An introduction to adiabatic quantum computation

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Outline

• Quantum computers
• Quantum computation and Hamiltonian dynamics
• The adiabatic theorem
• Adiabatic optimization
  ‣ Examples of success
  ‣ Example of failure
• Robustness of adiabatic QC
  ‣ Unitary control error
  ‣ Thermal noise
Why quantum computation?

Quantum computers can solve certain problems dramatically faster than classical computers can.

- Simulating quantum dynamics
- Factoring
- Discrete log
- Pell’s equation
- Abelian HSP
- Some nonabelian HSPs
- Shifted Legendre symbol/polynomial reconstruction
- Estimating Gauss sums
- Graph traversal
- Approximating Jones polynomial
- Counting solutions of finite field equations

Main questions:

- What other problems can we solve faster with a quantum computer?
- How can we actually build a quantum computer, despite the extreme sensitivity of quantum systems to noise?
Quantum circuits

- Prepare $n$ qubits in the state $|0 \cdots 0\rangle$
- Apply a sequence of poly($n$) unitary operations acting on one or two qubits at a time
- Measure in the computational basis to get the result
In the circuit model, we say a unitary operation can be implemented efficiently if it can be realized (approximately) by a short sequence of one- and two-qubit gates.

What Hamiltonian dynamics can be implemented efficiently?
Hamiltonian dynamics

\[ i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle \]

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What Hamiltonian dynamics can be implemented efficiently?

- Hamiltonians we can directly realize in the laboratory

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- Hamiltonians we can efficiently simulate using quantum circuits
Simulating Hamiltonian dynamics

**Definition.** A Hamiltonian $H$ acting on $n$ qubits can be *efficiently simulated* if for any error $\varepsilon > 0$ and time $t > 0$ there is a quantum circuit $U$ consisting of $\text{poly}(n, t, 1/\varepsilon)$ gates such that $\|U - e^{-iHt}\| < \varepsilon$. 
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Theorem. If $H$ is a sum of local terms, then it can be efficiently simulated. [Lloyd 1996]
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**Basic idea: Lie product formula**

$$e^{-i(H_1 + \cdots + H_k)t} = (e^{-iH_1 t/r} \cdots e^{-iH_k t/r})^r + O(k^2 t^2 \max \{\|H_j\|^2\}/r)$$
Sparse Hamiltonians

**Theorem.** Suppose that for any fixed $a$, we can efficiently compute all the nonzero values of $\langle a|H|b \rangle$. (In particular, there must be only polynomially many such values.) Then $H$ can be simulated efficiently. [Aharonov & Ta-Shma 2003, Childs et al. 2003, Ahokas et al. 2005]
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Basic idea: Color the interaction graph with a small number of colors and simulate each color separately

$$H = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
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\end{pmatrix}$$
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The adiabatic theorem

Let $\tilde{H}(s)$ be a smoothly varying Hamiltonian for $s \in [0,1]$

$\tilde{H}(s) = \sum_{j=0}^{D-1} E_j(s) |E_j(s)\rangle \langle E_j(s)|$

where $E_0(s) < E_1(s) \leq E_2(s) \leq \cdots \leq E_{D-1}(s)$
The adiabatic theorem

Let $\tilde{H}(s)$ be a smoothly varying Hamiltonian for $s \in [0, 1]$ where

$$T \text{ is the total run time}$$

$$H(0) = \tilde{H}(0)$$

$$H(t) = \tilde{H}(t/T)$$

$$H(T) = \tilde{H}(1)$$

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Suppose $|\psi(0)\rangle = |E_0(0)\rangle$

Then as $T \to \infty$, $|\langle E_0(1)|\psi(T)\rangle|^2 \to 1$
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For large $T$, $|\psi(T)\rangle \approx |E_0(1)\rangle$. But how large must it be?
Approximately adiabatic evolution

The total run time required for adiabaticity depends on the spectrum of the Hamiltonian.

**Gap:** \[ \Delta(s) = E_1(s) - E_0(s) , \quad \Delta = \min_{s \in [0,1]} \Delta(s) \]
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Rough estimates (see for example [Messiah 1961]) suggest the condition

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T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \| [\dot{H}(s)]^2 \| 
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Theorem. [Teufel 2003 + perturbation theory]

\[
T \geq \frac{4}{\epsilon} \left[ \frac{\| \dot{H}(0) \|}{\Delta(0)^2} + \frac{\| \dot{H}(1) \|}{\Delta(1)^2} + \int_0^1 ds \left( 10 \frac{\| \ddot{H} \|^2}{\Delta^3} + \frac{\| \dddot{H} \|}{\Delta} \right) \right]
\]

implies \( \| |\psi(T)\rangle - |E_0(1)\rangle \| \leq \epsilon \)
Satisfiability problems

• Given \( h: \{0,1\}^n \rightarrow \{0,1,2,\ldots\} \), is there a value of \( z \in \{0,1\}^n \) such that \( h(z) = 0 \)?

• Alternatively, what \( z \) minimizes \( h(z) \)?

• Example: 3SAT. \( (z_1 \lor z_2 \lor \overline{z}_3) \land \cdots \land (\overline{z}_{17} \lor z_{37} \lor \overline{z}_{42}) \)

\[
h(z) = \sum_c h_c(z)
\]

where \( h_c(z) = \begin{cases} 0 & \text{clause } c \text{ satisfied by } z \\ 1 & \text{otherwise} \end{cases} \)
Adiabatic optimization

• Define a problem Hamiltonian whose ground state encodes the solution:

\[ H_P = \sum_{z \in \{0,1\}^n} h(z)|z\rangle\langle z| \]

• Define a beginning Hamiltonian whose ground state is easy to create, for example

\[ H_B = -\sum_{j=1}^{n} \sigma_x^{(j)} \]

• Choose \( \tilde{H}(s) \) to interpolate from \( H_B \) to \( H_P \), for example

\[ \tilde{H}(s) = (1 - s)H_B + sH_P \]

• Choose total run time \( T \) so the evolution is nearly adiabatic

[Farhi et al. 2000]
Please mind the gap

Recall rough estimate:

\[ T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \| [\hat{\mathcal{H}}(s)]^2 \| \]

For \( \hat{\mathcal{H}}(s) = (1 - s)H_B + sH_P \),

\[ \| \hat{\mathcal{H}} \| = \| H_P - H_B \| \leq \| H_B \| + \| H_P \| \]

Crucial question: How big is \( \Delta \)?

- \( \geq 1 / \text{poly}(n) \): Efficient quantum algorithm
- \( 1 / \text{exp}(n) \): Inefficient quantum algorithm
Unstructured search

Finding a needle in a haystack: $h(z) = \begin{cases} 
0 & z = w \\
1 & z \neq w
\end{cases}$

(here $h: \{0, 1, \ldots, N-1\} \rightarrow \{0, 1\}$)
Unstructured search

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Query complexity (given black box for \( h \))

• Classically, \( \Theta(N) \) queries

• Quantumly, \( O(\sqrt{N}) \) queries are sufficient to find \( w \) \[ \text{Grover 1996} \]

• This cannot be improved: \( \Omega(\sqrt{N}) \) queries are necessary \[ \text{Bennett et al. 1997} \]
Example: Adiabatic unstructured search

\[ h(z) = \begin{cases} 
0 & z = w \\
1 & z \neq w 
\end{cases} \quad \Rightarrow \quad H_P = \sum_z h(z)|z\rangle\langle z| = 1 - |w\rangle\langle w| \]

Start in \[ |s\rangle = \frac{1}{\sqrt{N}} \sum_z |z\rangle \]

\[ H_B = 1 - |s\rangle\langle s| \]

\[ \tilde{H}(s) = (1 - s)H_B + s H_P \]
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\[ \tilde{H}(s) = [1 - f(s)]H_B + f(s)H_P \]

[Roland, Cerf 2002; van Dam et al. 2001]
Example: Transverse Ising model

\[ H_P = \sum_{j \in \mathbb{Z}_n} \frac{1}{2} \left( 1 - \sigma_z^{(j)} \sigma_z^{(j+1)} \right) \quad \text{“agree”} \]

\[ H_B = - \sum_{j=1}^{n} \sigma_x^{(j)} \quad \text{with ground state} \quad |s\rangle = | + \cdots + \rangle \]

\[ \tilde{H}(s) = (1 - s) H_B + s H_P \]

Diagonalize by fermionization (Jordan-Wigner transformation)

Result: \( \Delta \propto \frac{1}{n} \) (at critical point of quantum phase transition)

\[ |E_0(s \approx 0)\rangle \approx | + \cdots + \rangle \]

\[ |E_0(s \approx 1)\rangle \approx \frac{1}{\sqrt{2}} (|0 \cdots 0\rangle + |1 \cdots 1\rangle) \]  

[Farhi et al. 2000]
Example: The Fisher problem

\[ H_P = \sum_{j \in \mathbb{Z}_n} \frac{J_j}{2} \left( 1 - \sigma_z(j) \sigma_z(j+1) \right) \quad J_j = 1 \text{ or } 2, \text{ chosen randomly} \]

\[ H_B = -\sum_{j=1}^{n} \sigma_x(j) \]

Then typically \( \Delta \approx \exp(-c\sqrt{n}) \)
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\[ | + + + + + + + \rangle \]

\[ |00000000\rangle + |1111111\rangle + |1111111\rangle \]

[Reichardt 2004]
Robustness of adiabatic QC

Potential sources of error:

• Unitary control error

• Dephasing in instantaneous eigenstate basis

• Transitions between instantaneous eigenstates: thermal noise
Unitary control error

Adiabatic algorithm depends on going smoothly from $H_B$ to $H_P$, not on the particular path between them.

For smooth perturbations, we have no reason to expect the gap will become smaller rather than larger, even if the perturbation is not small (provided it is zero at the beginning and end of the evolution).

$$H'(t) = H(t) + K(t)$$

[Childs, Farhi, Preskill 2001]
Error in the final Hamiltonian

\[ \tilde{K}_1(s) = C_1 s \sum_{j=1}^{n} \hat{m}_j \cdot \vec{\sigma}(j) \]
Error in the interpolation

$$\tilde{K}_2(s) = C_2 \sin(\pi s) \sum_{j=1}^{n} \hat{m}_j \cdot \vec{\sigma}(j)$$

$n = 7$

$n = 10$
High frequency error

$$\tilde{K}_3(s) = \frac{1}{2} \sin(C_3 \pi s) \sum_{j=1}^{n} \hat{m}_j \cdot \vec{\sigma}(j)$$

\[ n = 8 \quad n = 8 \]

\[ n = 10 \quad n = 10 \]
Thermal noise

Efficient adiabatic quantum computation requires that the minimum gap $\Delta$ is not too small.

Provided $k_B T \ll \Delta$, thermal fluctuations are unlikely to drive the system out of the ground state. So a big gap not only allows for adiabaticity, but also provides protection against thermal noise!

Note: Here it is important that $H$ is the actual Hamiltonian of the quantum computer, not just a simulated Hamiltonian.

[Childs, Farhi, Preskill 2001]
Markovian master equations

\[ H = H_S + H_E + \lambda V \]

Weak coupling limit: \( \lambda \ll 1 \)

Product initial state: \( \rho(0) \otimes \rho_E \)

[Davies 1974]: \[
\frac{d\rho}{dt} = -i[H_S, \rho] + \lambda^2 K^\dagger \rho
\]

where

\[
K \rho = -\int_0^\infty dx \ \text{tr}_E[U(-x)VU(x), [V, \rho \otimes \rho_E]]
\]

\[
K^\dagger \rho = \lim_{x \to \infty} \frac{1}{x} \int_0^x dy \ U(-y) \{K[U(y)\rho U(-y)]\} U(y)
\]

\[
U(x) = e^{-ix(H_S+H_E)}
\]
Markovian master equation for thermal noise

Spins coupled to photons:

\[ V = \sum_i \int_0^\infty d\omega \left[ g(\omega) a_\omega \sigma_+^{(i)} + g^*(\omega) a_\omega^\dagger \sigma_-^{(i)} \right] \]

\[ \rho_E = \frac{e^{-\beta H_E}}{\text{tr} e^{-\beta H_E}} \]

Then we find

\[ \frac{d\rho}{dt} = -i[H_S, \rho] \]

\[ - \sum_{i,a,b} \left[ N_{ba} |g_{ba}|^2 \langle a | \sigma_-^{(i)} | b \rangle \langle b | \sigma_+^{(i)} | a \rangle + (N_{ab} + 1) |g_{ab}|^2 \langle b | \sigma_-^{(i)} | a \rangle \langle a | \sigma_+^{(i)} | b \rangle \right] \]

\[ \{ (|a\rangle \langle a| \rho) + (\rho | a \rangle \langle a |) - 2 |b\rangle \langle a | \rho | a \rangle \langle b | \} \]

\[ g_{ba} = \begin{cases} \lambda g(\omega_b - \omega_a) & \omega_b > \omega_a \\ 0 & \omega_b \leq \omega_a \end{cases} \]

\[ N_{ba} = \frac{1}{e^{\beta(\omega_b - \omega_a)} - 1} \]
Implications for adiabatic QC

Decoherence terms are suppressed by a factor

\[ N_{ba} = \frac{1}{e^{\beta(\omega_b-\omega_a)} - 1} \leq \frac{1}{e^{\beta \Delta} - 1} \]

which is very small provided \( \Delta \ll 1/\beta \).

(Note that this effect is difficult to see in simulations for two reasons:

- Simulating open quantum systems is very computationally intensive, so we can only consider small numbers of qubits.
- Cooling alone may be a good algorithm.)

Question: Is this good enough? I.e., is \( T = 1/poly(n) \) reasonable?
Some questions

• Can we better understand what problems have efficient adiabatic optimization algorithms?

• When can we improve the performance by choosing different interpolation paths?

• Can we increase the robustness of adiabatic quantum computers by careful encoding? In particular, can we make them robust against a small but $n$-independent temperature?