Making entanglement

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Outline

• The resource model of information theory
• Entanglement as a resource
• Some uses of entanglement
• How to (optimally) make entanglement
Resources in (quantum) information theory

*Information is a resource.*

- Physical
- Fungible

Examples:

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Quantum information theory is about the interconversion of informational resources.
What is entanglement?

Entangled pure state:
\[ |\psi\rangle_{AB} \neq |\phi\rangle_A |\eta\rangle_B \]

Canonical example: EPR pair
\[ |\Psi^\pm\rangle = \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}} \]

Entanglement = non-classical correlations

• Violation of Bell inequalities

• Can be used to perform classically impossible tasks!
Quantifying entanglement

Consider a bipartite state $|\psi\rangle$.

Any such state has a Schmidt decomposition:

$$|\psi\rangle = \sum_j \sqrt{p_j} |j\rangle_A |\tilde{j}\rangle_B$$

where $\sum_j p_j = 1$ and $\{|j\rangle_A\}$, $\{|\tilde{j}\rangle_B\}$ are orthonormal bases.

Entanglement:

$$E(|\psi\rangle) = -\sum_j p_j \log p_j$$

measured in ebits.

$$1 \text{ ebit} = E(|\Psi^+\rangle)$$
Entanglement is fungible

**Theorem.** Asymptotically, states with the same entanglement are interconvertible.

[Bennett et al. 95]

Entanglement concentration

\[ n \text{ copies of } |\psi\rangle \xrightarrow{\text{LO}} nE(|\psi\rangle) \text{ ebits} \]

Entanglement dilution

\[ nE(|\psi\rangle) \text{ ebits} \xrightarrow{\text{LOCC}} n \text{ copies of } |\psi\rangle \]
Entanglement: What is it good for?

- Superdense coding [Bennett, Wiesner 92]
- Quantum teleportation [Bennett et al. 93]
- Quantum key distribution [Lo, Chau 98]
- Entanglement-assisted classical communication
  - through unidirectional channels [Shor et al. 99]
  - through bidirectional channels [Bennett et al. 02]
- Remote state preparation [Lo 00, Bennett et al. 00]
- Data hiding [DiVincenzo et al. 00]
- Quantum Vernam cipher [Leung 00]

;
Superdense coding

[Bennett, Wiesner 92]

\[
1 \text{ ebit} + 1 \text{ qubit}_{A \rightarrow B} \quad \longrightarrow \quad 2 \text{ cbits}_{A \rightarrow B}
\]

- Alice and Bob share one ebit \( |\Psi^+\rangle \).
- Alice encodes two bits by choosing one of four unitary operators:

\[
\begin{align*}
00 & \quad I \\
01 & \quad X \\
10 & \quad Y \\
11 & \quad Z
\end{align*}
\]
- Alice applies this operator to her half of \( |\Psi^+\rangle \) and then sends her qubit to Bob. Bob gets one of four possible states:

\[
\begin{align*}
(I \otimes I)|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Psi^+\rangle \\
(X \otimes I)|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Phi^+\rangle \\
(Y \otimes I)|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Phi^-\rangle \\
(Z \otimes I)|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Psi^-\rangle
\end{align*}
\]

Note that these four states are orthogonal.
- Bob measures in the basis \( \{ |\Psi^\pm\rangle, |\Phi^\pm\rangle \} \) and acquires two bits of information.
Quantum teleportation

1 ebit + 2 cbits_{A\rightarrow B} \longrightarrow 1 \text{ qubit}_{A\rightarrow B}

- Alice and Bob share one ebit $|\Psi^+\rangle$.
- Alice has a qubit $|\eta\rangle = \alpha|0\rangle + \beta|1\rangle$. The joint state is
\[
|\eta\rangle|\Psi^+\rangle = \frac{1}{\sqrt{2}} (\alpha|0\rangle + \beta|1\rangle) (|00\rangle + |11\rangle)
= \frac{1}{2} (|\Psi^-\rangle |\eta\rangle \\
+ |\Phi^-angle X|\eta\rangle \\
+ |\Phi^+\rangle Y|\eta\rangle \\
+ |\Psi^+\rangle Z|\eta\rangle )
\]
where the first two qubits belong to Alice and the third belongs to Bob.

- Alice measures her two qubits in the basis $
\{|\Psi^\pm\rangle, |\Phi^\pm\rangle\}$ and sends the resulting two classical bits to Bob.

- Bob applies $I, X, Y, Z$ as appropriate to recover $|\eta\rangle$. 


**Physical systems**

- Adjacent quantum dots

![Adjacent quantum dots diagram](image)

- Distant labs connected by optical fiber

![Distant labs connected by optical fiber](image)

**General model:**

```
A'   O   B'
A    O   B
  
  
  
  
B'   O   A'
```
How to make entanglement

$$|\psi\rangle \left\{ \begin{array}{c} A' \\ A \\ B \\ B' \end{array} \right\} U \left\{ \begin{array}{c} \end{array} \right\} U|\psi\rangle$$

Choose $|\psi\rangle$ so that $U|\psi\rangle$ is more entangled than $|\psi\rangle$. 
Entanglement production cycle

Create initial entanglement (inefficiently)

Dilute $nE(|\psi\rangle)$ ebits

|\psi\rangle^n

Apply $U^n$

Concentrate $(U|\psi\rangle)^n$

$nE(U|\psi\rangle)$ ebits

Excess entanglement: $nE_U$ ebits
Entanglement generating capacity

\[ E_U = \sup_{|\psi\rangle \in AA'BB'} \left[ E(U|\psi\rangle) - E(|\psi\rangle) \right] \]

Three technical points:

- Mixed states
- Asymptotic vs. one-shot capacity
- Ancillary systems
Mixed states

**Theorem.** For unitary interactions, the optimal input state is always pure.

[Bennett, Harrow, Leung, Smolin 02]

Proof:

\[
E'_U = \sup_{\rho} [D(U \rho U^\dagger) - E_c(\rho)] \\
\leq \sup_{\rho} [E_c(U \rho U^\dagger) - E_c(\rho)] \\
= \sup_{\rho} \frac{1}{n} [E_f((U \rho U^\dagger)^\otimes n) - E_f(\rho^\otimes n)] + \epsilon \\
= \sup_{\rho} \frac{1}{n} \sum_i p_i [E((U|\psi_i\rangle)^\otimes n) - E(|\psi_i\rangle^\otimes n)] + \epsilon \\
= \sup_{\rho} \sum_i p_i [E(U|\psi_i\rangle) - E(|\psi_i\rangle)] \\
= \sup_{\rho,i} [E(U|\psi_i\rangle) - E(|\psi_i\rangle)] \\
= E'_U
\]
Asymptotic vs. one-shot

**Theorem.** $E_U^{(n)} = nE_U$

[Benett, Harrow, Leung, Smolin 02]

**Proof:**

The entanglement can only increase by application of $U$. For each use of $U$, the maximum increase is given by $E_U$. Thus $E_U^{(n)} \leq nE_U$.

By using the optimal input $n$ times, $E_U^{(n)} \geq nE_U$. \hfill $\square$
Using ancillas

Consider $U = \text{SWAP}$:

$$U|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle$$

Clearly $E(|\psi\rangle_{AB}) = E(U|\psi\rangle_{AB})$.

But:

In general, you can make more entanglement when ancillary systems are available. This makes it hard to compute $E_U$!
Entanglement capacity of a Hamiltonian

\[
E_H = \lim_{t \to 0} \left( E_{e^{-iHt}/t} \right) \\
= \sup_{\ket{\psi}} \left[ \frac{d}{dt} E(e^{-iHt}\ket{\psi}) \right]_{t=0}
\]

Using perturbation theory, we find

\[
E_{H,\psi} = \sum_{j,k} \sqrt{p_j p_k} \log(p_j/p_k) \ \text{Im} \langle j \tilde{j} | H | k \tilde{k} \rangle
\]

where

\[
\ket{\psi} = \sum_{j} \sqrt{p_j} \ket{j} \ket{j}_{AA'} \ket{\tilde{j}}_{BB'}
\]

This is...

- Zero for product states
- Zero for maximally entangled states
- Hard to optimize over \(|\psi\rangle|
Two-qubit Hamiltonians: Canonical form

A general two-qubit Hamiltonian has 15 real parameters. But only two of them are nonlocal!

**Fact:** Any two-qubit Hamiltonian $H$ is *locally equivalent* to a Hamiltonian of the form

$$\tilde{H} = a \, X \otimes X + b \, Y \otimes Y + c \, Z \otimes Z.$$

In other words, there are local Hamiltonians $H_A$, $H_B$ and local unitary operators $U, V$ so that

$$H = (U \otimes V) \tilde{H}(U^\dagger \otimes V^\dagger) + H_A + H_B.$$  

[Dür et al. 01]

\[
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
**Ising interaction**

Consider $H = Z \otimes Z$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

No ancillas:

$$E_{ZZ}^* = 2 \max_{p \in [0,1]} \sqrt{p(1 - p)} \log \frac{p}{1 - p}$$

$$\approx 1.9123$$

[Dür et al. 01]

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**Theorem.** $E_{ZZ} = 1.9123$

[Childs, Leung, Vidal, Verstraete 02]

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**Proof idea:** No pair of terms in the Schmidt decomposition with Schmidt coefficients $p_1, p_2$ can contribute more than $E_{ZZ}^*/(p_1 + p_2)$. □
\( \text{a } XX + b YY \)

**Upper bound:** Simulation.

\[
\text{a } X \otimes X + b Y \otimes Y \text{ can be } \text{simulated} \text{ using} \ (a + b) \ Z \otimes Z.
\]

There exist unitaries \( H, K \) so that

\[
HZH^\dagger = X \quad \quad KZK^\dagger = Y
\]

Simulation uses the Lie product formula:

\[
e^{i(H_1+H_2)t} = \lim_{n \to \infty} (e^{iH_1 t/n}e^{iH_2 t/n})^n
\]

Therefore \( E_{aXX+bYY} \leq (a + b)E_{ZZ} \).

**Lower bound:** By an explicit protocol (with no ancillas), \( E_{aXX+bYY} \geq (a + b)E_{ZZ} \). [Dür et al. 01]
Summary of known capacities

Gates:

\[ E_{\text{CNOT}} = 1 \]
\[ E_{\text{SWAP}} = 2 \]

Hamiltonians:

\[ E_{aXX+bYY} = 1.9123(a + b) \]

In general, there may be no closed-form expression for the capacity of a given interaction.

Conjecture:

\[ E_{a(XX+YY)+ZZ} = 2 \sup \left[ \sqrt{p_1p_2} \log(p_1/p_2) (\sin n + a \sin(m - l)) \right. \\
+ \left. \sqrt{p_1p_4} \log(p_1/p_4) a \sin l \right. \\
+ \left. \sqrt{p_2p_4} \log(p_2/p_4) (\sin m + a \sin(n - l)) \right] \\
\]

where \( p_1, p_2, p_4 > 0 \) and \( p_1 + 2p_2 + p_4 = 1 \).
Open problems

• Calculate capacities for two-qubit gates

• Find an upper bound on the optimal ancilla dimension for a $d_A \times d_B$ dimensional gate or Hamiltonian

• Study entanglement generation by nonunitary quantum operations

• Inverse problem: How much entanglement is needed to simulate a gate (or Hamiltonian)?

\[ E_U \leq \text{ebits needed to simulate } U \]

When is this achievable?