Spatial search by quantum walk

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Unstructured search

- $N$ items $\{1,2,\ldots,N\}$
- One "marked item" $w$
- Query: "is $w=x$?"
  I.e., black box function $f(x) = \begin{cases} 
0 & x \neq w \\
1 & x = w
\end{cases}$

- Classical: $\Theta(N)$
- Grover 1996: $O(N^{1/2})$ quantum algorithm
- BBBV 1996: This is optimal
Combinatorial search vs. spatial search

- **Combinatorial search**: $f(x)$ is an efficiently computable function

- **Spatial search**: $N$ items distributed in space (e.g., a physical database)

Model: $N$-vertex graph $G$

Algorithm must be *local* with respect to this graph.
Grover’s algorithm in $d$ dimensions

• One dimension: no speedup; $\Theta(N)$

• Benioff 00: searching a $d$-dimensional grid with a “quantum robot”
  – Each iteration takes $O(N^{1/d})$ steps to traverse the grid
  – $N^{1/2}$ Grover iterations $\Rightarrow O(N^{1/2+1/d})$ algorithm

• Can we do better?
Aaronson-Ambainis algorithm

- Recursive search of subcubes with amplitude amplification

- Results
  - $d > 2$: $O(N^{1/2})$ algorithm
  - $d = 2$: $O(N^{1/2} \log^2 N)$ algorithm
Quantum walk search algorithm

- Simple Hamiltonian dynamics
- Applicable to any graph $G$
- Results
  - Complete graph=“analog analogue” [FG96]; run time $O(N^{1/2})$
  - Hypercube: $O(N^{1/2})$ by previous results
  - $d$-dimensional lattice
    - $d>4$: $O(N^{1/2})$
    - $d=4$: $O(N^{1/2} \log^{3/2} N)$
    - $d<4$: no speedup
Graphs and matrices

- Undirected graph $G$ with no self loops

- **Adjacency matrix**: $A_{jk} = \begin{cases} 1 & (j, k) \in G \\ 0 & \text{otherwise} \end{cases}$

- **Laplacian**: $L = A - D$
  $D$ diagonal, $D_{jj} = \text{deg}(j)$
<table>
<thead>
<tr>
<th><strong>Random walk</strong></th>
<th><strong>Quantum walk</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>State space</strong></td>
<td></td>
</tr>
<tr>
<td>$N$ vertices $j=1,...,N$</td>
<td>$N$ basis states $</td>
</tr>
<tr>
<td>$p_j = \text{probability of being at vertex } j$</td>
<td>$q_j = \langle j</td>
</tr>
<tr>
<td></td>
<td>be at vertex $j$</td>
</tr>
<tr>
<td><strong>Differential equation</strong></td>
<td></td>
</tr>
<tr>
<td>$\frac{dp_j}{dt} = \gamma \sum_k L_{jk} p_k$</td>
<td>$i \frac{dq_j}{dt} = \sum_k H_{jk} q_k$</td>
</tr>
<tr>
<td><strong>Generator</strong></td>
<td></td>
</tr>
<tr>
<td>$L = \text{Laplacian of } G$</td>
<td>Can choose $H = -\gamma L$</td>
</tr>
<tr>
<td><strong>Probability conservation</strong></td>
<td></td>
</tr>
<tr>
<td>$\sum_j L_{jk} = 0 \Rightarrow \frac{d}{dt} \sum_j p_j = 0$</td>
<td>$H = H^\dagger \Rightarrow \frac{d}{dt} \sum_j</td>
</tr>
</tbody>
</table>
Quantum walk search algorithm

- Marked state identified by "oracle Hamiltonian" $H_w = -|w\rangle\langle w|$

**Algorithm**

- Start in state $|s\rangle = \frac{1}{\sqrt{N}} \sum_j |j\rangle$
- Schrödinger evolve for time $T$ using Hamiltonian $H = -\gamma L + H_w$
- Measure position

- Goal: Choose $\gamma, T$ so that $|\langle w|e^{-iHT}|s\rangle|^2$ is as close to 1 as possible (for $T$ not too big)
Why might this work?

\[ H = -\gamma L - |w\rangle \langle w| \]

critical \( \gamma \)

- ground state \( \sim |s\rangle + |w\rangle \)
- first excited state \( \sim |s\rangle - |w\rangle \)
- time \( \sim 1/(E_1 - E_0) \)

\( \gamma \rightarrow 0 \)

- \( H \sim -|w\rangle \langle w| \)
- ground state \( \sim |w\rangle \)
- first excited state \( \sim |s\rangle \)

\( \gamma \rightarrow \infty \)

- \( H \sim -\gamma L \)
- ground state \( \sim |s\rangle \)
Complete graph

\[ L + NI = N |s\rangle \langle s| = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \]

\[ H = -\gamma N |s\rangle \langle s| - |w\rangle \langle w| \]

\( \gamma N = 1 \) is the “analog analogue” of Grover’s algorithm

Eigenstates \( \sim |s\rangle \pm |w\rangle \)

Gap \( 2N^{-1/2} \)
Complete graph

\[ |\langle w | \psi_0 \rangle|^2 \]

\[ |\langle s | \psi_0 \rangle|^2 \]

\[ E_1 - E_0 \]

\[ |\langle w | \psi_1 \rangle|^2 \]

\[ |\langle s | \psi_1 \rangle|^2 \]

\[ N = 1024 \]
Hypercube

Vertices labelled by $n$-bit strings $N=2^n$

Adjacency matrix: $A = \sum_{j=1}^{n} \sigma_x^{(j)}$

Hamiltonian: $H = -\gamma A - |w\rangle \langle w|$

Analyze using total spin operators [FGGS00]
$N = 2^{10} = 1024$
\textit{d-dimensional lattice}

- Periodic cubic lattice with $N$ sites, size $N^{1/d}$ in each dimension
- Exact eigenstates/eigenvalues of $-L$

\[
\left| \phi(\vec{k}) \right\rangle = \frac{1}{\sqrt{N}} \sum_{\vec{x}} e^{i \vec{k} \cdot \vec{x}} \left| \vec{x} \right\rangle
\]

\[
\mathcal{E}(\vec{k}) = 2 \left( d - \sum_{j=1}^{d} \cos k_j \right)
\]

\[
k_j = \frac{2\pi m_j}{N^{1/d}}, \quad m_j = 0, 1, \ldots, N^{1/d} - 1
\]
(d>4)-dimensional lattice

Critical region

\[ \gamma = \gamma^* \pm O(N^{-1/2}) \]

\[ \gamma < \gamma^* \]

\[ |s\rangle \sim |\psi_1\rangle \]

\[ \gamma > \gamma^* \]

\[ |s\rangle \sim |\psi_0\rangle \]

\[ \gamma \sim \gamma^* \]

\[ E_1 - E_0 = O(N^{-1/2}) \]

\[ |\psi_{0,1}\rangle \sim |s\rangle \pm O(1) |w\rangle \]

Run time \( O(N^{1/2}) \)

\[ d=5 \]

\[ N=4^5=1024 \]
4-dimensional lattice

Critical region

\[ \gamma = \gamma^* \pm O\left(\sqrt{\log \frac{N}{N}}\right) \]

\[ \gamma < \gamma^* \]

\[ |s\rangle \sim |\psi_1\rangle \]

\[ \gamma > \gamma^* \]

\[ |s\rangle \sim |\psi_0\rangle \]

\[ \gamma \sim \gamma^* \]

\[ E_1 - E_0 = O\left(\frac{1}{\sqrt{N \log N}}\right) \]

\[ |\psi_{0,1}\rangle \sim |s\rangle \pm O\left(\frac{1}{\sqrt{\log N}}\right) |w\rangle \]

Run time \( O(\sqrt{N} \log^{3/2} N) \)

\[ d=4 \]
[\[ N=6^4=1296 \]
**3-dimensional lattice**

$$d=3$$
$$N=10^{3}=1000$$

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<tr>
<td>$\gamma = \gamma^* \pm O\left(\frac{1}{N^{1/3}}\right)$</td>
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<tr>
<td>$\gamma &lt; \gamma^*$</td>
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<tr>
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<td>$</td>
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<tr>
<td>$\gamma \sim \gamma^*$</td>
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$$E_1 - E_0 = O\left(\frac{1}{N^{2/3}}\right)$$

$$|\psi_{0,1}\rangle \sim |s\rangle \pm O\left(\frac{1}{N^{1/6}}\right) |w\rangle$$

Run time $O(N)$
**2-dimensional lattice**

Critical region

\[ \gamma = \gamma^* \log N \pm O(1) \]

\[ \gamma < \gamma^* \log N \]

\[ |s\rangle \sim |\psi_1\rangle \]

\[ \gamma > \gamma^* \log N \]

\[ |s\rangle \sim |\psi_0\rangle \]

\[ \gamma \sim \gamma^* \log N \]

\[ E_1 - E_0 = O\left(\frac{\log N}{N}\right) \]

\[ |\psi_{0,1}\rangle \sim |s\rangle \pm O\left(\sqrt{\frac{\log N}{N}}\right) |w\rangle \]

Run time \( O(N^2/\log^2 N) \)

\[ d=2 \]
\[ N=32^2=1024 \]
Related algorithms

- Shenvi, Kempe, Whaley 02: discrete time quantum walk search algorithm on hypercube, $O(N^{1/2})$

  Behavior in finite dimensions?

- Adiabatic evolution [RC01, vDMV01]

- Measurement [CDFGGS02]
Open questions

- Find more applications of quantum walks
- What is the actual complexity of the search problem in d=2?