Universal computation by quantum walk

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Quantum walk algorithms

Exponential speedups

- Black box graph traversal [CCDFGS 03]
- Hidden sphere problem [CSV 07]

Polynomial speedups

- Search on graphs [Shenvi, Kempe, Whaley 02], [CG 03, 04], [Ambainis, Kempe, Rivosh 04]
- Element distinctness [Ambainis 03]
- Triangle finding [Magniez, Santha, Szegedy 03]
- Checking matrix multiplication [Buhrman, Špalek 04]
- Testing group commutativity [Magniez, Nayak 05]
- Formula evaluation [Farhi, Goldstone, Gutmann 07], [ACRŠZ 07], [Cleve, Gavinsky, Yeung 08], [Reichardt, Špalek 08]
- Unstructured search [Grover 96] (+ many applications)

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Ex: Adjacency matrix. $H_{kj} = A_{kj} = \begin{cases} 1 & (j,k) \in E \\ 0 & (j,k) \notin E \end{cases}$

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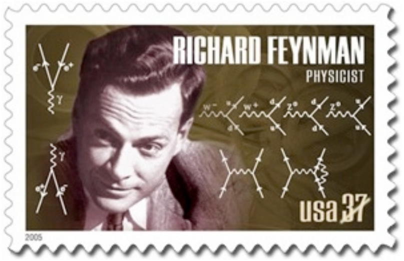
Loosely interpreted (any fixed Hamiltonian): Yes! [Feynman 85]



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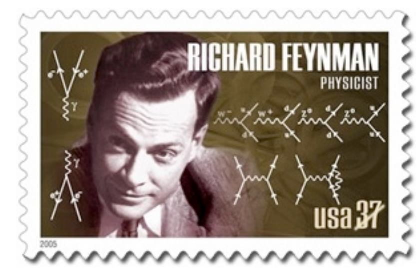

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max degree of $G =$ constant
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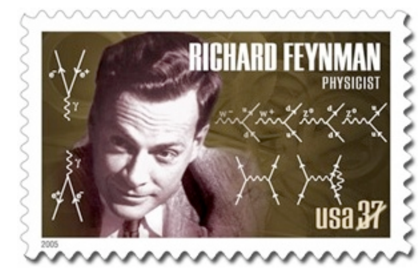

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The resulting construction also suggests an approach to quantum walk algorithms.

The plan

- Scattering theory on graphs
- Gate widgets
- Simplifying the initial state: Momentum filtering and separation
- Toward scattering algorithms

Scattering theory

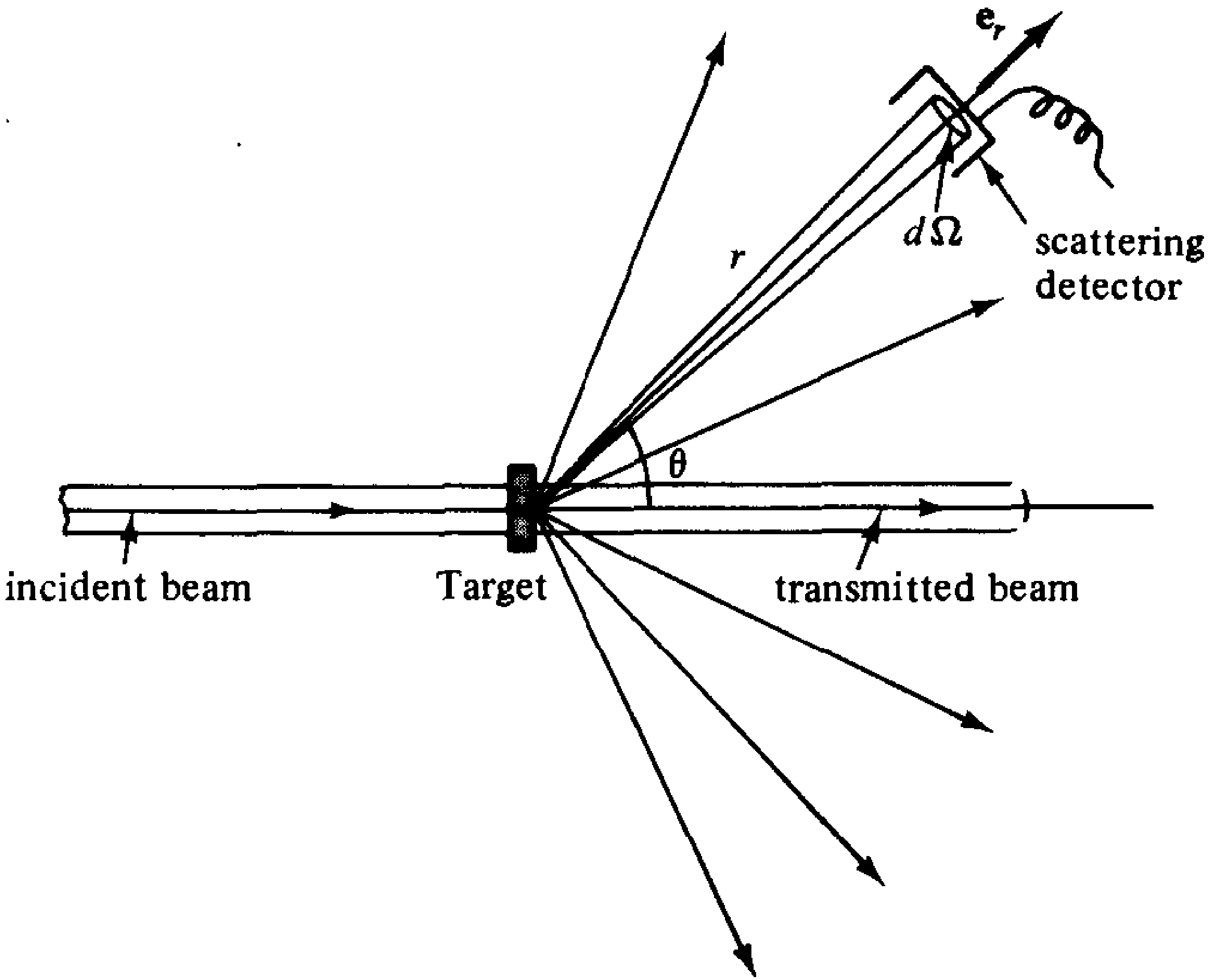
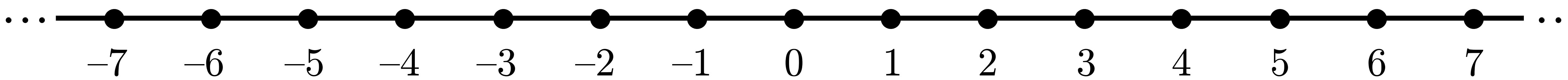


FIGURE 14.1 Scatterhıg configuration.

[Liboff, *Introductory Quantum Mechanics*]

Momentum states

Consider an infinite line:



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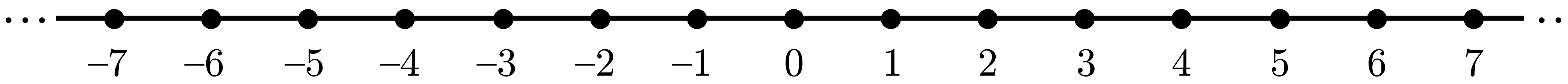

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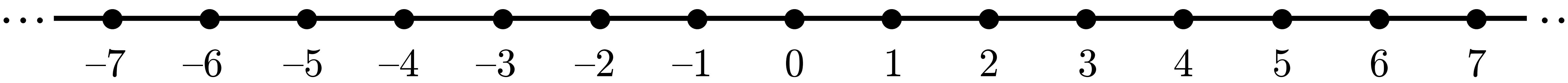

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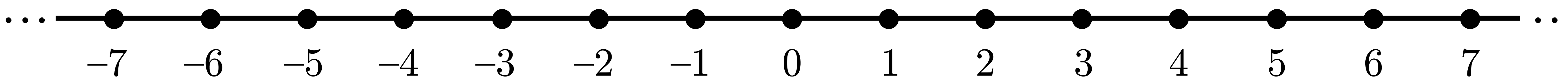

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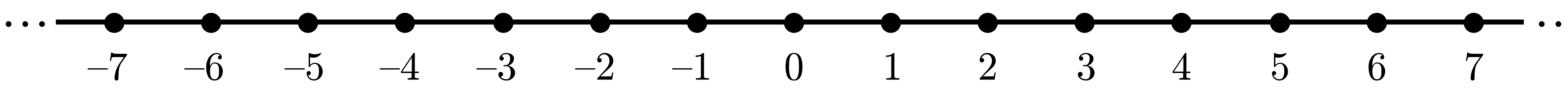

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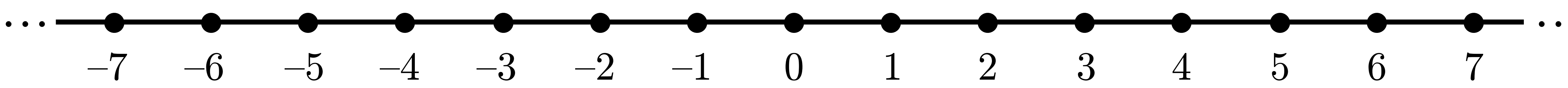

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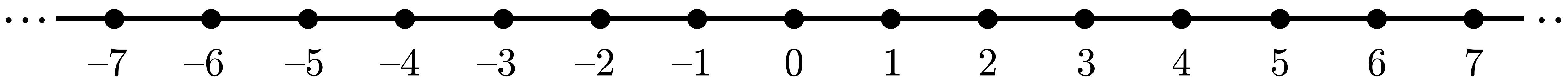

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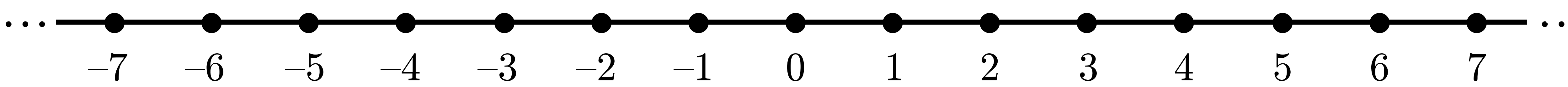

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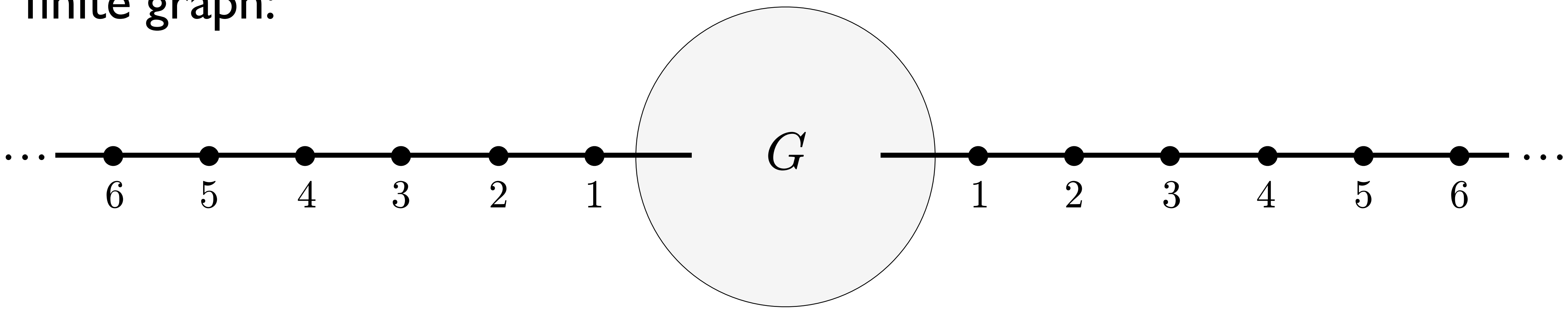
so this is an eigenstate with eigenvalue $2\cos k$.

Scattering on graphs

Now consider adding semi-infinite lines to two vertices of an arbitrary finite graph:



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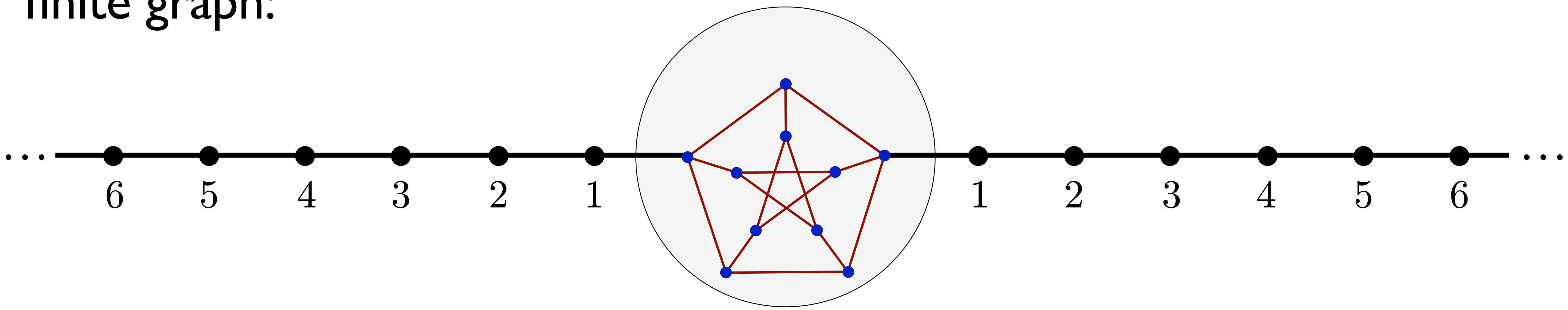

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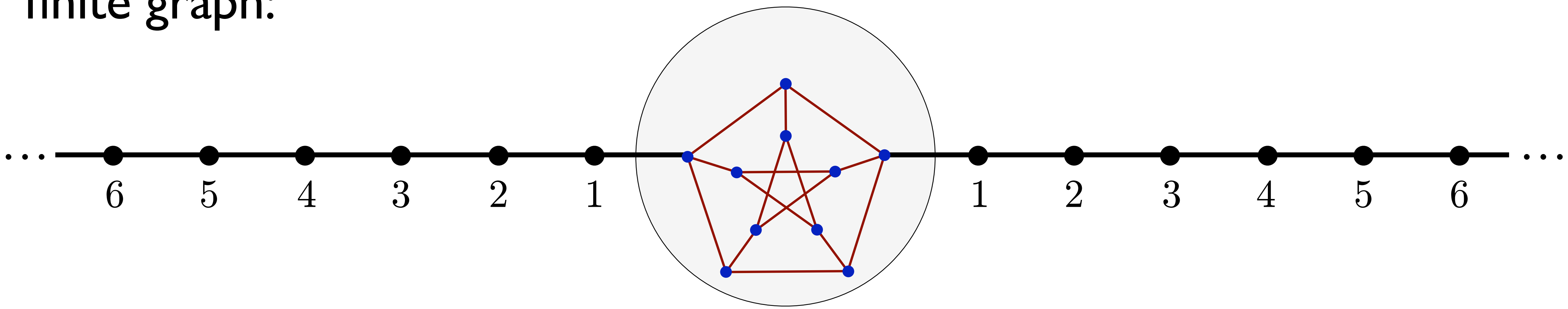

Three kinds of eigenstates:

$$\langle x, \text{left}|\tilde{k}, \text{sc}_{\text{left}}^{\rightarrow}\rangle = e^{-ikx} + R(k)e^{ikx} \qquad \langle x, \text{right}|\tilde{k}, \text{sc}_{\text{left}}^{\rightarrow}\rangle = T(k)e^{ikx}$$

$$\langle x, \text{left}|\tilde{k}, \text{sc}_{\text{right}}^{\rightarrow}\rangle = \bar{T}(k)e^{ikx} \qquad \langle x, \text{right}|\tilde{k}, \text{sc}_{\text{right}}^{\rightarrow}\rangle = e^{-ikx} + \bar{R}(k)e^{ikx}$$

$$\langle x, \text{left}|\tilde{\kappa}, \text{bd}^{\pm}\rangle = (\pm e^{-\kappa})^x \qquad \langle x, \text{right}|\tilde{\kappa}, \text{bd}^{\pm}\rangle = B^{\pm}(\kappa)(\pm e^{-\kappa})^x$$

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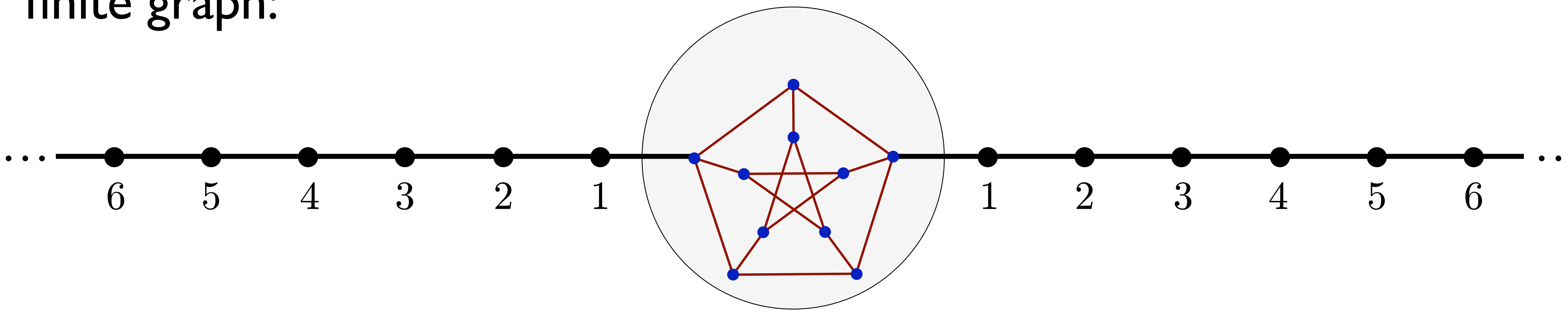

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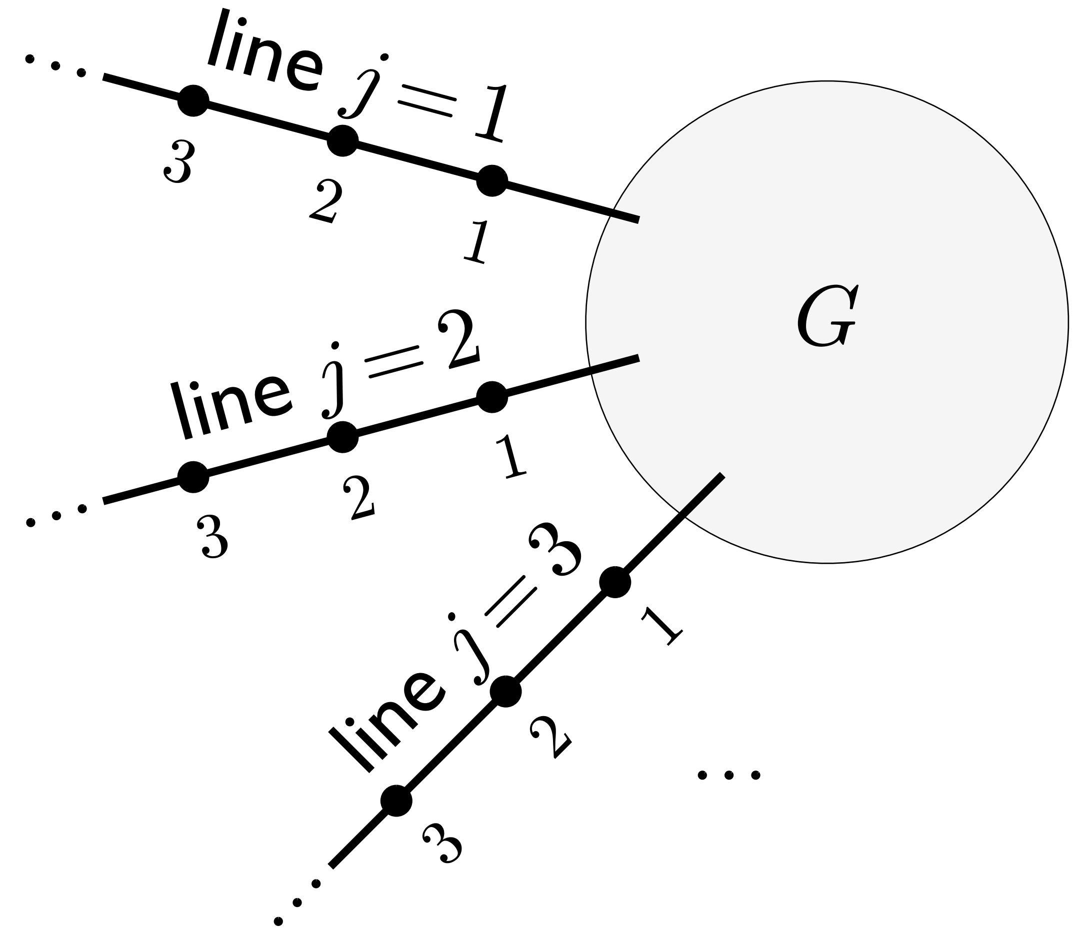
It can be shown that these states form a complete, orthonormal basis of the Hilbert space, where $k \in [-\pi, 0]$ and $\kappa > 0$ takes certain discrete values.

Scattering on graphs

This generalizes to any number of semi-infinite lines attached to any finite graph.



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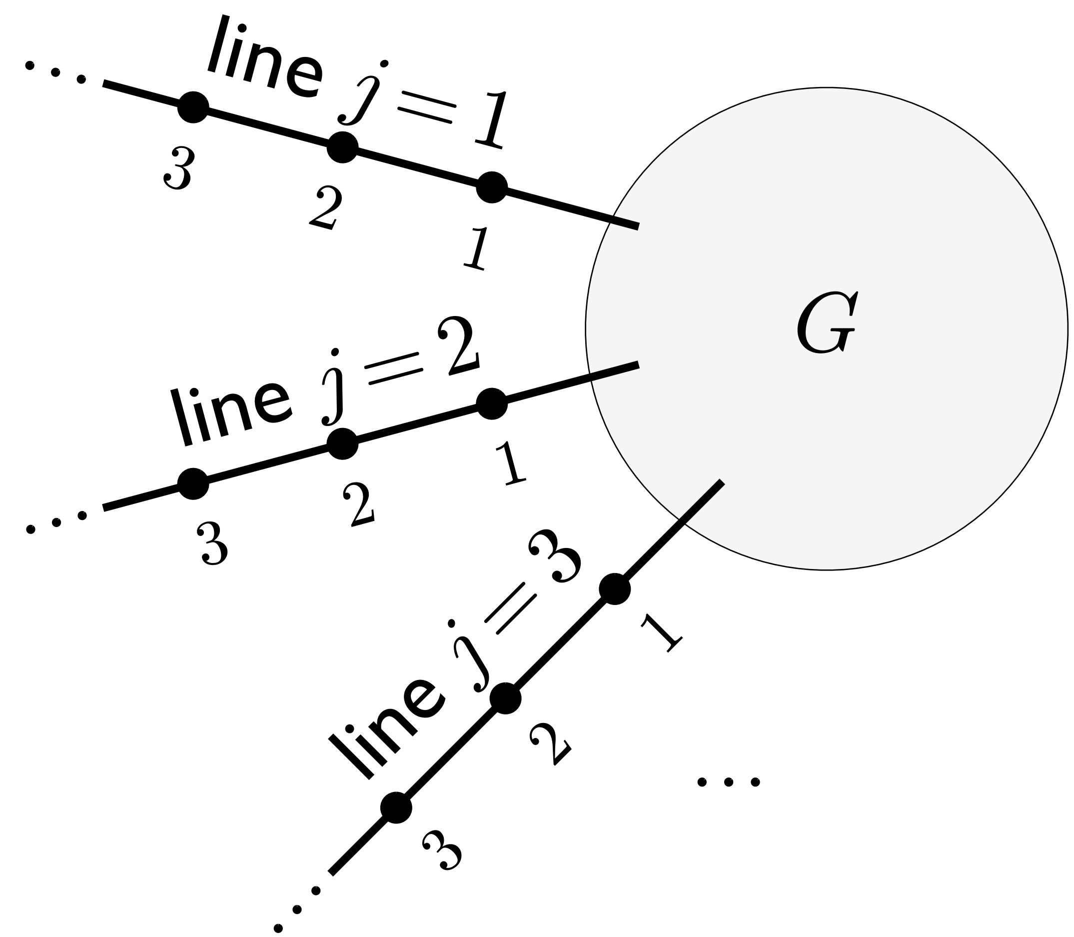

Incoming scattering states:

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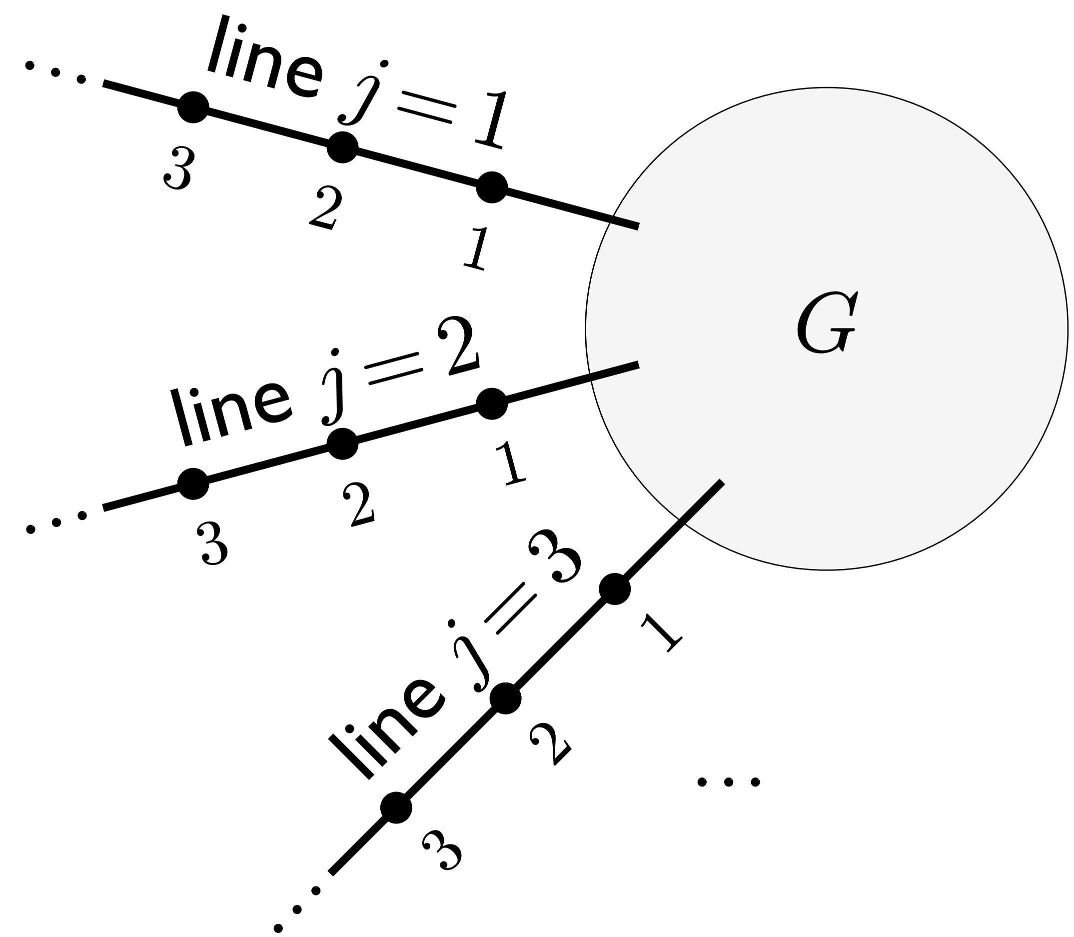

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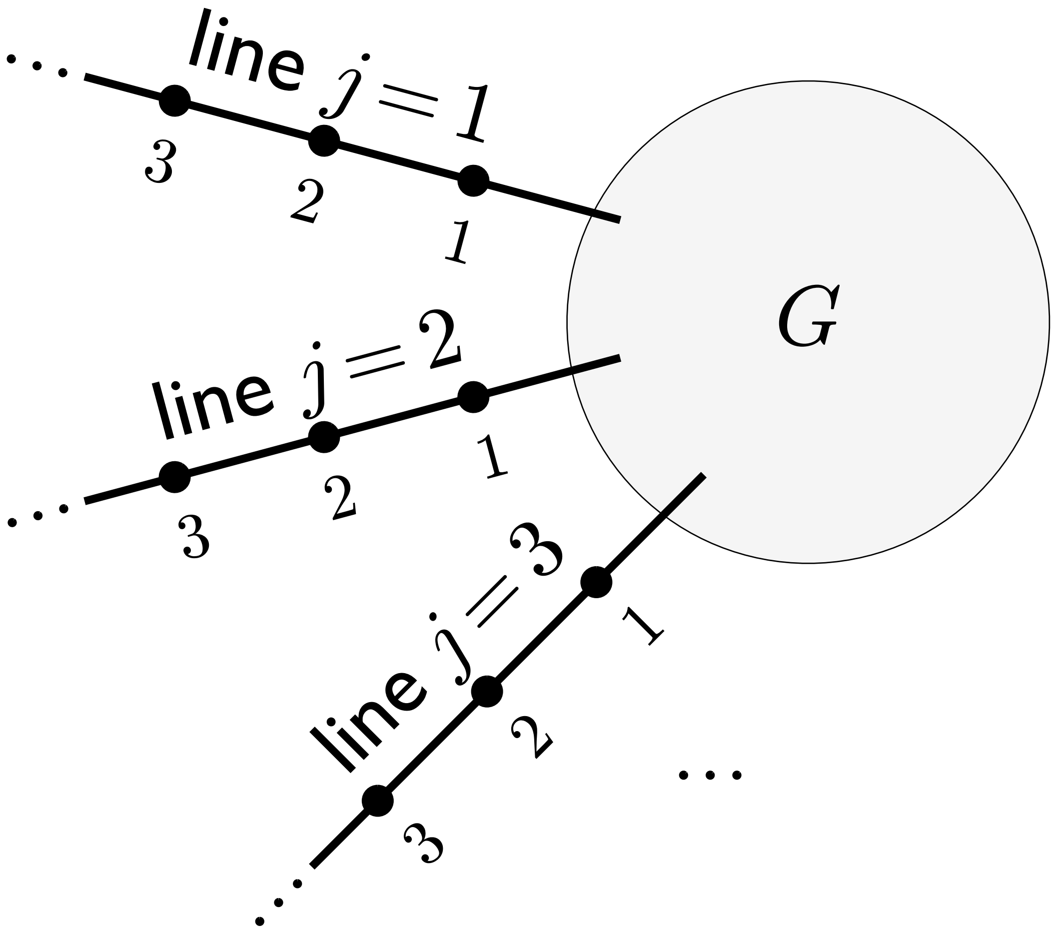
Bound states:

$$\langle x, j|\tilde{\kappa}, \mathrm{bd}^{\pm}\rangle = B_j^{\pm}(\kappa)\,(\pm e^{-\kappa})^x$$

The S-matrix

Scattering states characterize asymptotic transformations from incoming waves to outgoing waves:

$$S(k) = \begin{pmatrix} R_1(k) & T_{1,2}(k) & \cdots & T_{1,N}(k) \\ T_{2,1}(k) & R_2(k) & & T_{2,N}(k) \\ \vdots & & \ddots & \vdots \\ T_{N,1}(k) & T_{N,2}(k) & \cdots & R_N(k) \end{pmatrix}$$



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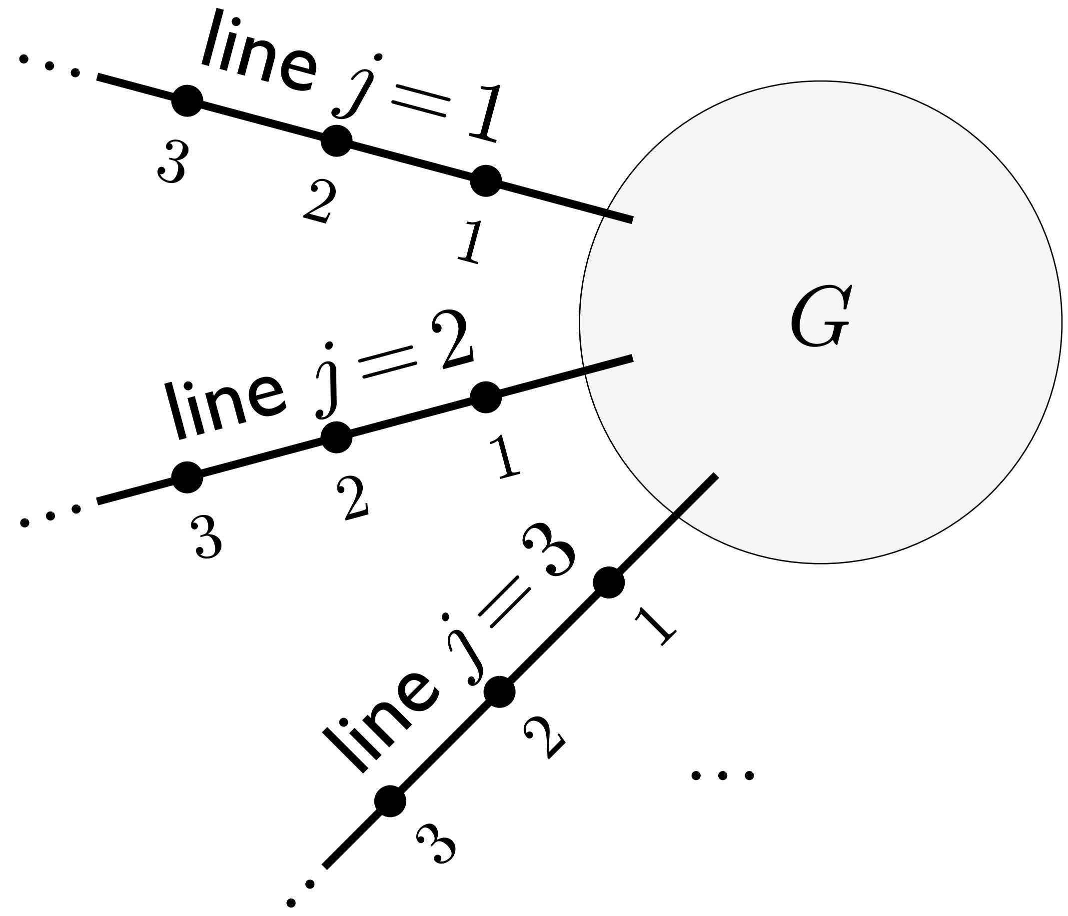

To understand the dynamics in general, expand the Hamiltonian in a basis of scattering states and compute integrals by the method of stationary phase.

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Solution of the quantum walk equation:

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The method of stationary phase

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$$\int e^{\mathrm{i}x\phi(k)} a(k) \mathrm{d}k$$

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The phase is stationary for k satisfying $x + y + \ell_{j,j'}(k) = v(k)t$

$$v(k) := \frac{\mathrm{d}}{\mathrm{d}k} 2\cos k = -2\sin k \qquad \text{group velocity}$$

$$\ell_{j,j'}(k) := \frac{\mathrm{d}}{\mathrm{d}k} \arg T_{j,j'}(k) \qquad \text{effective length}$$

Finite lines suffice

To obtain a finite graph, truncate the semi-infinite lines at a length $O(t)$, where t is the total evolution time.

This gives nearly the same behavior since the quantum walk on a line has a maximum propagation speed of 2.

Computation by scattering

Encode quantum circuits into graphs.

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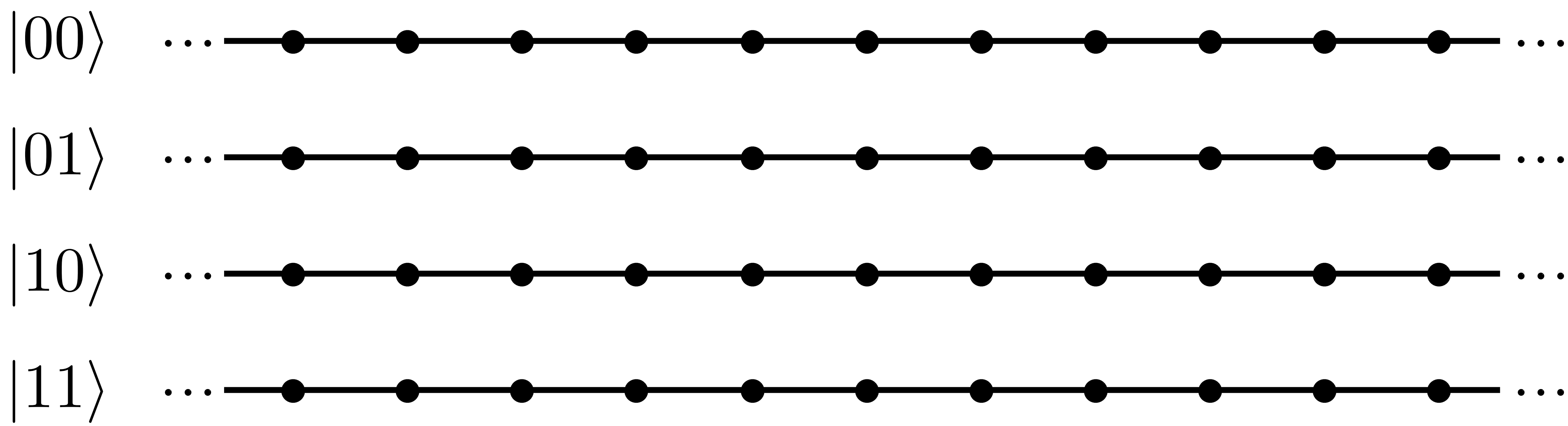

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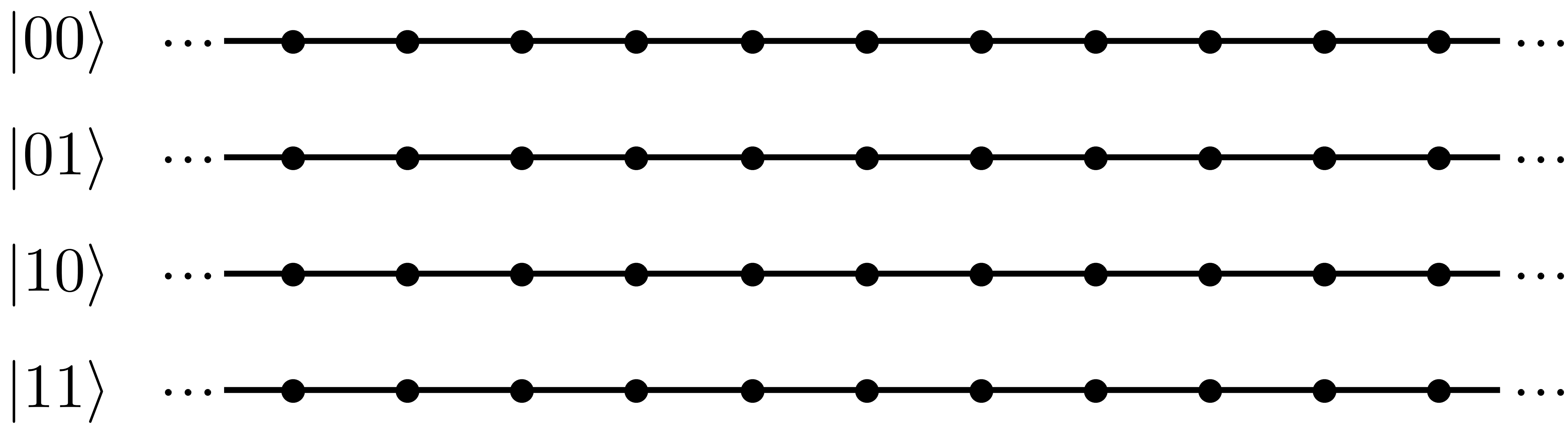

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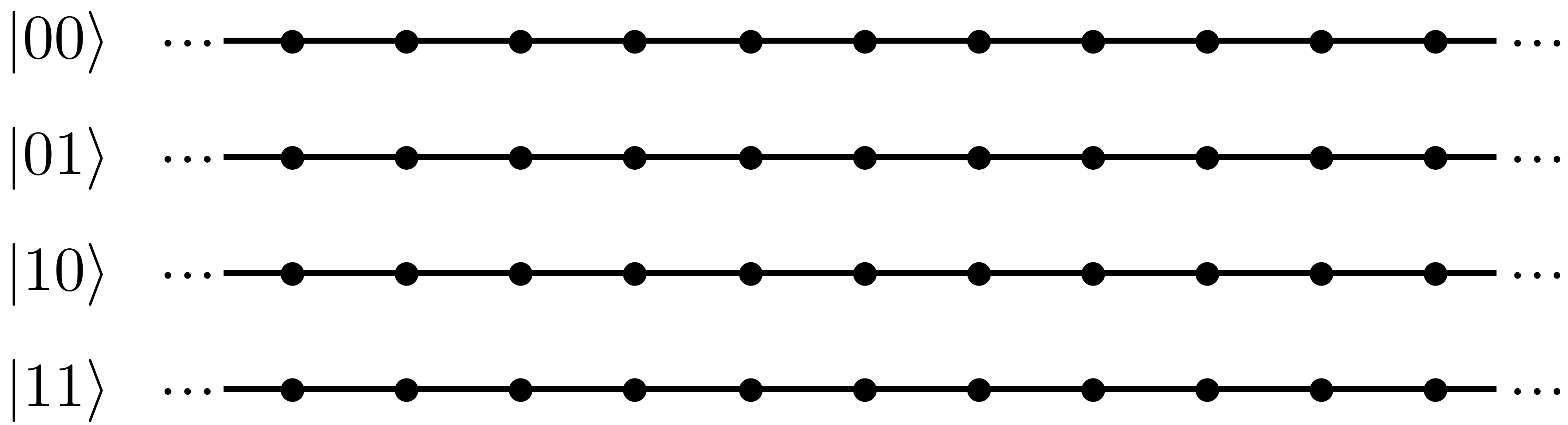

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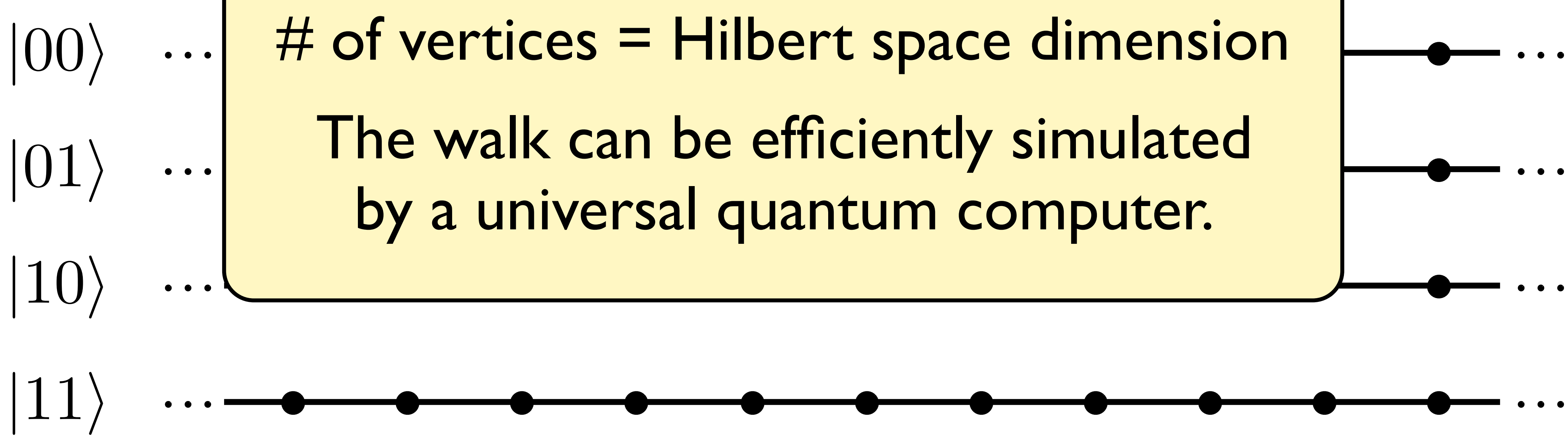

Note: This is not extravagant.

# of vertices = Hilbert space dimension

The walk can be efficiently simulated
by a universal quantum computer.

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A universal gate set

Theorem. Any unitary operation on n qubits can be approximated arbitrarily closely by a product of gates from the set

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\mathrm{i}} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

[Boykin et al. 00]

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We can implement these elementary gates (and indeed, any product of these gates) by scattering on graphs.

Controlled-not

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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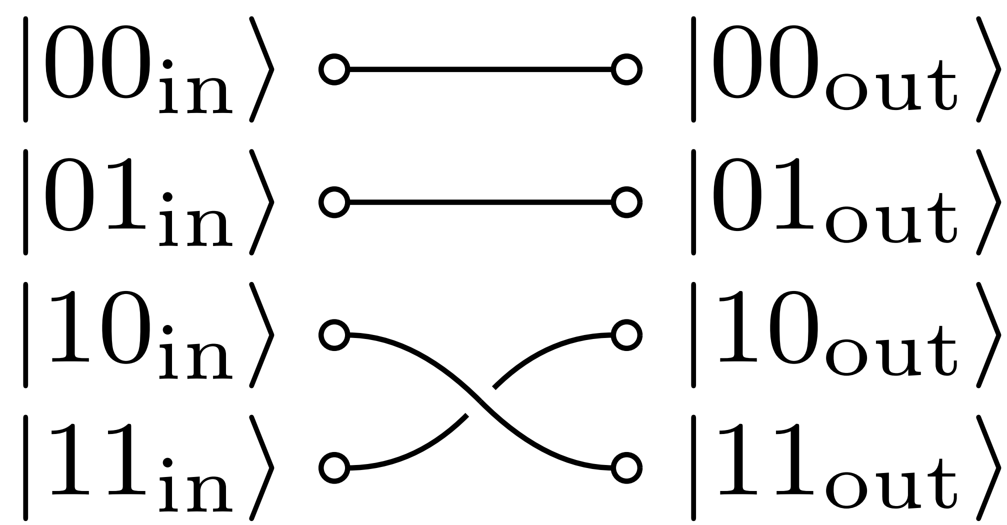

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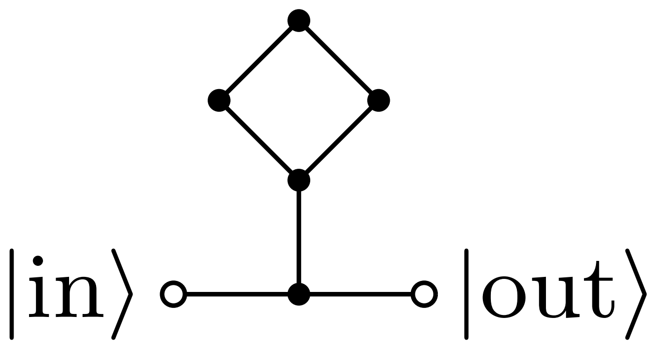
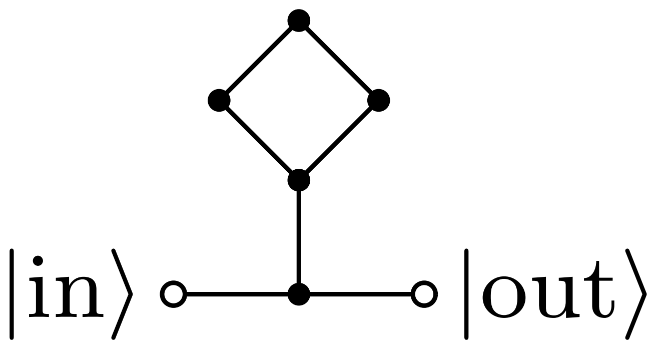
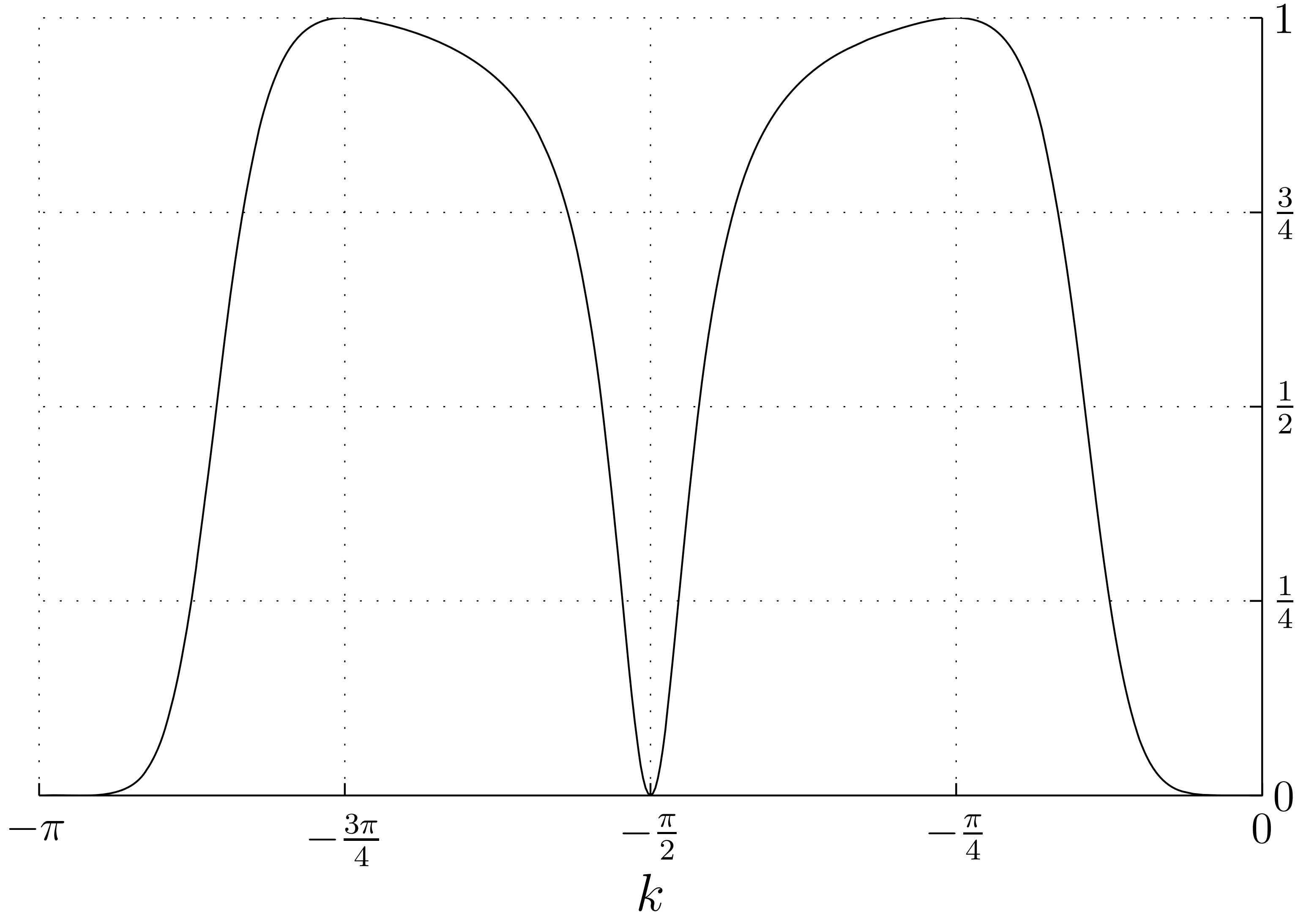


Phase

$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\mathrm{i}} \end{pmatrix}$$

$$T_{\mathrm{in,out}}(k) = \frac{8}{8 + \mathrm{i}\cos 2k \csc^3 k \sec k}$$





A basis-changing gate

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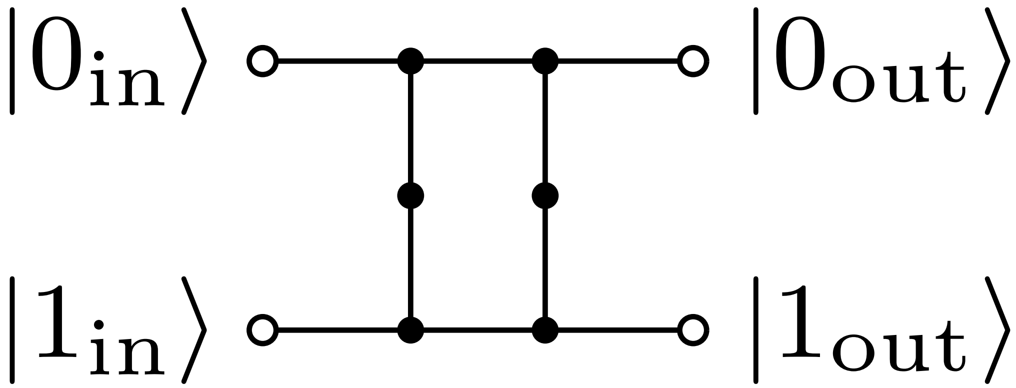

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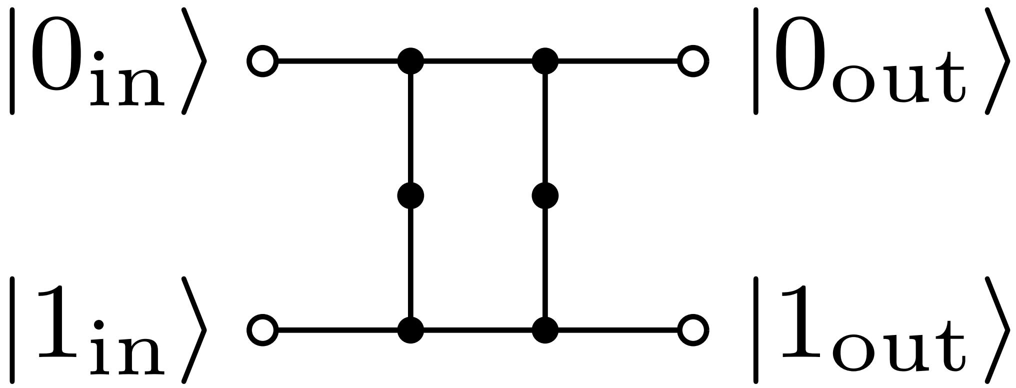

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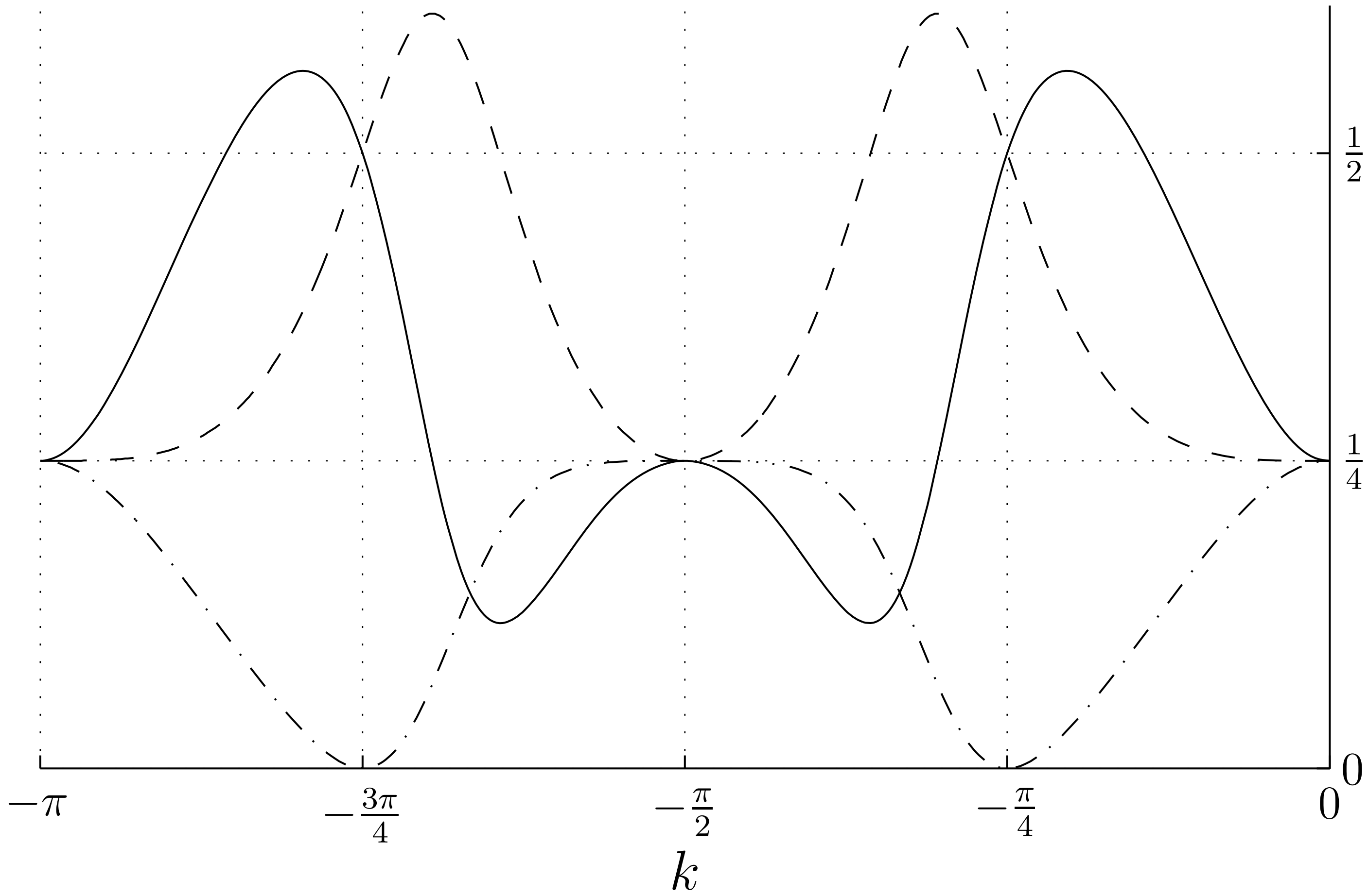
$$T_{0_{\text{in}},1_{\text{out}}}(k) = -\frac{1}{2\cos k + \mathrm{i}(\sin 3k - \sin k)}$$

$$R_{0_{\text{in}}}(k) = T_{0_{\text{in}},1_{\text{in}}}(k) = -\frac{e^{\mathrm{i}k}\cos 2k}{2\cos k + \mathrm{i}(\sin 3k - \sin k)}$$



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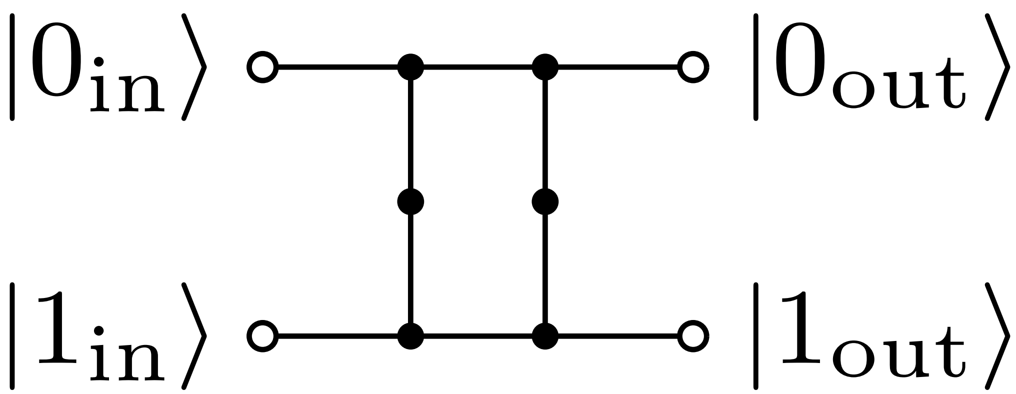

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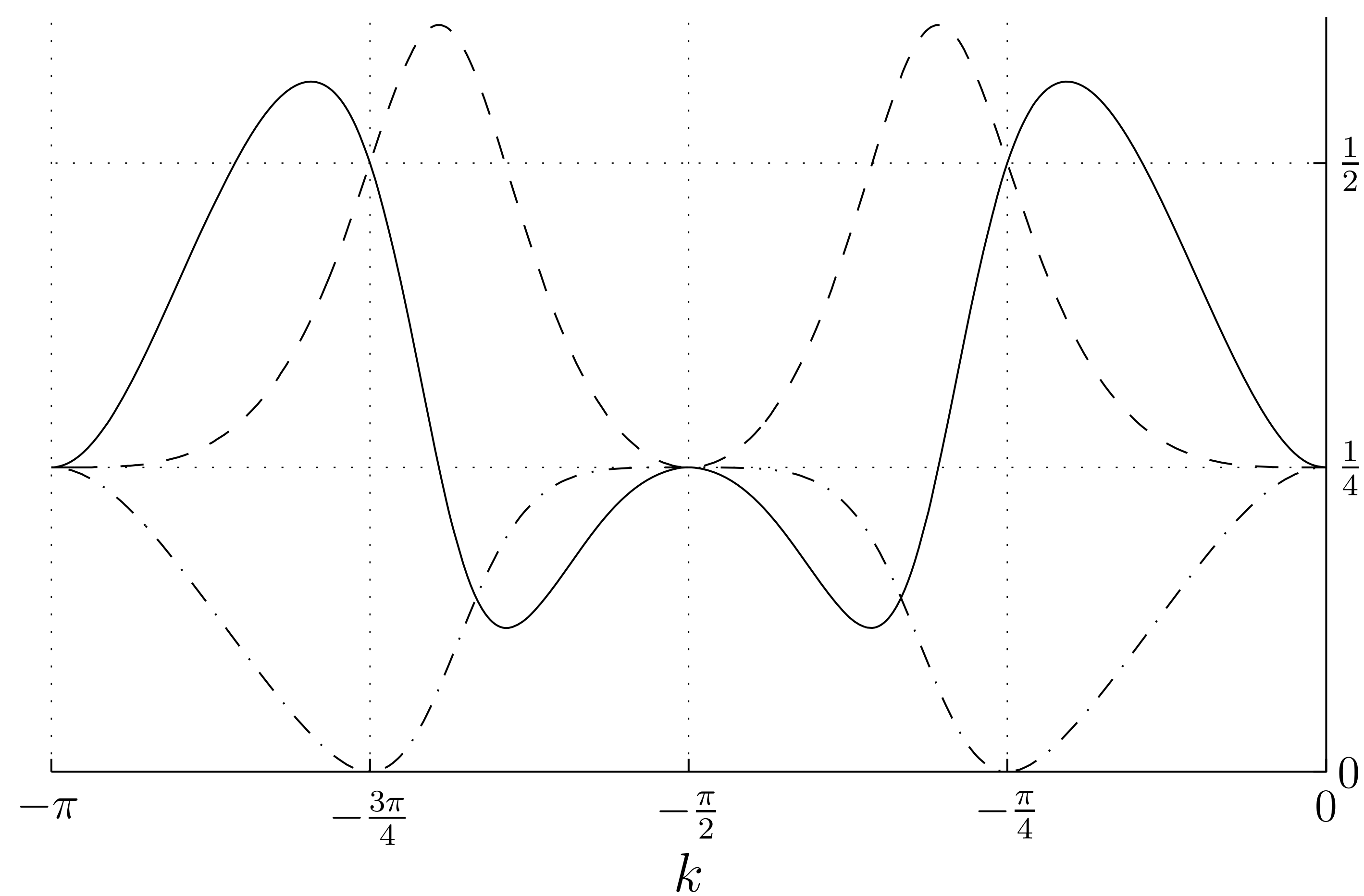
At $k = -\pi/4$ this implements the unitary transformation

$$U = -\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix}$$

from inputs to outputs



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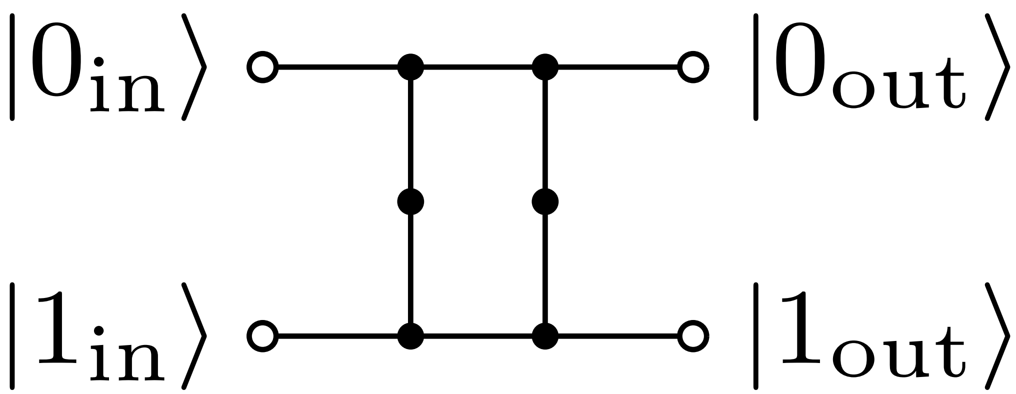

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$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} = e^{\mathrm{i}\phi} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

At $k = -$

impleme

unitary transformation

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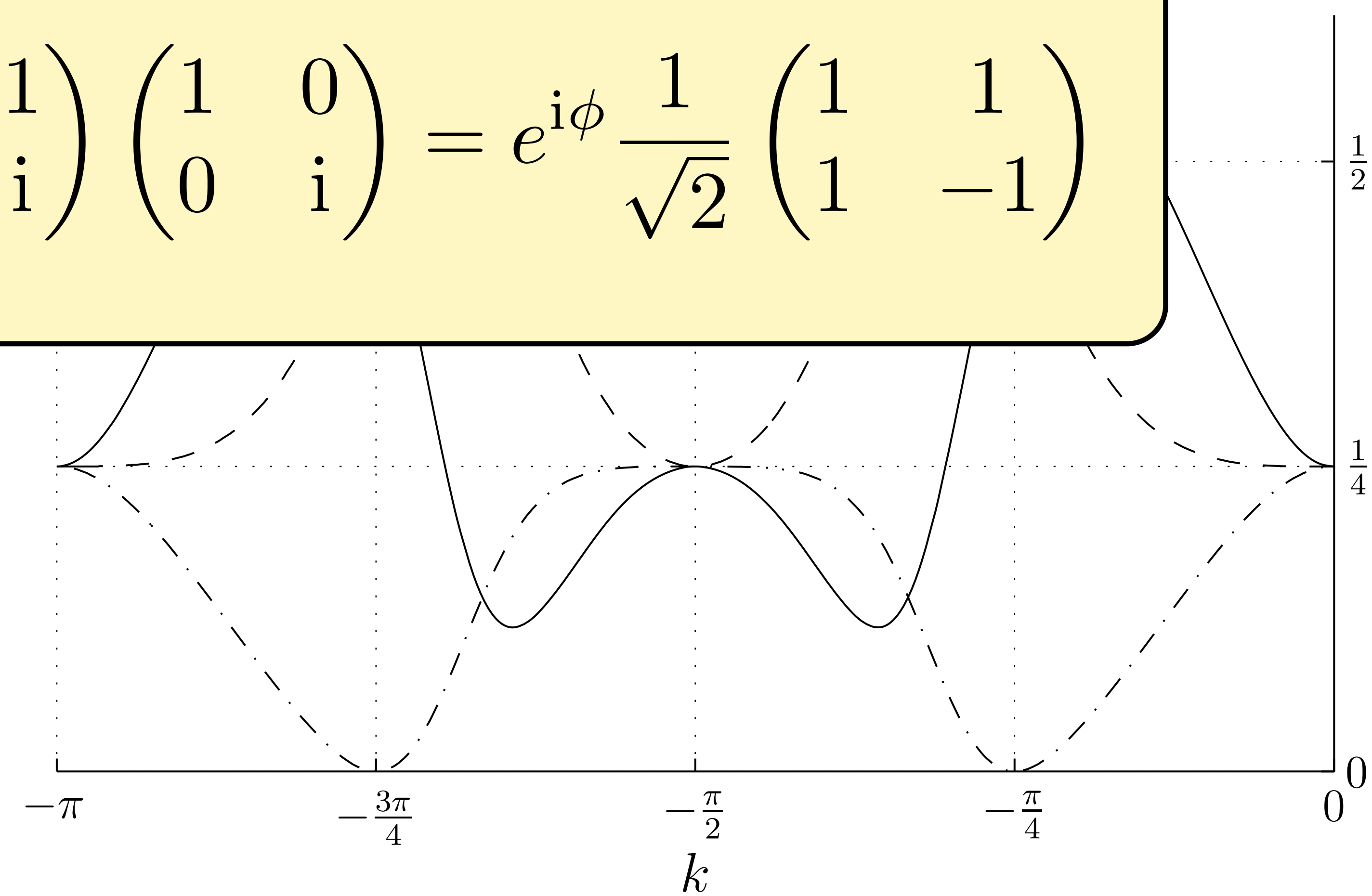

Tensor product structure

To embed an m-qubit gate in an n-qubit system, simply include the gate widget 2^{n-m} times, once for every possible computational basis state of the $n-m$ qubits not acted on by the gate.

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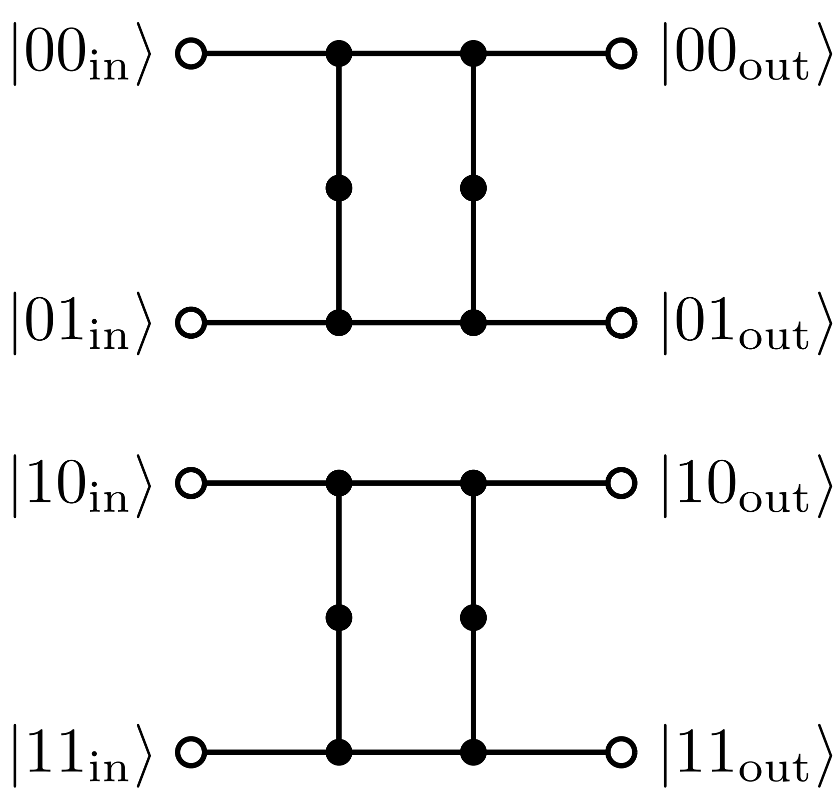
Ex:



Composition law

To perform a sequence of gates, simply connect the output wires to the next set of input wires.

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Arrange the transmission/reflection coefficients as transformations from inputs to outputs:

$$\mathcal{T}_{j,j'} = T_{j_{\text{in}},j'_{\text{out}}} \qquad \mathcal{R}_{j,j'} = \begin{cases} R_{j_{\text{in}}} & j = j' \\ T_{j_{\text{in}},j'_{\text{in}}} & j \neq j' \end{cases}$$

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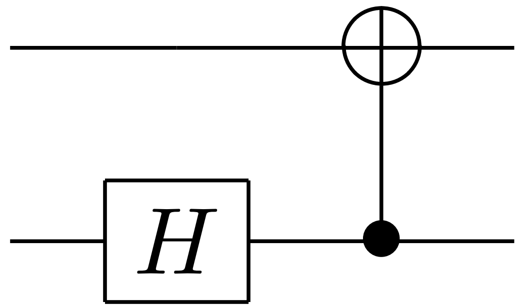
Then we have

$$\begin{aligned} \mathcal{T}_{12} &= \mathcal{T}_1 (1 - \mathcal{R}_2 \bar{\mathcal{R}}_1)^{-1} \mathcal{T}_2 \\ \mathcal{R}_{12} &= \mathcal{R}_1 + \mathcal{T}_1 (1 - \mathcal{R}_2 \bar{\mathcal{R}}_1)^{-1} \mathcal{R}_2 \bar{\mathcal{T}}_1 \\ \bar{\mathcal{T}}_{12} &= \bar{\mathcal{T}}_2 (1 - \bar{\mathcal{R}}_1 \mathcal{R}_2)^{-1} \bar{\mathcal{T}}_1 \\ \bar{\mathcal{R}}_{12} &= \bar{\mathcal{R}}_2 + \bar{\mathcal{T}}_2 (1 - \bar{\mathcal{R}}_1 \mathcal{R}_2)^{-1} \bar{\mathcal{R}}_1 \mathcal{T}_2 \end{aligned}$$

Example



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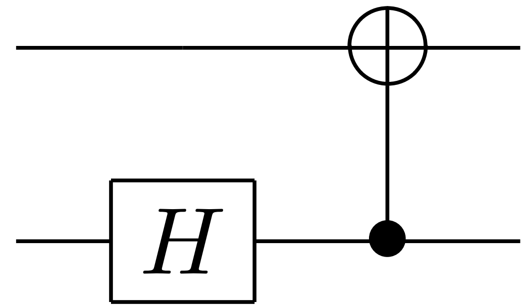
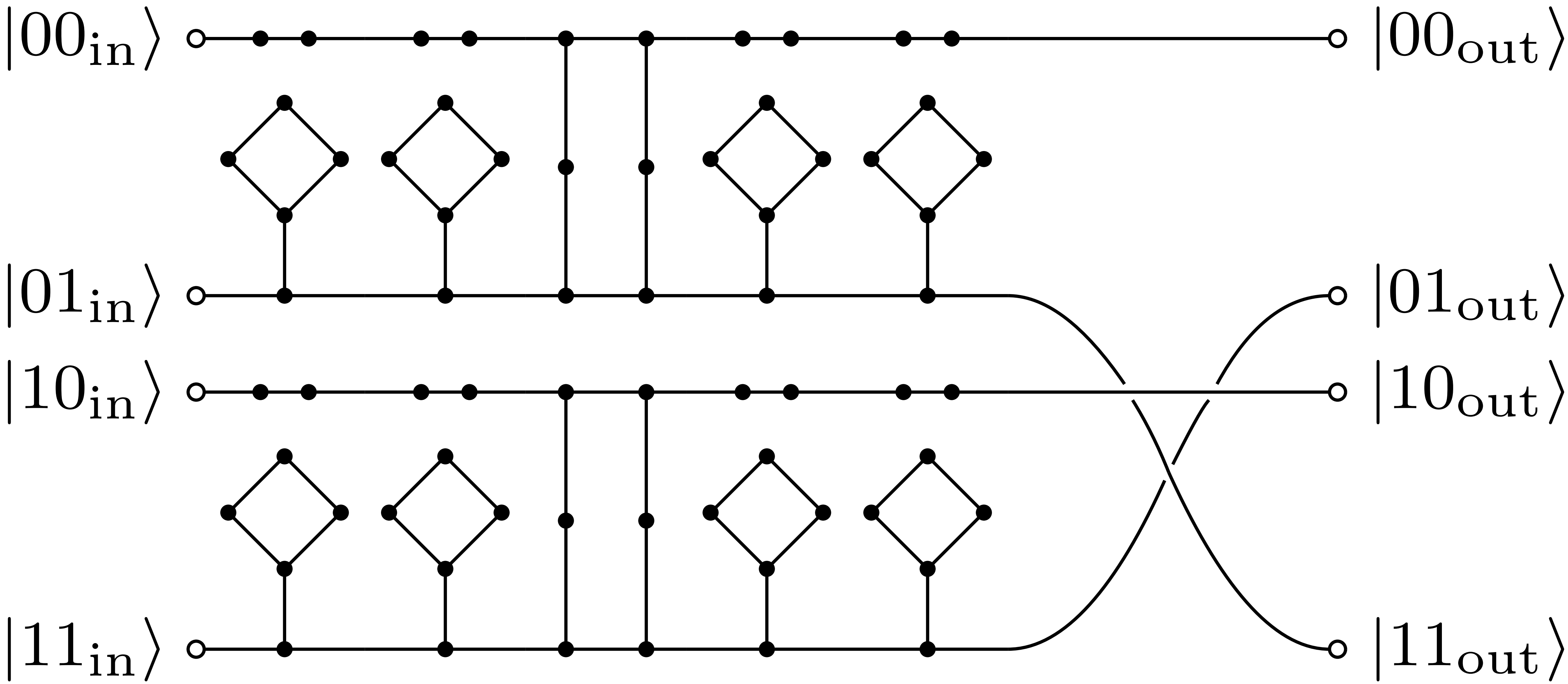
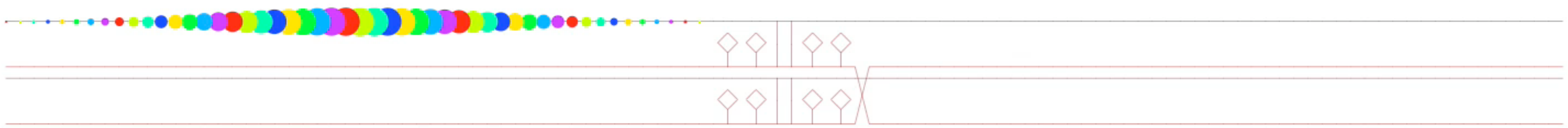

H


$|00_{\mathrm{in}}\rangle$
$|01_{\mathrm{in}}\rangle$
$|10_{\mathrm{in}}\rangle$
$|11_{\mathrm{in}}\rangle$
$|00_{\mathrm{out}}\rangle$
$|01_{\mathrm{out}}\rangle$
$|10_{\mathrm{out}}\rangle$
$|11_{\mathrm{out}}\rangle$

Example in action



Simplifying the initial state

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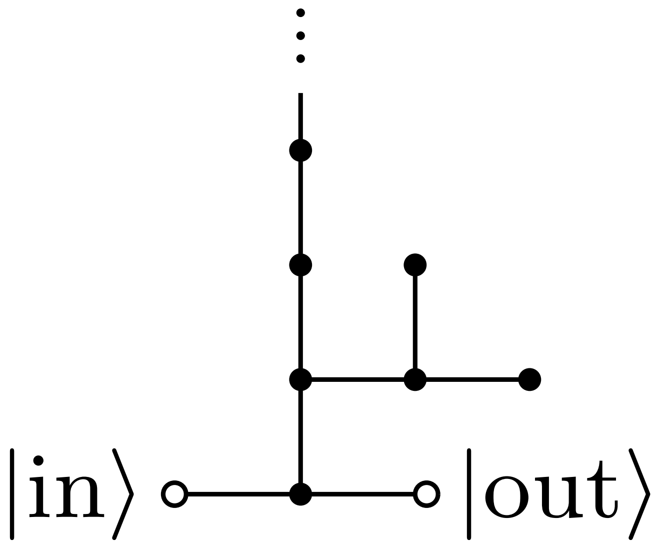
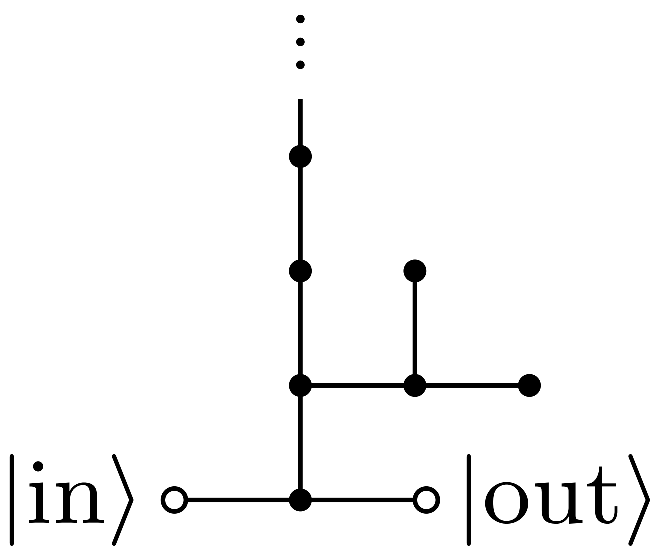
Can we relax this restriction? Start from a single vertex of the graph?

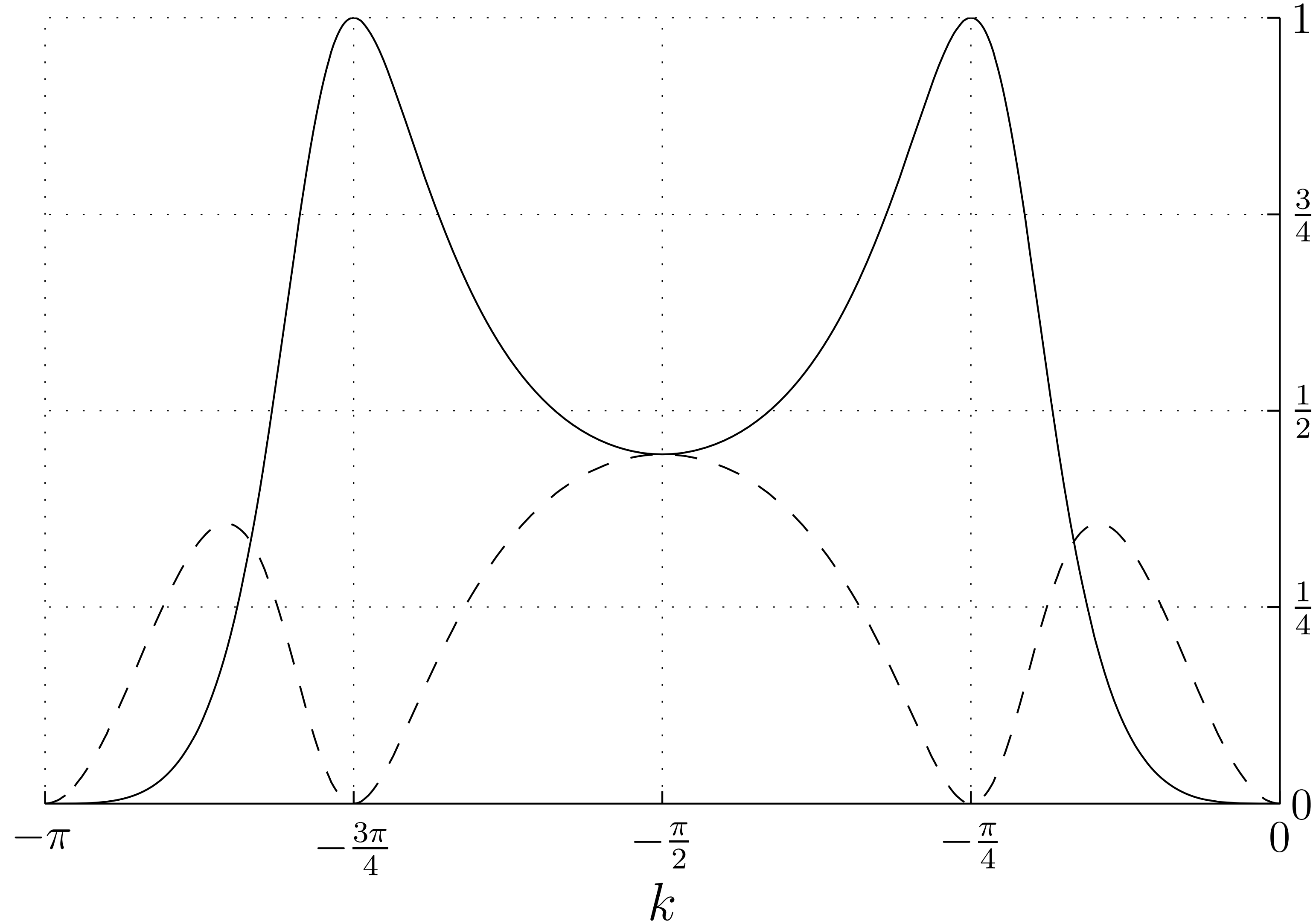
Idea: A single vertex has equal amplitudes for all momenta. Filter out momenta except within $1/\mathrm{poly}(n)$ of $k = -\pi/4$.

Momentum filter



Momentum filter





The curse of symmetry

Problem: Our filter passes $k = -3\pi/4$ in addition to $k = -\pi/4$.

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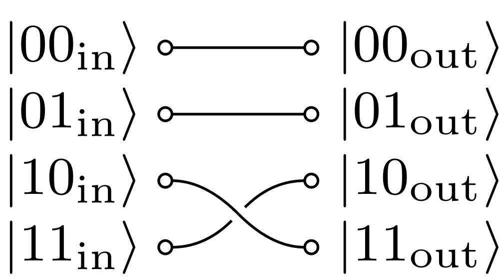
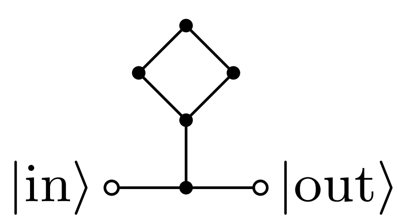
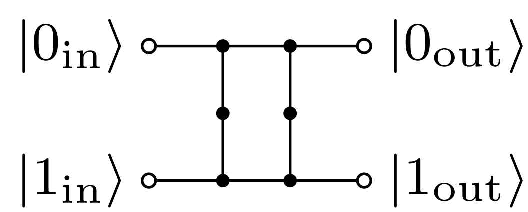
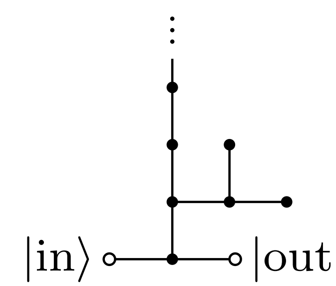
In fact, all widgets so far have a symmetry under $k \to -\pi - k$.









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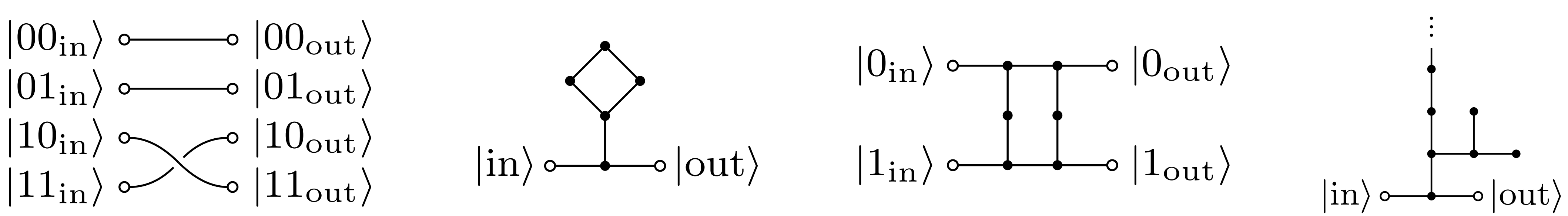
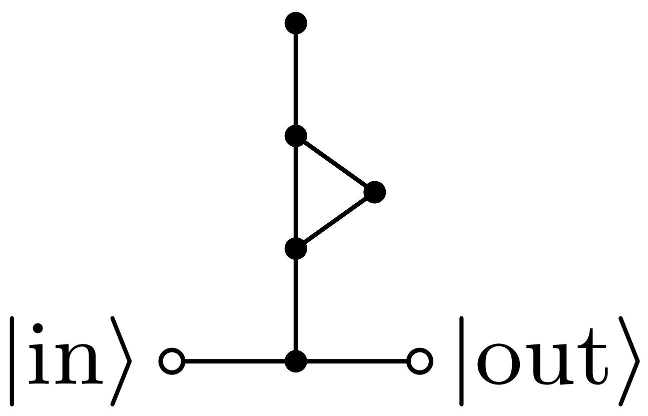


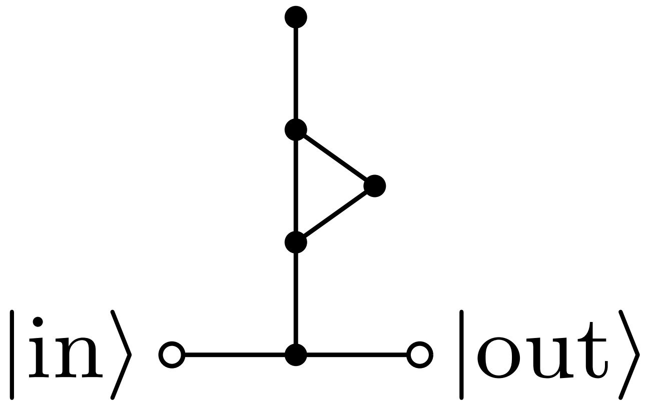
This is because they are all bipartite. [Goldstone]

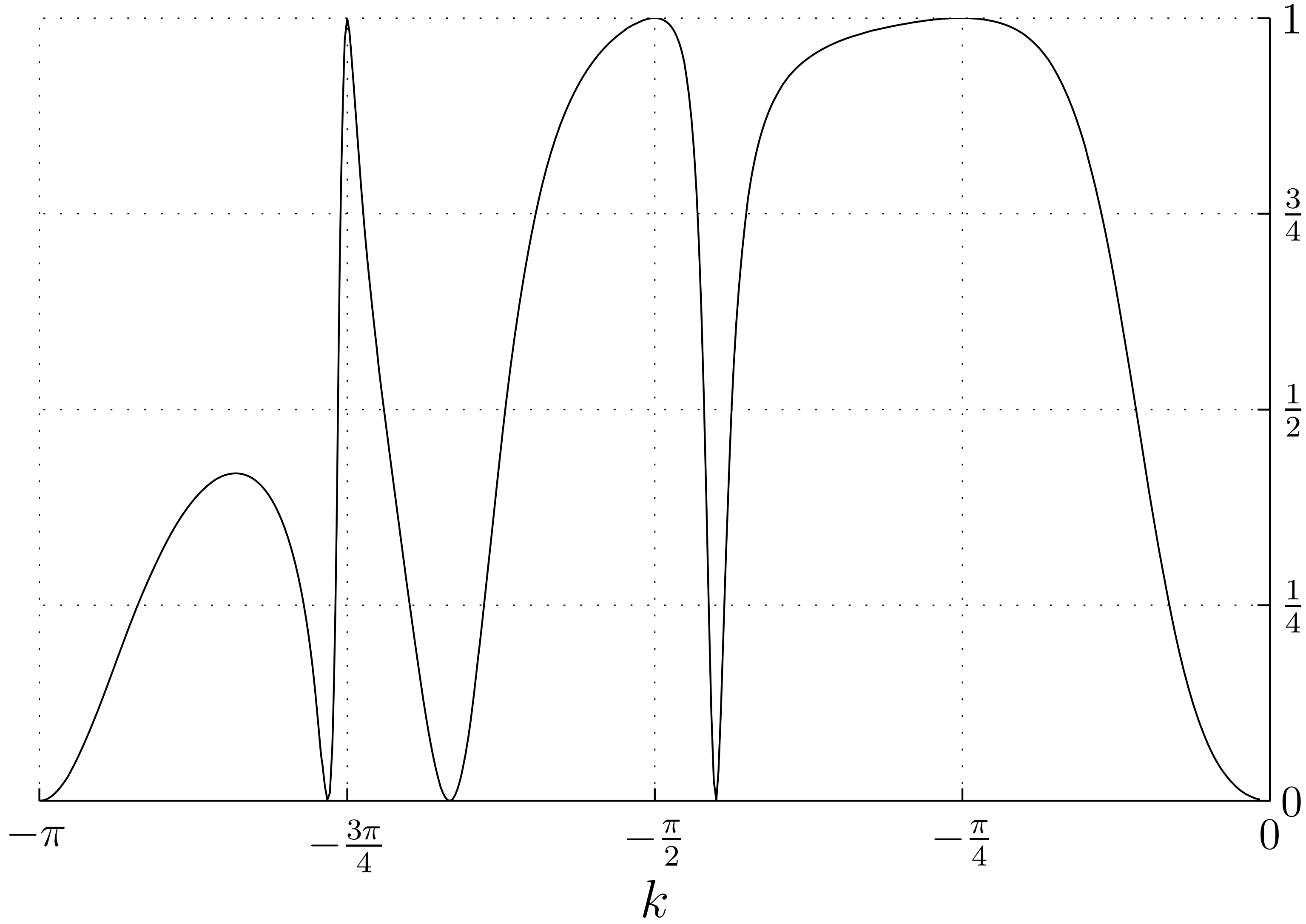
Momentum separator



Momentum separator

$$T_{\mathrm{in,out}}(k) = \left[1 + \frac{\mathrm{i}(\cos k + \cos 3k)}{\sin k + 2\sin 2k + \sin 3k - \sin 5k}\right]^{-1}$$





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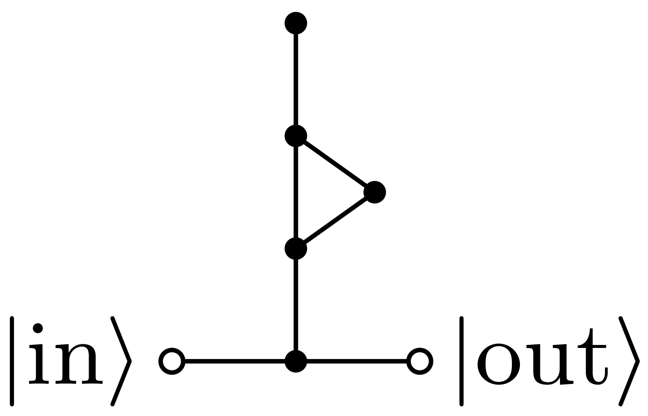
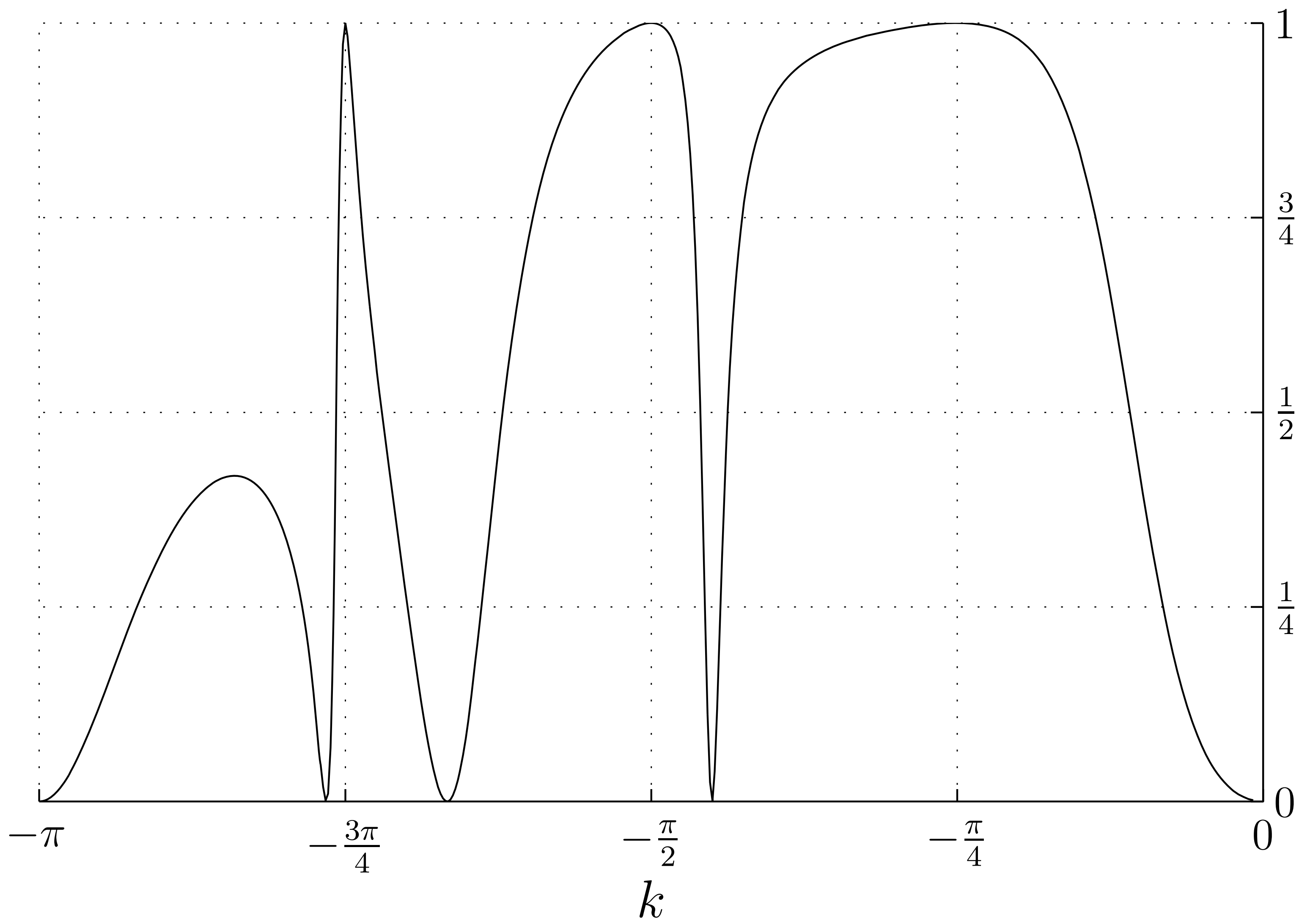
$$\ell_{\mathrm{in,out}}(-\pi/4) = 4(3 - 2\sqrt{2}) \approx 0.686$$

$$\ell_{\mathrm{in,out}}(-3\pi/4) = 4(3 + 2\sqrt{2}) \approx 23.3$$





A universal computer

Consider an m-gate quantum circuit (unitary transformation U).

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Graph:

- $\log \Theta(m^2)$ filter widgets on input line $00\ldots0$
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- Widgets for m gates in the circuit
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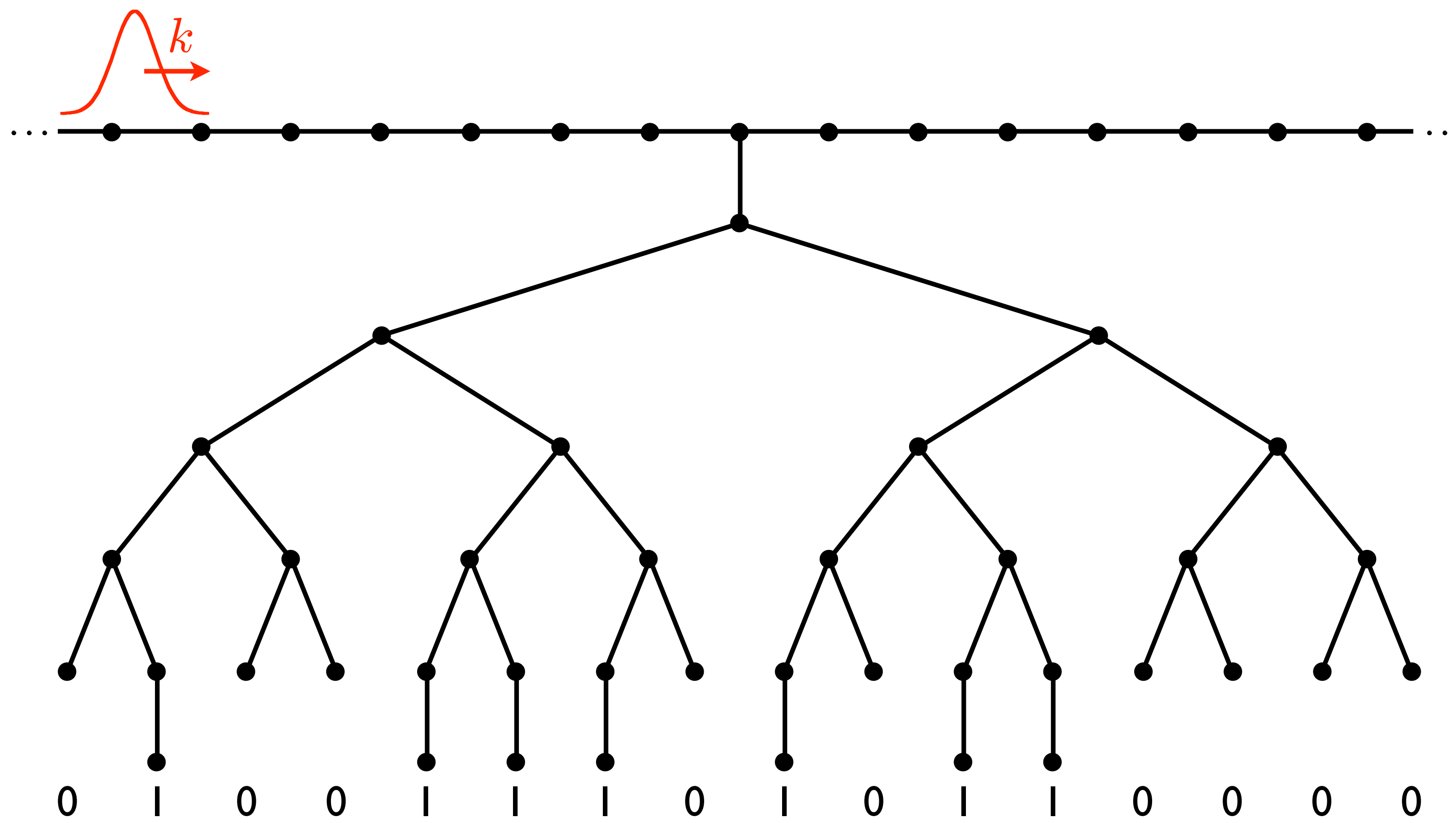
Simulation:

- Start at vertex $x = \Theta(m^2)$ on input line $00\ldots0$
- Evolve for time $t = \pi\lfloor(x+\ell)/\sqrt{2}\pi\rfloor = O(m^2)$
- Measure in the vertex basis
- Conditioned on reaching vertex 0 on some output line s (which happens with probability $\Omega(1/m^2)$), the distribution over s is approximately $|\langle s|U|00\ldots0\rangle|^2$

Toward scattering algorithms

Query algorithm for a decision problem: [Farhi, Goldstone, Gutmann 07]

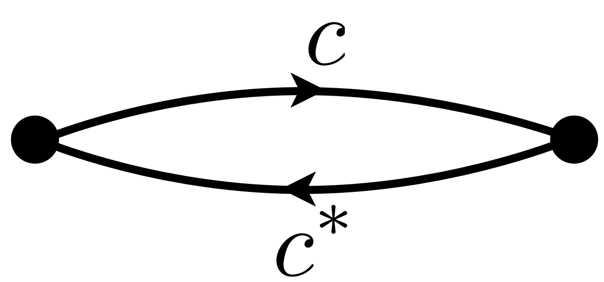


- Can we solve other problems by scattering?
- Can we implement quantum transforms (e.g., the Fourier transform) more directly than by a circuit decomposition?

joint work with Gorjan Alagic, Aaron Denney, and Cris Moore

Relaxing the model

- Arbitrary edge weights (in complex conjugate pairs)



$$\begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$$

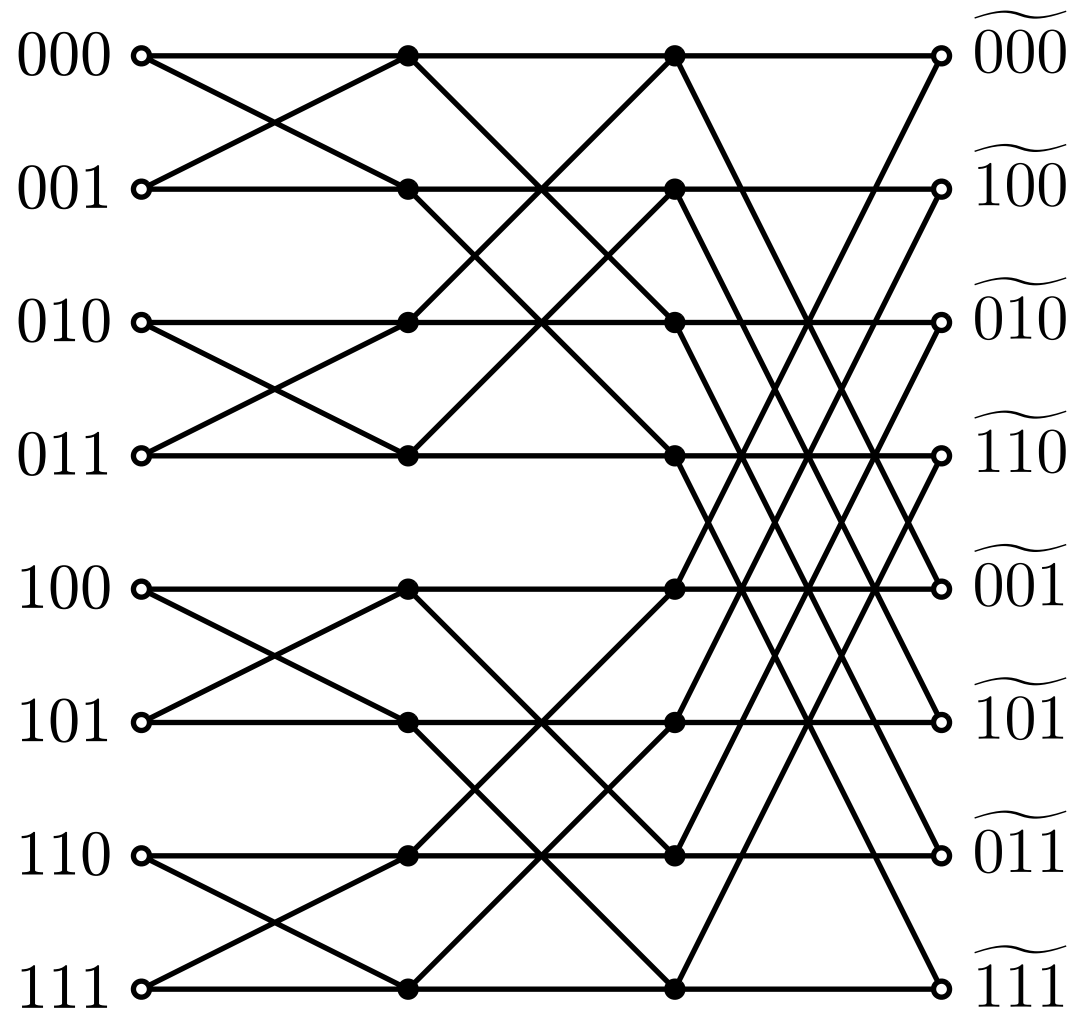
- Let input/output states be wave packets (encoding/decoding can be performed efficiently)

- Output wires can be separate from, or identical to, input wires

joint work with Gorjan Alagic, Aaron Denney, and Cris Moore

QFT over $\mathbb{Z}_{2^n}$

"Butterfly network":



With appropriate choice of weights, $S(k_0) = \mathrm{QFT}(\mathbb{Z}_{2^n})$.

Can we get further away from the circuit model?

joint work with Gorjan Alagic, Aaron Denney, and Cris Moore

Supplementary material

Random walk

A Markov process on a graph $G = (V, E)$.

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In discrete time:

Stochastic matrix $W \in \mathbb{R}^{|V| \times |V|}$ $(\sum_k W_{kj} = 1)$

with $W_{kj} \neq 0$ iff $(j, k) \in E$

probability of taking a step from j to k

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Ex: Simple random walk. $W_{kj} = \begin{cases} \frac{1}{\deg j} & (j,k) \in E \\ 0 & (j,k) \notin E \end{cases}$

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In continuous time:

Generator matrix $M \in \mathbb{R}^{|V| \times |V|}$ ($\sum_k M_{kj} = 0$)

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Ex: Laplacian walk. $M_{kj} = L_{kj} = \begin{cases} -\deg j & j = k \\ 1 & (j,k) \in E \\ 0 & (j,k) \notin E \end{cases}$

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Ex: Adjacency matrix. $H_{kj} = A_{kj} = \begin{cases} 1 & (j,k) \in E \\ 0 & (j,k) \notin E \end{cases}$

Aside: Discrete-time quantum walk

We can also define a quantum walk that proceeds by discrete steps.

[Watrous 99]

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We must enlarge the state space: $\mathbb{C}^{|V|} \otimes \mathbb{C}^{|V|}$ instead of $\mathbb{C}^{|V|}$.

Unitary operator U with $U_{(k,j),(j,\ell)} \neq 0$ iff $(j,k) \in E$

Aside: Discrete-time quantum walk

We can also define a quantum walk that proceeds by discrete steps.

[Watrous 99]

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In this talk we will focus on the continuous-time model.

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Write $\begin{pmatrix} \langle x+1|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \\ \langle x|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \end{pmatrix} = M \begin{pmatrix} \langle x|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \\ \langle x-1|\tilde{k}, \mathrm{sc}_{\mathrm{in}}^{\rightarrow}\rangle \end{pmatrix}$

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$$M^m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$T = \frac{2\mathrm{i}e^{-\mathrm{i}km}\sin k}{-ae^{-\mathrm{i}k} - b + c + de^{\mathrm{i}k}}$$

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Eigenvalues of M