Universal computation
by multi-particle quantum walk

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Quantum walk

Quantum analog of a random walk on a graph.

Idea: Replace probabilities by quantum amplitudes. Interference can produce radically different behavior!

classical

quantum
Quantum walk algorithms

Quantum walk is a major tool for quantum algorithms (especially query algorithms with polynomial speedup).

- Exponential speedup for black-box graph traversal [CCDFGS 02]
- Quantum walk search framework [Szegedy 05], [Magniez et al. 06]
  - Spatial search [Shenvi-Kempe-Whaley 02], [CG 03, 04], [Ambainis-Kempe-Rivosh 04]
  - Element distinctness [Ambainis 03]
- Subgraph finding [Magniez, Santha, Szegedy 03], [CK 10]
- Matrix/group problems [Buhrman, Špalek 04], [Magniez, Nayak 05]
- Evaluating formulas/span programs
  - AND-OR formula evaluation [Farhi, Goldstone, Gutmann 07], [ACRŠZ 07]
  - Span programs for general query problems [Reichardt 09]
  - Learning graphs [Belovs 11] → new upper bounds (implicitly, quantum walk algorithms), new kinds of quantum walk search
Universality of quantum walk

Quantum walk can be efficiently simulated by a universal quantum computer.

\[ N \text{-vertex graph} \quad \Rightarrow \quad \text{circuit with} \quad \text{poly}(\log N) \text{ qubits} \]
\[ \text{max degree poly}(\log N) \quad \Rightarrow \quad \text{poly}(\log N) \text{ gates} \]
\[ \text{efficiently computable neighbors} \quad \Rightarrow \quad \text{poly}(\log N) \text{ gates} \]

Conversely, quantum walk is a universal computational primitive: any quantum circuit can be simulated by a quantum walk. [C 09]

\[ N \text{ dimensions} \quad \Rightarrow \quad \text{graph with} \quad \text{poly}(N, g) \text{ vertices} \]
\[ g \text{ gates} \quad \Rightarrow \quad \text{max degree 3} \]
\[ \text{walk for time poly}(g) \]

Note: The graph is necessarily exponentially large in the number of qubits! Vertices represent basis states.
Quantum walk experiments

Experimental realization of a quantum quincunx by use of linear optical elements

Michael Karski, Sourish Tewari, Jin-Won Choi, Arash Fallahzadeh, William A. Itano, Dawn Braakman, Andrew Scherer, and Ion� Kominek

The authors report the experimental realization of a quantum quincunx, a quantum analog of the classical Galton quincunx. They use linear optical elements to realize the quantum walk, demonstrating the quantum nature of the system.

Quantum Walk in Position Space with Single Optically Trapped Atoms


The authors demonstrate a quantum walk in position space using a single optically trapped atom. This experiment provides insights into quantum mechanics and opens up possibilities for quantum information processing.

Realization of Quantum Walks with Negligible Decoherence in Waveguide Lattices

Hage D. P. Santos, Y. Lemaitre, F. Fingerold, M. S. Z. Hong, R. F. M. Mendez, and Y. S. Ilie

The authors report the experimental realization of a quantum walk in a waveguide lattice with negligible decoherence, demonstrating the potential for quantum information processing applications.

In this section, we discuss the realization of a quantum quincunx and its implications for quantum information processing. The quantum quincunx is a quantum analog of the classical Galton quincunx, which is a statistical concept used to model random processes.

The quantum quincunx is realized using linear optical elements, demonstrating the quantum nature of the system. This experiment provides insights into quantum mechanics and opens up possibilities for quantum information processing.

The quantum walk is the quantum analog of the well-known random walk, which forms the basis for many applications, such as quantum cellular automata. Quantum walks have been suggested for the implementation of quantum algorithms, providing a new approach to solving problems that are intractable for classical computers.

In conclusion, the experimental realization of a quantum quincunx and the demonstration of quantum walks in position space with single optically trapped atoms pave the way for future advancements in quantum information processing. These experiments highlight the potential of quantum technologies for solving complex problems and advancing our understanding of fundamental physics.
Multi-particle quantum walk

With many walkers, the Hilbert space can be much bigger.  
m distinguishable particles on an \( n \)-vertex graph: \( n^m \) dimensions  
(similar scaling for indistinguishable bosons/fermions)

Main result: Any \( n \)-qubit, \( g \)-gate quantum circuit can be simulated by a 
multi-particle quantum walk of \( n + 1 \) particles interacting for time  
\( \text{poly}(n, g) \) on a graph with \( \text{poly}(n, g) \) vertices.

Consequences:

• Architecture for a quantum computer with no time-dependent control

• Simulating interacting many-body systems is BQP-hard (e.g., Bose-Hubbard model on a sparse, unweighted, planar graph)
Outline

• Scattering theory on graphs
• Single-particle universality (review)
• Multi-particle universality
• Proof ideas
• Refinements and extensions
• Open questions
Scattering theory on graphs

[Liboff, Introductory Quantum Mechanics]
Quantum walk

Quantum analog of a random walk on a graph $G = (V, E)$.

Idea: Replace probabilities by quantum amplitudes.

$$|\psi(t)\rangle = \sum_{v \in V} a_v(t) |v\rangle$$

amplitude for vertex $v$ at time $t$

Define time-homogeneous, local dynamics on $G$.

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

Adjacency matrix: $H = \sum_{(u,v) \in E(G)} |u\rangle \langle v|$
Momentum states

Consider an infinite path:

... -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 ...

Hilbert space: \( \text{span}\{ |x\rangle : x \in \mathbb{Z} \} \)

Eigenstates of the adjacency matrix: \( |\tilde{k}\rangle \) with
\[
\langle x | \tilde{k} \rangle := e^{ikx} \quad k \in [-\pi, \pi)
\]

We have \( \langle x | A | \tilde{k} \rangle = \langle x - 1 | \tilde{k} \rangle + \langle x + 1 | \tilde{k} \rangle \)
\[
= e^{ik(x-1)} + e^{ik(x+1)}
\]
\[
= (2 \cos k) \langle x | \tilde{k} \rangle
\]
so this is an eigenstate with eigenvalue \( 2 \cos k \).
Wave packets

A wave packet is a normalized state with momentum concentrated near a particular value $k$.

Example: \[ \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{-i k x} |x\rangle \quad \text{(large $L$)} \]

Propagation speed: \[ \left| \frac{dE}{dk} \right| = 2|\sin k| \]
Scattering on graphs

Now consider adding semi-infinite lines to two vertices of an arbitrary finite graph.

Before:

After:
The S-matrix

This generalizes to any number $N$ of semi-infinite paths attached to any finite graph.

Incoming wave packets of momentum near $k$ are mapped to outgoing wave packets (of the same momentum) with amplitudes corresponding to entries of an $N \times N$ unitary matrix $S(k)$, called the S-matrix.
Single-particle universality

Physical Review Letters 102, 180501 (2009)
Encoding a qubit

Encode quantum circuits into graphs.

Computational basis states correspond to paths ("quantum wires").

For one qubit, use two wires ("dual-rail encoding"): 

Fix some value of the momentum (e.g., $k = \pi/4$).

Quantum information propagates from left to right at constant speed.
Implementing a gate

To perform a gate, design a graph whose $S$-matrix implements the desired transformation $U$ at the momentum used for the encoding.

$$S(k) = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$$
Universal set of single-qubit gates

\[
\begin{pmatrix}
1 & 0 \\
0 & \sqrt{i}
\end{pmatrix}
\]

\[
-\frac{1}{\sqrt{2}} \begin{pmatrix}
i & 1 \\
1 & i
\end{pmatrix}
\]

momentum for logical states: \( k = \pi/4 \)
Single-particle universality construction

With an appropriate encoding of $n$-qubit states, two-qubit gates are trivial.

Implement sequences of gates by concatenation.

Any $n$-qubit circuit can be simulated by some graph. The number of vertices is (necessarily) exponential in $n$. 

![Diagram of quantum circuit](image-url)
Multi-particle universality
Multi-particle quantum walk

With $m$ distinguishable particles:

states: $|v_1, \ldots, v_m\rangle \quad v_i \in V(G)$

Hamiltonian: $H_G^{(m)} = \sum_{i=1}^{m} \sum_{(u,v) \in E(G)} |u\rangle\langle v|_i + \mathcal{U}$

Indistinguishable particles:
- bosons: symmetric subspace
- fermions: antisymmetric subspace

Many possible interactions:
- on-site: $\mathcal{U} = J \sum_{v \in V(G)} \hat{n}_v (\hat{n}_v - 1)$
- nearest-neighbor: $\mathcal{U} = J \sum_{(u,v) \in E(G)} \hat{n}_u \hat{n}_v$

states: $|v_1, \ldots, v_m\rangle \quad v_i \in V(G)$
Logical encoding

Encode each qubit in a single-particle state as before:

\[ \begin{array}{c}
\text{encoded } |0\rangle \\
\text{encoded } |1\rangle
\end{array} \]

For \( n \) qubits, use \( n \) particles on \( 2n \) paths.

Perform single-qubit gates as before.

How to perform interactions?
Two-particle scattering

In general, multi-particle scattering is complicated.

But scattering of indistinguishable particles on an infinite path is simple.

Before:

\( k \rightarrow p \)

After:

\( e^{i\theta} \times \)

Phase \( \theta \) depends on momenta and interaction details.
Momentum switch

To selectively induce the two-particle scattering phase, we route particles depending on their momentum.

Particles with momentum $\pi/4$ follow the single line.

Particles with momentum $\pi/2$ follow the double line.
Controlled phase gate

Computational qubits have momentum $\pi/4$. Introduce a “mediator qubit” with momentum $\pi/2$. We can perform an entangling gate with the mediator qubit.

$$C\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}$$
Canonical form of a circuit

For almost all $\theta$, we can use $C\theta$ to do

$$CP = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}.$$

Any circuit can be implemented using

- Single-qubit gates on computational qubits
- $CP$ gates between a computational and the mediator qubit
- Hadamard gates on the mediator qubit

Proof:

\[ |0\rangle \quad \quad |0\rangle \quad \quad |0\rangle \quad \quad H \quad H \quad H \quad H \quad |0\rangle \]
Hadamard on mediator qubit

[Blumer-Underwood-Feder 11]
Example
Error bound

Initial state: each particle is a square wave packet of length $L$

Consider a $g$-gate, $n$-qubit circuit:

2($n + 1$) paths, $O(gL)$ vertices on each path
Evolution time $O(gL)$
Total # of vertices $O(ngL)$

Theorem: The error can be made arbitrarily small with $L = \text{poly}(n, g)$.

Example: For Bose-Hubbard model, $L = O(n^{12}g^4)$ suffices.
Proof ideas
Approximating wave packet scattering

Analysis of single-particle construction: method of stationary phase

Instead, we directly prove that, with long enough incoming square wave packets (length $L$), the outgoing wave packets are well-approximated by the effect of the S-matrix (error $O(L^{-1/4})$).

Theorem 1:

Theorem 2:
Truncation

**Truncation Lemma:** If, for all times of interest, the state is well-approximated by one that is well-localized to some region, then changing how the Hamiltonian acts outside that region has little effect.

This lets us

- approximate finite graphs by infinite ones
- approximate the evolution by piecewise scattering through separate gate gadgets
- focus exclusively on one- and two-particle scattering

**Example:** For small enough times,
Refinements and extensions
Planarity

Planar Hadamard:

Put a mediator between every pair of computational qubits and unwind pairs of $C\theta$ gates:
Distinguishable particles

For two distinguishable particles on a long path, reflected and transmitted states are distinct:

\[ k \rightarrow \quad \leftarrow p \quad \rightarrow T \quad \leftarrow p \quad \rightarrow k \rightarrow \quad \leftarrow p \quad k \rightarrow \]

Then the effect of scattering on a long path is not just to accumulate a phase.

But with an appropriate choice of parameters, can ensure (say) \( R = 0 \).
Open questions

• Improved error bounds
• Simplified initial state
• Are generic interactions universal for distinguishable particles?
• New quantum algorithms
• Experiments
• Fault tolerance