $ANN(r \cap r') \subseteq ANN(r) \cap ANN(r')$.
   Let $at'' = (d, t, L, U) \in ANN(r \cap r')$. We show that $at'' \in ANN(r) \cap ANN(r')$. By definition of ANN, $at'' \in ANN(r \cap r')$ implies that there exists a tp-tuple $tp'' \in r \cap r'$ such that:

   $tp'' = (d, \gamma''); \gamma'' = \gamma''_1, \ldots \gamma''_k$
   $(\forall 1 \leq i \leq k)\ \gamma''_i = \langle C''_i, D''_i, L''_i, U''_i, \delta''_i \rangle$ **and**
   $(\exists i \in \{1, \ldots k\})(t \in sol(C''_i) \wedge [L, U] = [\delta''_i(D''_i, t) \cdot L''_i, \delta''_i(D''_i, t) \cdot U''_i])$.

As $tp'' \in r \cap r'$, by definition of intersection of two tp-relations, there exist tp-tuples $tp = (d, \gamma) \in r)$ and $tp' = (d, \gamma') \in r')$ where:

$$\begin{aligned} \gamma &= \gamma_1, \ldots, \gamma_n \\ \gamma' &= \gamma'_1, \ldots, \gamma_{n'} \\ \gamma_i &= \langle C_i, D_i, L_i, U_i, \delta_i \rangle \\ \gamma'_i &= \langle C'_i, D'_i, L'_i, U'_i, \delta'_i \rangle \end{aligned}$$

In addition, there exist integers $i_1 \in \{1, \ldots, n\}$ and $i_2 \in \{1, \ldots, n'\}$ such that either

$$\begin{aligned} C''_i &= (C_{i_1} \wedge C'_{i_2}) \text{ and} \\ D''_i &= D_{i_1} \text{ and} \\ \delta''_i &= \delta_{i_1} \text{ and} \\ [L''_i, U''_i] &= [L_{i_1}, U_{i_1}]\,. \end{aligned}$$

or

$$\begin{aligned} C''_i &= (C_{i_1} \wedge C'_{i_2}) \text{ and} \\ D''_i &= D_{i_2} \text{ and} \\ \delta''_i &= \delta_{i_2} \text{ and} \\ [L''_i, U''_i] &= [L_{i_2}, U_{i_2}]\,. \end{aligned}$$

Without loss of generality, assume that the first case holds (the reasoning for the other case is symmetric). As $t \in sol(C_{i_1})$ and $t \in sol(C'_{i_2})$, by definition of ANN, we may say that $ANN(tp) \ni at = (d, t, L_1, U_1)$ and $ANN(tp) \ni at' = (d, t, L_2, U_2)$ where $[L_1, U_1] = [\delta(D_{i_1}, t) \cdot L_{i_1}, \delta(D_{i_1}, t) \cdot U_{i_1}]$ and $[L_2, U_2] = [\delta({}'_{i_2} D_{i_1}, t) \cdot L_{i_1}, \delta'_{i_2}(D_{i_2}, t) \cdot U_{i_2}]$ Therefore, by definition of intersection of two annotated relations, we conclude that $\{at, at'\} \subseteq ANN(r) \cap ANN(r')$. It follows that $at'' = (d, t, L, U) = at = (d, t, L_{i_1}, U_{i_1})$, because $[L, U] = [L_{i_1}, U_{i_1}]$.

2. *Claim 2:* $ANN(r \cap r') \supseteq ANN(r) \cap ANN(r')$. Suppose $at = (d, t, L, U) \in ANN(r) \cap ANN(r')$. Hence, by definition of intersection there exists an annotated tuple $at' = (d, t, L', U') \in ANN(r) \cap ANN(r')$, such that

   - **either** $at \in ANN(r); at' \in ANN(r')$
   - **or** $at' \in ANN(r); at \in ANN(r')$

   Without loss of generality we will assume that $at \in ANN(r)$; $at' \in ANN(r')$ (the other case is symmetric).

   As $at \in ANN(r)$ we conclude by definition of ANN that there exists a tp-tuple $tp = (d, \gamma) \in r$ such that $\gamma = \gamma_1, \ldots, \gamma_n$
   $(\forall 1 \leq i \leq n)(\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle)$
   and $(\exists i \in \{1, \ldots, n\})(t \in sol(C_i) \wedge [L, U] = [\delta_i(D_i, t) L_i, \delta_i(D_i, t) U_i])$.

   Now from $at' \in ANN(r')$, we know that there exists a tp-tuple $tp' = (d, \gamma') \in r'$ such that $\langle tp, tp' \rangle \in m$ and
   $\gamma' = \gamma'_1, \ldots, \gamma'_n$
   $(\forall 1 \leq i \leq n')(\gamma'_i = \langle C'_i, D'_i, L'_i, U'_i, \delta'_i \rangle)$
   and $(\exists j \in \{1, \ldots, n'\})(t \in sol(C'_j) \wedge [L', U'] = [\delta'_j(D'_j, t) L'_j, \delta'_j(D'_j, t) U'_j])$.

As tp and tp' are *data identical*, there exists such a tp-tuple $tp'' = (d, \gamma'') \in r \cap r'$ that
$\gamma'' = \gamma''_1, \ldots, \gamma''_{n''}$
and $(\exists 1 \leq k \leq n'')(\gamma''_k = \langle C''_k, D''_k, L''_k, U''_k, \delta''_k \rangle \wedge C''_k = (C_i \wedge C'_j) \wedge D''_k = D_i \wedge [L''_k, U''_k] = [L_i, U_i] \wedge \delta''_k = \delta_i)$.

As $t \in sol(C_i)$ and $t \in sol(C'_j)$, $t \in sol(C_i \wedge C'_j) = C''_k$. Therefore, $ANN(t'')$ will contain a tuple $at'' = (d, t, L'', U'')$ where
$[L'', U''] = [\delta''_k(D''_k, t) \cdot L''_k, \delta''_k(D''_k, t) \cdot U''_k] = [\delta_i(D_i, t) \cdot L_i, \delta_i(D_i, t) \cdot U_i]$.

We can see that $[L'', U''] = [L, U]$ and therefore, $at'' = at$ and therefore
$at \in ANN(tp'')$, from which it follows that $at \in ANN(r \cap_m r')$.

3. *Claim 3:* $ANN(r \cap_\chi r') = ANN(r) \cap_\chi ANN(r')$.
By definition of the intersection of two TP relations $ANN(r \cap_\chi r') = ANN(\kappa_\chi(r \cap r'))$. As $\langle \kappa^\cap_\chi(.)$, $\kappa^\cap_\chi(\mathrm{ANN}(.)) \rangle$ is a *compatible pair*,we have $ANN(\kappa^\cap_\chi(r \cap r')) = \kappa^\cap_\chi(ANN(r \cap r')) = \kappa^\cap_\chi(ANN(r) \cap ANN(r')) = ANN(r) \cap_\chi ANN(r')$. □

Proof of Theorem 12.
Let r and r' be two tp-relations and χ be a combination function. By definition of union $r \cup_\chi r' = \kappa_\chi(r \cup r')$ and $ANN(r) \cup_\chi ANN(r') = \kappa_\chi(ANN(r) \cup ANN(r'))$. Since $\kappa_c hi(r)$ and $\kappa_\chi(ANN(r))$ is a compatible pair of compaction operations, it is sufficient to prove that $ANN(r \cup r') = ANN(r) \cup ANN(r')$.

- $\underline{ANN(r \cup r') \subseteq ANN(r) \cup ANN(r')}$. Let $at \in ANN(r \cup r')$. Then there exists a tp-tuple $tp \in r \cup r'$, such that $at \in ANN(tp)$. By definition of multiset union of two tp-relations, either $tp \in r$ or $tp \in r'$. If the former holds, $at \in ANN(r)$ and if the latter holds, $at \in ANN(r')$. In either case, $at \in ANN(r) \cup ANN(r')$ as $ANN(r) \cup ANN(r')$ consists of all annotated tuples in $ANN(r)$ and all annotated tuples in $ANN(r')$.

- $\underline{ANN(r \cup r') \supseteq ANN(r) \cup ANN(r')}$. Let $at \in ANN(r) \cup ANN(r')$. Then, either $at \in ANN(r)$ or $at \in ANN(r)$. Assume the former is true (the other case is symmetric). Then, by def. of annotation operation, there exists a tp-tuple $tp \in r$, such that $at \in ANN(tp)$. But, since by definition of multiset union of two tp-relations $tp \in r \cup r'$, we also get $at \in ANN(r \cup r')$. □

Proof of Theorem 13.
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two atomic selection constraints. We will consider a number of cases, each for each pair of constraint types.

- If **both** $\mathcal{C}_1$ and $\mathcal{C}_2$ are *data* constraints, then the statement of the theorem follows from the similar result in relational algebra.

- If **both** $\mathcal{C}_1$ and $\mathcal{C}_2$ are *temporal* constraints, the statement of the theorem will be true because of commutativity of the conjunction of boolean *temporal* constraints.

- If **both** $\mathcal{C}_1$ and $\mathcal{C}_2$ are *probabilistic* constraints, the statement of the theorem will be true because of commutativity of the conjunction of boolean *probabilistic* constraints.

- Let $\mathcal{C}_1$ be a *data* constraint and $\mathcal{C}_2$ be a *temporal* constraint.

  First we prove $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) \subseteq sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$. Let $tp \in \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$. Then, (a) tp satisfies the data constraint $\mathcal{C}_1$ and (b) $tp \in \sigma_{\mathcal{C}_2}(r)$. By definition of selection on atomic *temporal* constraint, there exists a tp-tuple $tp' \in r$ such that
  $tp = (d, \gamma), \gamma = \gamma_1, \ldots, \gamma_k, \gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq k;$
  $tp' = (d, \gamma'), \gamma' = \gamma'_1, \ldots, \gamma'_m, \gamma'_i = \langle C'_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq m$ and

1. $m \geq k$;
2. and there exists a mapping $f : \{1, \ldots, k\} \to \{1, \ldots, m\}$ such that $f(i) = f(j)$ **iff** $i = j$ and $C_i = C'_{f(i)} \wedge \mathcal{C}_2$.
3. If for some $1 \leq l \leq m$, $C'_l \wedge \mathcal{C}_2$ is consistent, then there exists such number $1 \leq j \leq k$, that $f(j) = l$.

Since tp and tp' are data-identical, $tp' \in \sigma_{\mathcal{C}_1}(r)$, since tp' must also satisfy $\mathcal{C}_1$. But then, by the definition of selection on temporal constraint $sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$ will contain the tuple tp'' defined as follows: $tp'' = (d, \gamma'')$, $\gamma'' = \gamma''_1, \ldots \gamma''_n$, $\gamma''_i = \langle C''_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq n$ and for each $1 \leq j \leq m$ such that $C'_j \wedge \mathcal{C}_2$ is consistent, there exists a unique $1 \leq i \leq n$ such that $C''_i = C'_j \wedge \mathcal{C}_2$.

But the latter description is equivalent to the description of tp, i.e. $tp'' = tp$, i.e. $tp \in \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$.

Now we prove that $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) \supseteq \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$. Let $tp \in \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$. This means that $(\sigma_{\mathcal{C}_1}(r))$ contains a tuple tp' such that $tp = (d, \gamma), \gamma = \gamma_1, \ldots, \gamma_k$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq k$;
$tp' = (d, \gamma'), \gamma' = \gamma'_1, \ldots, \gamma'_m$, $\gamma'_i = \langle C'_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq m$ and

1. $m \geq k$;
2. and there exists a mapping $f : \{1, \ldots, k\} \to \{1, \ldots, m\}$ such that $f(i) = f(j)$ **iff** $i = j$ and $C_i = C'_{f(i)} \wedge \mathcal{C}_2$.
3. If for some $1 \leq l \leq m$, $C'_l \wedge \mathcal{C}_2$ is consistent, then there exists such number $1 \leq j \leq k$, that $f(j) = l$.

Since $tp' \in \sigma_{\mathcal{C}_1}(r)$ and $\mathcal{C}_1(r)$, $tp' \in r$. Then $\sigma_{\mathcal{C}_2}(r)$ will contain a tuple tp'' such that $tp'' = (d, \gamma'')$, $\gamma'' = \gamma''_1, \ldots \gamma''_n$, $\gamma''_i = \langle C''_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq n$ and for each $1 \leq j \leq m$ such that $C'_j \wedge \mathcal{C}_2$ is consistent, there exists a unique $1 \leq i \leq n$ such that $C''_i = C'_j \wedge \mathcal{C}_2$. Clearly, $tp = tp''$. Since tp'' and tp' are data-identical, $tp'' \in \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$, i.e., $tp \in \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$.

The proof for the case when $\mathcal{C}_1$ is a temporal constraint and $\sigma_{\mathcal{C}_2}$ is a data constraint is symmetric.

- $\mathcal{C}_1$ is a *data* constraint and $\sigma_{\mathcal{C}_2}$ is a *probabilistic* constraint.

  First we prove $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) \subseteq \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$. Let $tp \in \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$. Then, (a) tp satisfies the data constraint $\mathcal{C}_1$ and (b) $tp \in \sigma_{\mathcal{C}_2}(r)$. By definition of selection on atomic *probabilistic* constraint, there exists a tp-tuple $tp' \in r$ such that
  $tp = (d, \gamma), \gamma = \gamma_1, \ldots, \gamma_k$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq k$;
  $tp' = (d, \gamma'), \gamma' = \gamma'_1, \ldots, \gamma'_m$, $\gamma'_i = \langle C'_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq m$ and

  1. $m \geq k$;
  2. and there exists a mapping $f : \{1, \ldots, k\} \to \{1, \ldots, m\}$ such that $f(i) = f(j)$ **iff** $i = j$ and $C_i = TP - filter(\gamma_{f(i)}, \mathcal{C}_2)$.
  3. If for some $1 \leq l \leq m$, $sol(TP - filter(\gamma_{f(i)}, \mathcal{C}_2)) \neq \emptyset$ then there exists such number $1 \leq j \leq k$, that $f(j) = l$.

  Then by definition of selection on atomic *data* constraint, $\sigma_{\mathcal{C}_1}(r)$ will contain tp' as tp and tp' are data-identical and tp satisfies $\mathcal{C}_1$. Therefore, $\sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$ will contain a tp-tuple $tp'' = (d, \gamma'')$ constructed as follows: for each $\gamma'_i \in \gamma'$ such that sol$(TP - filter(\gamma'_i, \mathcal{C}_2)) \neq \emptyset$, there will be a case $\langle TP - filter(\gamma'_i, \mathcal{C}_2), D_i, L_i, U_i, \delta_i \rangle$ in γ'' and there will be no other cases in γ''. But it is clear that in this case $tp'' = tp$ and therefore $tp \in \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$.

  To prove that $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) \supseteq \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$ now let $tp \in \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$.

In this case, $\sigma_{\mathcal{C}_1}(r)$ will contain a tp-tuple tp' such that
$tp = (d, \gamma), \gamma = \gamma_1, \ldots, \gamma_k, \gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq k$;
$tp' = (d, \gamma'), \gamma' = \gamma'_1, \ldots, \gamma'_m, \gamma'_i = \langle C'_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq m$ and

1. $m \geq k$;
2. and there exists a mapping $f : \{1, \ldots, k\} \rightarrow \{1, \ldots, m\}$ such that $f(i) = f(j)$ **iff** $i = j$ and $C_i = TP - filter(\gamma_{f(i)}, \mathcal{C}_2)$.
3. If for some $1 \leq l \leq m$, $sol(TP - filter(\gamma'_{f(i)}, \mathcal{C}_2)) \neq \emptyset$ then there exists such number $1 \leq j \leq k$, that $f(j) = l$.

Then by definition of selection on atomic *data* constraint, tp' must satisfy $\mathcal{C}_1(r)$ and therefore $tp' \in r$. But then $\mathcal{C}_2(r)$ will contain tp (see the construction of tp'' above to see why this is true). And since tp and tp' are data-identical, tp will satisfy $\mathcal{C}_1(r)$ and therefore $tp \in \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$.

The proof for the case when $\mathcal{C}_1$ is a probabilistic constraint and $\mathcal{C}_2$ is a data constraint is symmetric.

- $\mathcal{C}_1$ is a *temporal* constraint and $\mathcal{C}_2$ is a *probabilistic* constraint.

  First we prove $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) \subseteq \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$. Let $tp \in \sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r))$. Then, by definition of a selection on an atomic *temporal* condition, $\sigma_{\mathcal{C}_2}(r)$ will contain a tuple tp' such that
  $tp = (d, \gamma), \gamma = \gamma_1, \ldots, \gamma_k, \gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq k$;
  $tp' = (d, \gamma'), \gamma' = \gamma'_1, \ldots, \gamma'_m, \gamma'_i = \langle C'_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq m$ and

  1. $m \geq k$;
  2. and there exists a mapping $f : \{1, \ldots, k\} \rightarrow \{1, \ldots, m\}$ such that $f(i) = f(j)$ **iff** $i = j$ and $C_i = {C'}_{f(i)} \wedge \mathcal{C}_2$.
  3. If for some $1 \leq l \leq m$, $C'_l \wedge \mathcal{C}_2$ is consistent, then there exists such number $1 \leq j \leq k$, that $f(j) = l$.

  Since $tp' \in \sigma_{\mathcal{C}_2}(r)$, by definition of selection on atomic *probabilistic* condition there exists a tuple $tp'' \in r$ such that
  $tp'' = (d, \gamma''), \gamma'' = \gamma''_1, \ldots, \gamma''_n, \gamma''_i = \langle C''_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq n$ and

  1. $n \geq m$;
  2. and there exists a mapping $g : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ such that $g(i) = g(j)$ **iff** $i = j$ and $C'_i = TP - filter(\gamma''_{g(i)}, \mathcal{C}_2)$.
  3. If for some $1 \leq l \leq n$, $sol(TP - filter(\gamma_{g(i)}, \mathcal{C}_2)) \neq \emptyset$ then there exists such number $1 \leq j \leq k$, that $g(j) = l$.

  From the above we obtain that $C_i = TP - filter(\gamma(g(f(i)), \mathcal{C}_2) \wedge \mathcal{C}_1$.

  Since $tp'' \in r$, we know that $\sigma_{\mathcal{C}_1}(r)$ will contain a tp-tuple $tp''' = (d, \gamma''')$ such that $\gamma''' = \gamma'''_1, \ldots \gamma'''_r$, $\gamma'''_i = \langle C'''_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq r$ and for each $1 \leq j \leq n$ such that $C''_j \wedge \mathcal{C}_1$ is consistent, there exists a unique $1 \leq i \leq r$ such that $C'''_i = C''_j \wedge \mathcal{C}_2$.

  Finally, $\sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$ will contain a tp-tuple $tp^* = (d, \gamma^*)$ such that for each $\gamma'''_i \in \gamma'''$ such that sol$(TP - filter(\gamma'''_i, \mathcal{C}_2)) \neq \emptyset$, there will be a case $\langle TP - filter(\gamma'''_i, \mathcal{C}_2), D_i, L_i, U_i, \delta_i \rangle$ in γ^* and there will be no other cases in γ^*.

  Now we will show that tp indeed is equal to tp^*. As we noticed, every case in tp had its C constraint have a form $C_i = TP - filter(\gamma''_g(f(i)), \mathcal{C}_2) \wedge \mathcal{C}_1$.

Now, it is easy to notice that since every constraint in tp''' had a form $C'''_j = C''_l \wedge \mathcal{C}_1$, every constraint is tp^* will be of the form $C^*_i = TP-filter(\gamma'''_j, \mathcal{C}_2) = TP-filter(\langle C''_l \wedge \mathcal{C}_1, D''_l, L''_l, U''_l, \delta''_l\rangle, \mathcal{C}_2)$.

Clearly, if for some $1 \leq j \leq n$, there was such $1 \leq i \leq k$ that $C_i = TP-filter(\gamma''_j, \mathcal{C}_2) \wedge \mathcal{C}_1$ then $TP-filter(\langle C''_j \wedge \mathcal{C}_1, D''_j, L''_j, U''_j, \delta''_j\rangle, \mathcal{C}_2)$ will be the value of some C^*_l from γ^*.

We now show that $TP-filter(\gamma''_j, \mathcal{C}_2) \wedge \mathcal{C}_1 = TP-filter(\langle C''_j \wedge \mathcal{C}_1, D''_j, L''_j, U''_j, \delta''_j\rangle, \mathcal{C}_2))$.

Let $t \in sol(TP-filter(\gamma''_j, \mathcal{C}_2) \wedge \mathcal{C}_1)$. In this case, $t \in sol(C''_j)$ and $\delta''_j(D''_j, t)$ satisfies $\mathcal{C}_2$. Also, t satisfies $\mathcal{C}_1$. But then, t satisfies $C''_j \wedge \mathcal{C}_1$. Since this does not affect the probability estimate for t, we obtain that $t \in sol(TP-filter(\langle C''_j \wedge \mathcal{C}_1, D''_j, L''_j, U''_j, \delta''_j\rangle, \mathcal{C}_2))$.

Conversely, if $t \in sol(TP-filter(\langle C''_j \wedge \mathcal{C}_1, D''_j, L''_j, U''_j, \delta''_j\rangle, \mathcal{C}_2))$ then (a) $\delta''_j(D''_j, t)$ satisfies $\mathcal{C}_2$ and (b) t satisfies $C''_j \wedge \mathcal{C}_1$. Therefore t satisfies C''_j and t satisfies $\mathcal{C}_1$. Then clearly, $t \in sol(TP-filter(\langle C''_j, D''_j, L''_j, U''_j, \delta''_j\rangle \mathcal{C}_2))$. Therefore, $t \in sol(TP-filter(\langle C''_j, D''_j, L''_j, U''_j, \delta''_j\rangle \mathcal{C}_2)) \cap sol(\mathcal{C}_1)$, i.e. $t \in sol(TP-filter(\gamma''_j, \mathcal{C}_2) \wedge \mathcal{C}_1)$.

This proves the desired inclusion.

The proof that $\sigma_{\mathcal{C}_1}(\sigma_{\mathcal{C}_2}(r)) \supseteq \sigma_{\mathcal{C}_2}(\sigma_{\mathcal{C}_1}(r))$ is symmetric to the proof above. Similarly, the proof of the statement of the theorem for the case when $\mathcal{C}_1$ is a probabilistic condition and $\mathcal{C}_2$ is a temporal condition is symmetric to the proof above.

The list enumerates all possible pairs of types of constraints, therefore the theorem is proven. $\square$

Proof of Theorem 14.

We will prove this theorem for the case when $\mathcal{C}$ is an atomic predicate (constraint). Then, by theorem 13 this theorem will be true for arbitrary constraints as well.

So, assume $\mathcal{C}$ is atomic predicate. We have to consider three cases:

1. $\mathcal{C}$ is a predicate on the data part of the relational schema of r.

   In this case we notice that $\sigma_{\mathcal{C}}(r) \subseteq r$, i.e. only tuples from r can be found in $\sigma_{\mathcal{C}}(r)$.

   Let $at = (d, t, L, U) \in ANN(\sigma_{\mathcal{C}}(r))$. $\mathcal{C}$. By definition of annotation operation, there must be a tuple $tp = (d, \gamma) \in \sigma_{\mathcal{C}}(r)$, such that $at \in ANN(tp)$. Since $\mathcal{C}$

   is a predicate on the data part, it must be the case that d satisfies $\mathcal{C}$.

   But since $\sigma_{\mathcal{C}}(r) \subseteq r$, we know that $tp \in r$, therefore, since $at \in ANN(tp)$, $at \in ANN(r)$. Finally, as d satisfies $\mathcal{C}$ we get $at \in \sigma_{\mathcal{C}}(ANN(r))$.

   To prove the inclusion the other way around, let us consider the tuple $at = (d, t, L, U \in \sigma_{\mathcal{C}}(ANN(r))$. Since $\mathcal{C}$ is a predicate on the data part d satisfies $\mathcal{C}$.

   By definition of selection on annotated tuples, $at \in ANN(r)$. But then, there must be a tp-tuple $tp = (d, \gamma) \in r$ such that $at \in ANN(tp)$. since d satisfies $\mathcal{C}$, $tp \in \sigma_{\mathcal{C}}(r)$, and therefore, $at \in ANN(\sigma_{\mathcal{C}}(r))$.

2. $\mathcal{C}$ is a temporal predicate.

   - $ANN(\sigma_{\mathcal{C}}(r)) \subseteq \sigma_{\mathcal{C}}(ANN(r))$
     Let $at = (d, t, L, U) \in ANN(\sigma_{\mathcal{C}}(r))$. We show that $at \in \sigma_{\mathcal{C}}(ANN(r))$.
     $at \in ANN(\sigma_{\mathcal{C}}(r))$ implies that there exists a tp-tuple $tp = (d, \gamma) \in \sigma_{\mathcal{C}}(r)$, such that
     $\gamma = \gamma_1, \ldots, \gamma_n$;
     $at \in ANN(tp)$ and
     $(\exists i \in \{1, \ldots, n\})(\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i\rangle \wedge t \in sol(C_i) \wedge [L, U] = [\delta_i(D_i, t)L_i, \delta_i(D_i, t)U_i]$.

Since $tp \in \sigma_{\mathcal{C}}(r)$, by definition of selection on a TP-relation, there must exist a tp-tuple $tp' \in r$, such that:
$tp' = (d, \gamma'); \gamma' = \gamma'_1, \dots, \gamma'_k$ and
$(\exists j \in \{1, \dots, k\})(\gamma'_j = \langle C'_j, D'_j, L'_j, \delta'_j \rangle)$ such that
$\langle D'_j, L'_j, U'_j, \delta'_j \rangle = \langle D_i, L_i, U_i, \delta_i \rangle$ and
$C_i = C'_j \wedge \mathcal{C}$.
Since $t \in sol(C_i)$, $t \in sol(C'_j)$ and $t \in sol(\mathcal{C})$. The former means that $at \in ANN(tp')$, i.e., $at \in ANN(r)$. The latter means that by definition of selection on annotated relation, $at \in \sigma_{\mathcal{C}}(ANN(r))$.

- $ANN(\sigma_{\mathcal{C}}(r)) \supseteq \sigma_{\mathcal{C}}(ANN(r))$
  Let $at = (d, t, L, U) \in \sigma_{\mathcal{C}}(ANN(r))$. We show that $at \in ANN(\sigma_{\mathcal{C}}(r))$.
  Since $at \in \sigma_{\mathcal{C}}(ANN(r))$, by definition of selection on annotated relations
  (i) $at \in ANN(r)$
  (ii) $t \in sol(\mathcal{C})$
  Since $at \in ANN(r)$, there exists a tp-tuple $tp = (d, \gamma) \in r$, such that $\gamma = \gamma_1, \dots, \gamma_n$;
  $at \in ANN(tp)$ and
  $(\exists i \in \{1, \dots, n\})(\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle \wedge t \in sol(C_i) \wedge [L, U] = [\delta_i(D_i, t)L_i, \delta_i(D_i, t)U_i]$.
  We know that $t \in sol(C_i)$ and $t \in sol(\mathcal{C})$, therefore $t \in sol(C_i \wedge \mathcal{C})$. This means that there exists a tp-tuple $tp' \in \sigma_{\mathcal{C}}(r)$ such that,
  $tp' = (d, \gamma'); \gamma' = \gamma'_1, \dots, \gamma'_k$ and
  $(\exists j \in \{1, \dots, k\})(\gamma'_j = \langle C'_j, D'_j, L'_j, \delta'_j \rangle)$ such that
  $\langle D'_j, L'_j, U'_j, \delta'_j \rangle = \langle D_i, L_i, U_i, \delta_i \rangle$ and
  $C'_j = C_i \wedge \mathcal{C}$.
  $t \in sol(C'_j)$ and therefore, $at \in ANN(tp')$, i.e., $at \in ANN(\sigma_{\mathcal{C}}(r))$.

3. $\mathcal{C}$ is a probabilistic predicate.

- $ANN(\sigma_{\mathcal{C}}(r)) \subseteq \sigma_{\mathcal{C}}(ANN(r))$
  Let $at = (d, t, L, U) \in ANN(\sigma_{\mathcal{C}}(r))$. We show that $at \in \sigma_{\mathcal{C}}(ANN(r))$.
  $at \in ANN(\sigma_{\mathcal{C}}(r))$ implies that there exists a tp-tuple $tp = (d, \gamma) \in \sigma_{\mathcal{C}}(r)$, such that
  $\gamma = \gamma_1, \dots, \gamma_n$;
  $at \in ANN(tp)$ and
  $(\exists i \in \{1, \dots, n\})(\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle \wedge t \in sol(C_i) \wedge [L, U] = [\delta_i(D_i, t)L_i, \delta_i(D_i, t)U_i])$.
  Since $tp \in \sigma_{\mathcal{C}}(r)$, by definition of selection on a TP-relation, there must exist a tp-tuple $tp' \in r$, such that:
  $tp' = (d, \gamma'); \gamma' = \gamma'_1, \dots, \gamma'_k$ and
  $(\exists j \in \{1, \dots, k\})(\gamma'_j = \langle C'_j, D'_j, L'_j, \delta'_j \rangle)$ such that
  $\langle D'_j, L'_j, U'_j, \delta'_j \rangle = \langle D_i, L_i, U_i, \delta_i \rangle$ and
  $C_i =$ TP-filter$(\gamma'_j, \mathcal{C})$
  Since $t \in sol(C_i)$, i.e, $t \in sol($TP-filter$(\gamma'_j, \mathcal{C}))$, we have, $[L, U] \in sol(\mathcal{C})$ and also, $t \in sol(C_j)$
  The latter means that $at \in ANN(tp')$, i.e., $at \in ANN(r)$. The former means that by definition of selection on annotated relation, $at \in \sigma_{\mathcal{C}}(ANN(r))$.
- $ANN(\sigma_{\mathcal{C}}(r)) \supseteq \sigma_{\mathcal{C}}(ANN(r))$
  Let $at = (d, t, L, U) \in \sigma_{\mathcal{C}}(ANN(r))$. We show that $at \in ANN(\sigma_{\mathcal{C}}(r))$.
  Since $at \in \sigma_{\mathcal{C}}(ANN(r))$, by definition of selection on annotated relations
  (i) $at \in ANN(r)$
  (ii) $[L, U] \in sol(\mathcal{C})$.

Since $at \in ANN(r)$, there exists a tp-tuple $tp = (d, \gamma) \in r$, such that $\gamma = \gamma_1, \ldots, \gamma_n$;
$at \in ANN(tp)$ and
$(\exists i \in \{1, \ldots, n\})(\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle \wedge t \in sol(C_i) \wedge [L, U] = [\delta_i(D_i, t)L_i, \delta_i(D_i, t)U_i])$.
Let us consider tuple $tp' = \sigma_{\mathcal{C}}(tp)$ i.e.,
$tp' = (d, \gamma'); \gamma' = \gamma'_1, \ldots, \gamma'_k$ and
$(\exists j \in \{1, \ldots, k\})(\gamma'_j = \langle C'_j, D'_j, L'_j, \delta'_j \rangle)$ such that
$\langle D'_j, L'_j, U'_j, \delta'_j \rangle = \langle D_i, L_i, U_i, \delta_i \rangle$ and
$C'_j =$ TP-filter$(\gamma_i, \mathcal{C})$

Since $[L, U] \in sol(\mathcal{C})$, and $t \in sol(C_i)$, we know that $t \in sol($TP-filter$(\gamma_i, \mathcal{C}))$ and therefore, $at \in ANN(tp')$, i.e., $at \in ANN(\sigma_{\mathcal{C}}(r))$. □

Proof of Theorem 15.
We break the proof into two parts:

1. $\underline{\mathrm{ANN}(r - r') \subseteq \mathrm{ANN}(r) - \mathrm{ANN}(r')}$.

   Let $at = (d, t, L, U) \in ANN(r - r')$. We will show that $at \in \mathrm{ANN}(r) - \mathrm{ANN}(r')$. As $at \in ANN(r - r')$, by definition of ANN, there exists such a tuple $tp'' = (d, \gamma'') \in r - r'$ such that $at \in ANN(tp)$. Two cases have to be considered here:

   (a) tp'' is in $(r - r')$ because $tp'' \in r$ and there is no tp-tuple in r' which is data identical to it. In this case, there is no annotated tuple in $ANN(r')$ which is data identical to $ANN(tp^p p)$ and hence, $at \in ANN(r) - ANN(r')$.

   (b) Otherwise, tp'' is in $(r - r')$ because there is a tp-tuples $tp = (d, \gamma) \in r$ and a $tp' = (d, \gamma')$ in r' and tp'' is constructed from these two tp-tuples using the construction shown in Definition 6.17. Let (tp, tp') be any such pair of tp-tuples.

   As $at = (d, t, L, U) \in ANN(r - r')$, it follows that there is a unique integer i such that the constraint of the form $C_i \wedge \neg C'$ in Step (2) of Definition 6.17 is satisfied. This means that the time point t is a solution of one of the C-constraints of tp and none of the C-constraints of tp'. This holds for all tp-tuples in r' that have an annotated tuple of the form $(d, t, -, -)$ in their annotated expansion. Hence, independently of how we choose a tuple tp° from r', if tp and tp° are data-identical, then no tuple of the form $(d, t, -, -)$ can be in $ANN(tp) - ANN(tp^\circ)$. It follows that there can be no annotated tuple of the form $(d, t, -, -)$ can be in $ANN(r')$. If the C-constraint of tp alluded to above is $\langle C_i, D_i, L_i, U_i, \delta_i \rangle$, then we know that $(d, t, L, U) \in ANN(r)$ where $L = \delta_i(D_i, t) \cdot L_i$ and $U = \delta_i(D_i, t) \cdot U_i$. Hence, $at \in \mathrm{ANN}(r) - \mathrm{ANN}(r')$.

2. $\underline{\mathrm{ANN}(r) - \mathrm{ANN}(r') \subseteq \mathrm{ANN}(r - r')}$.

   Let $at = (d, t, L, U) \in \mathrm{ANN}(r) - \mathrm{ANN}(r')$. By definition, $at \in ANN(r)$ and there is no $at' = (d', t', L', U') \in ANN(r')$ such that $d = d'$ and $t = t'$. It follows by definition of ANN that there exists $tp \in r$ such that $at \in ANN(tp)$ and there is no $tp' \in r'$ such that $ANN(tp')$ is of the form $(d, t, -, -)$. It follows immediately from the construction of $r - r'$ (Definition 6.17) that $at \in ANN(r - r')$.

Proof of Theorem 16.

The proof will consist of two parts

1. $\underline{ANN(r \times_\alpha r') \subseteq ANN(r) \times_\alpha ANN(r')}$.

   Let $at'' = (d'', t'', L'', U'') \in ANN(r \times_\alpha r')$. We will show that $at'' \in ANN(r) \times_\alpha ANN(r')$. By the definition of an annotated relation, there exists a tp-tuple $tp'' \in r \times_\alpha r'$ such that $at'' \in ANN(tp'')$. This means that $tp'' = (d'', \gamma'')$, $\gamma'' = \gamma''_1, \ldots, \gamma''_n$, $\gamma''_i = \langle C''_i, D''_i, L''_i, U''_i, \delta''_i \rangle$, $1 \leq i \leq n$ and there exists a number $1 \leq j \leq n$ such that $sol(C''_j) = \{t''\}$ and $[L'', U''] = [L''_j, U''_j]$.

   Since $tp'' \in r \times_\alpha r'$, by the definition of cartesian product of two tp-relations, there exists such a tp-tuple $tp \in r$ and a tp-tuple $tp' \in r'$ that:

   $tp = (d, \gamma)$ $\qquad tp' = (d', \gamma')$

   $$d'' = d, d'$$

   $\gamma = \gamma_1, \ldots \gamma_k$ $\qquad \gamma' = \gamma'_1, \ldots \gamma'_m$

   $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle, 1 \leq i \leq k$ $\qquad \gamma'_i = \langle C'_i, D'_i, L'_i, U'_i, \delta'_i \rangle, 1 \leq i \leq m$

   $(\exists h \in \{1, \ldots k\})(t'' \in sol(C_h))$ $\qquad (\exists l \in \{1, \ldots m\})(t'' \in sol(C'_l))$

   and $[L''_j, U''_j] = [L'', U''] = [\delta_h(D_h, t'') \cdot L_h, \delta_h(D_h, t'') \cdot U_h] \otimes_\alpha [\delta'_l(D'_l, t'') \cdot L'_l, \delta'_l(D'_l, t'') \cdot U'_l]$.

   But in this case, by the definition of annotated relation, there will be an annotated tuple $at = (d, t'', \delta_h(D_h, t'') \cdot L_h, \delta_h(D_h, t'') \cdot U_h) \in ANN(tp) \subseteq ANN(r)$ and an annotated tuple $at' = (d', t'', \delta'_l(D'_l, t'') \cdot L'_l, \delta'_l(D'_l, t'') \cdot U'_l) \in ANN(tp') \subseteq ANN(r')$. Then by the definition of a cartesian product of annotated relations and since $[L'', U''] = [\delta_h(D_h, t'') \cdot L_h, \delta_h(D_h, t'') \cdot U_h] \otimes_\alpha [\delta'_l(D'_l, t'') \cdot L'_l, \delta'_l(D'_l, t'') \cdot U'_l]$, $at'' = (d, d', t'', L'', U'') \in ANN(r) \times_\alpha ANN(r')$.

2. $\underline{ANN(r) \times_\alpha ANN(r') \subseteq ANN(r \times_\alpha r')}$.

   Let $at'' = (d'', t'', L'', U'') \in ANN(r) \times_\alpha ANN(r')$. We will show that $at'' \in ANN(r \times_\alpha r')$. By the definition of cartesian product of two annotated relations there exists an annotated tuple $at = (d, t'', L, U) \in ANN(r)$ and an annotated tuple $at' = (d', t'', L', U') \in ANN(r')$ such that $d'' = d, d'$ and $[L'', U''] = [L, U] \otimes_\alpha [L', U']$.

   Since $at \in ANN(r)$, there exists a tp-tuple $tp \in r$ such that $at \in ANN(tp)$, i.e., $tp = (d, \gamma)$, $\gamma = \gamma_1, \ldots \gamma_k$, $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$, $1 \leq i \leq k$ and $(\exists h \in \{1, \ldots k\})(t'' \in sol(C_h))$ such that $[L, U] = [\delta_h(D_h, t'') \cdot L_h, \delta_h(D_h, t'') \cdot U_h]$.

   Similarly, there exists a tp-tuple $tp' \in r'$ such that $at' \in ANN(tp')$, i.e., $tp' = (d'\gamma')$, $\gamma' = \gamma'_1, \ldots \gamma'_m$, $\gamma'_i = \langle C'_i, D'_i, L'_i, U'_i, \delta'_i \rangle$, $1 \leq i \leq m$ and $(\exists l \in \{1, \ldots m\})(t'' \in sol(C'_l))$ and $[L', U'] = [\delta'_l(D'_l, t'') \cdot L'_l, \delta'_l(D'_l, t'') \cdot U'_l]$.

   Since $t'' \in C_h$ and $t'' \in C'_l$, by definition of cartesian product of two tp-relations, $r \times_\alpha r'$ will contain a tp-tuple $tp'' = (d, d', \gamma'')$, $\gamma'' = \gamma''_1, \ldots, \gamma''_n$, $\gamma''_i = \langle C''_i, D''_i, L''_i, U''_i, \delta''_i \rangle$, $1 \leq i \leq n$ such that there exists a number $1 \leq j \leq n$ such that $sol(C''_j) = \{t''\}$ and $[L''_j, U''_j] = [L, U] \otimes_\alpha [L', U'] = [L'', U'']$. But then $ANN(r \times_\alpha r')$ will contain the tuple $at'' = ((d, d'), t'', L'', U'')$. $\square$

Proof of Theorem 17.
As usual the proof has two parts.

1. $\underline{ANN(\pi_{\mathcal{F},\rho}(r)) \subseteq \pi_{\mathcal{F},\rho}(\mathrm{ANN}(r))}$.

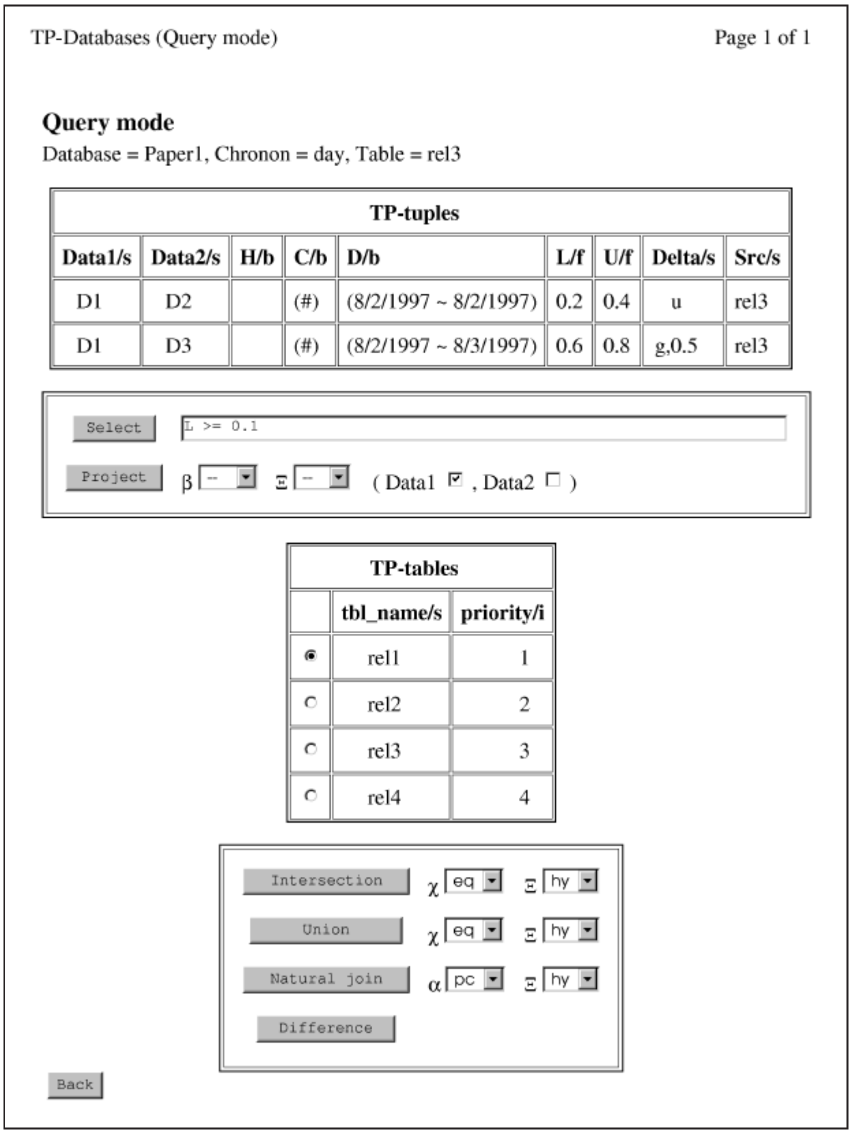
   Suppose $at = (d, t, L, U) \in ANN(\pi_{\mathcal{F},\rho}(r))$. By definition, there is a tp-tuple in $\pi_{\mathcal{F},\rho}(r)$ of the form $tp = (d, \gamma)$ in $\pi_{\mathcal{F},\rho}(r)$ such that $at \in ANN(tp)$. Let $\gamma = \gamma_1, \ldots, \gamma_n$ and let $\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle$ — hence, there is a unique integer $1 \leq j \leq n$ such that $t \in \mathrm{sol}(D_j)$, and $L = \delta_j(D_j, t) \cdot L_j$ and $U = \delta_j(D_j, t) \cdot U_j$. As TP-projection is a multiset operation, there is a unique tuple, $tp^\star$ in r such that $tp.H = tp^\star.H \parallel$ "$A_1 : d.\mathrm{A}_1, ..., A_n : d.\mathrm{A}_\mathrm{n}$" and for all attributes in F, $tp^\star$'s attribute values and those of tp coincide. Hence, $at \in \pi_{\mathcal{F},\rho}(\mathrm{ANN}(tp^\star))$.

2. $\underline{\pi_{\mathcal{F},\rho}(\mathrm{ANN}(r)) \subseteq ANN(\pi_{\mathcal{F},\rho}(r))}$.

Suppose $at = (d, t, L, U) \in \pi_{\mathcal{F},\rho}(\mathrm{ANN}(r))$. Then there is an annotated tuple, $at^\star \in ANN(r)$ such that $at = \pi_{\mathcal{F},\rho}(\mathrm{ANN}(at^\star))$. But then there is a TP-tuple $tp^\star$ in r such that $at^\star \in tp^\star$. Hence, $at \in \pi_{\mathcal{F},\rho}(\mathrm{ANN}(tp^\star))$. $\square$

Proof of Theorem 18.
As join is a derived operation defined in terms of selection, projection and cartesian product operations, the result follows immediately from Theorems 14, 17 and 16. $\square$

D Appendix: TP-Database Query Interface



The figure above shows a screen shot of a query interface of a prototype TP Database implementation described in Section 7.