Universal computation by multi-particle quantum walk

Andrew Childs   David Gosset   Zak Webb

Institute for Quantum Computing

University of Waterloo

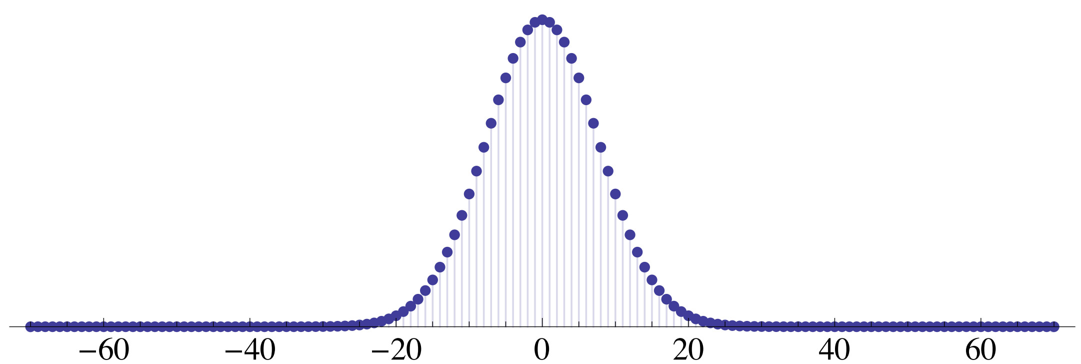
arXiv:1205.3782

Science 339, 791-794 (2013)

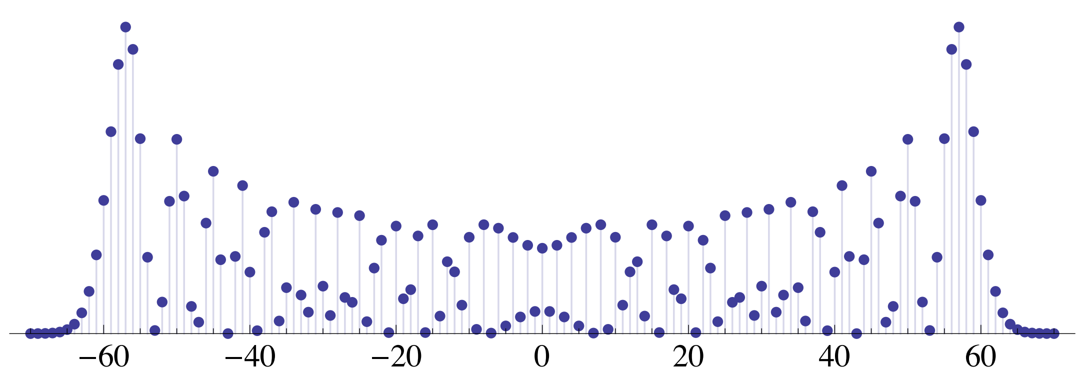
Quantum walk

Quantum analog of a random walk on a graph.

Idea: Replace probabilities by quantum amplitudes.
Interference can produce radically different behavior!



classical



quantum

Quantum walk algorithms

Quantum walk is a major tool for quantum algorithms (especially query algorithms with polynomial speedup).

- Exponential speedup for black-box graph traversal [CCDFGS 02]
- Quantum walk search framework [Szegedy 05], [Magniez et al. 06]
  - Spatial search [Shenvi-Kempe-Whaley 02], [CG 03, 04], [Ambainis-Kempe-Rivosh 04]
  - Element distinctness [Ambainis 03]
  - Subgraph finding [Magniez, Santha, Szegedy 03], [CK 10]
  - Matrix/group problems [Buhrman, Špalek 04], [Magniez, Nayak 05]
- Evaluating formulas/span programs
  - AND-OR formula evaluation [Farhi, Goldstone, Gutmann 07], [ACRŠZ 07]
  - Span programs for general query problems [Reichardt 09]
  - Learning graphs [Belovs 11] → new upper bounds (implicitly, quantum walk algorithms), new kinds of quantum walk search

Universality of quantum walk

Quantum walk can be efficiently simulated by a universal quantum computer.

N-vertex graph max degree $\mathrm{poly}(\log N)$ efficiently computable neighbors	$\Rightarrow$	circuit with $\mathrm{poly}(\log N)$ qubits $\mathrm{poly}(\log N)$ gates

Conversely, quantum walk is a *universal computational primitive*: any quantum circuit can be simulated by a quantum walk. [C 09]

N dimensions g gates	$\Rightarrow$	graph with $\mathrm{poly}(N, g)$ vertices max degree 3 walk for time $\mathrm{poly}(g)$

Note: The graph is necessarily exponentially large in the number of qubits! Vertices represent basis states.

Quantum walk experiments

Multi-particle quantum walk

With many walkers, the Hilbert space can be much bigger.

m distinguishable particles on an n-vertex graph: n^m dimensions

(similar scaling for indistinguishable bosons/fermions)

Main result: Any n-qubit, g-gate quantum circuit can be simulated by a multi-particle quantum walk of $n+1$ particles interacting for time $\mathrm{poly}(n,g)$ on a graph with $\mathrm{poly}(n,g)$ vertices.

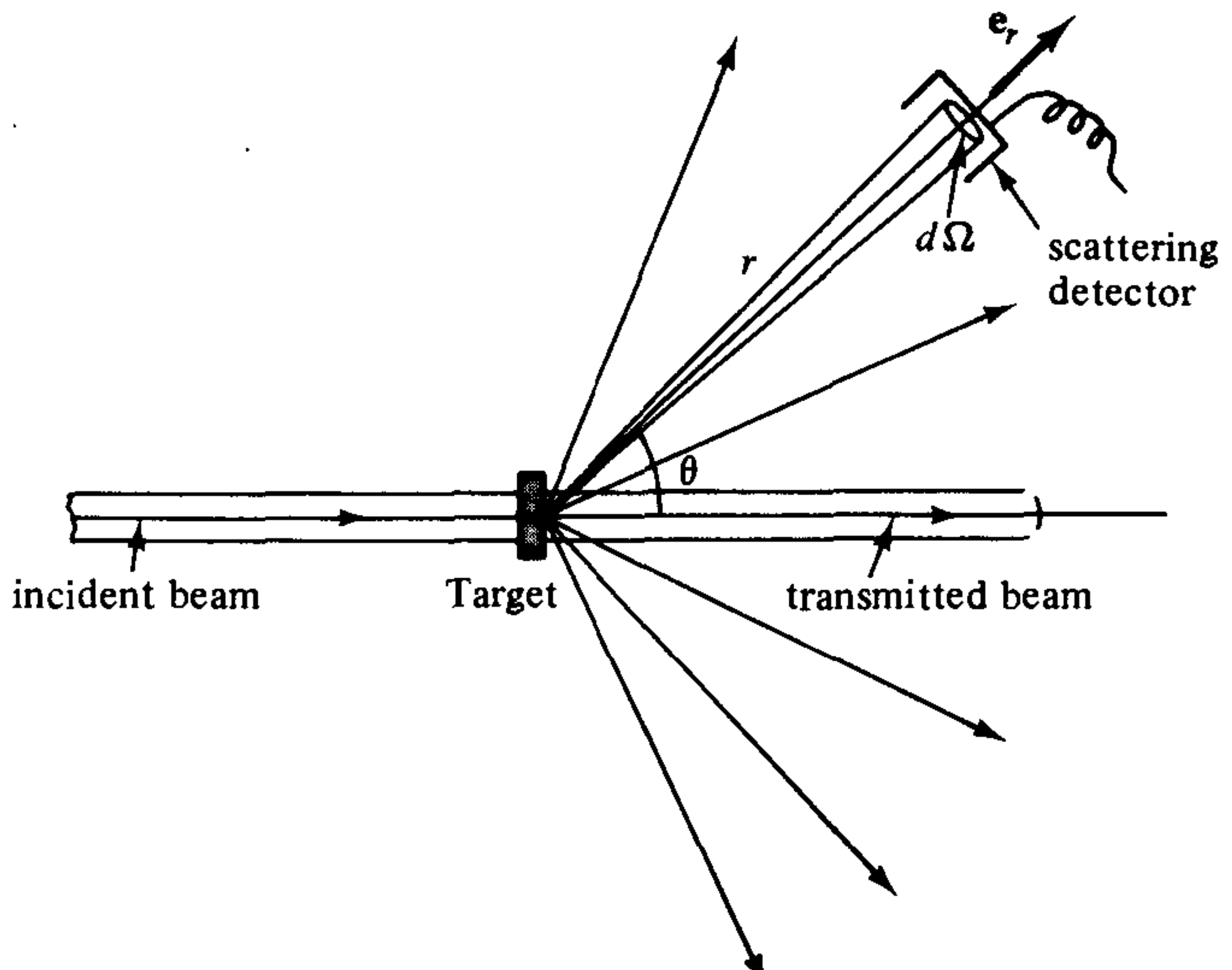
Consequences:

- Architecture for a quantum computer with no time-dependent control
- Simulating interacting many-body systems is BQP-hard (e.g., Bose-Hubbard model on a sparse, unweighted, planar graph)

Outline

- Scattering theory on graphs
- Single-particle universality (review)
- Multi-particle universality
- Proof ideas
- Refinements and extensions
- Open questions

Scattering theory on graphs



[Liboff, *Introductory Quantum Mechanics*]

Quantum walk

Quantum analog of a random walk on a graph $G = (V, E)$.

Idea: Replace probabilities by quantum amplitudes.

$$|\psi(t)\rangle = \sum_{v \in V} a_v(t) |v\rangle$$

amplitude for vertex v at time t

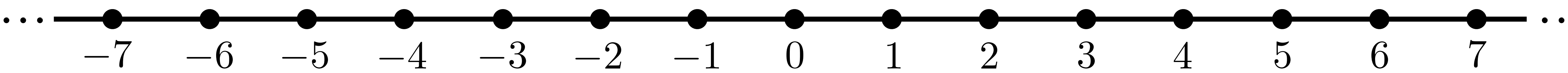
Define time-homogeneous, local dynamics on G.

$$\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H |\psi(t)\rangle$$

Adjacency matrix: $H = \sum_{(u,v) \in E(G)} |u\rangle\langle v|$

Momentum states

Consider an infinite path:



Hilbert space: $\mathrm{span}\{|x\rangle : x \in \mathbb{Z}\}$

Eigenstates of the adjacency matrix: $|\tilde{k}\rangle$ with

$$\langle x|\tilde{k}\rangle := e^{\mathrm{i}kx} \qquad k \in [-\pi, \pi)$$

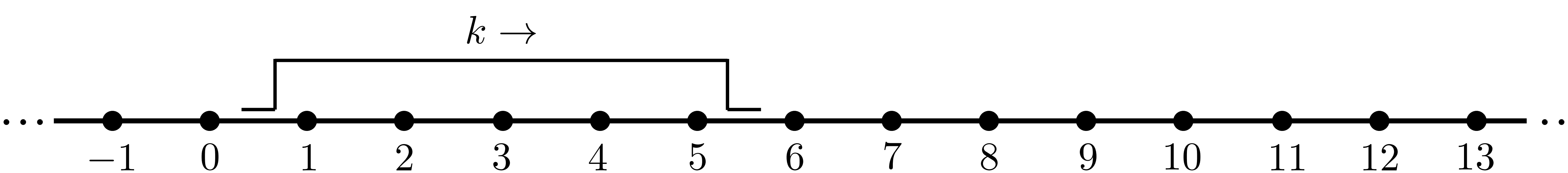
We have
$$\begin{aligned} \langle x|A|\tilde{k}\rangle &= \langle x-1|\tilde{k}\rangle + \langle x+1|\tilde{k}\rangle \\ &= e^{\mathrm{i}k(x-1)} + e^{\mathrm{i}k(x+1)} \\ &= (2\cos k)\langle x|\tilde{k}\rangle \end{aligned}$$

so this is an eigenstate with eigenvalue $2\cos k$.

Wave packets

A *wave packet* is a normalized state with momentum concentrated near a particular value k.

Example: $$\frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{-\mathrm{i}kx} |x\rangle \qquad \text{(large } L\text{)}$$

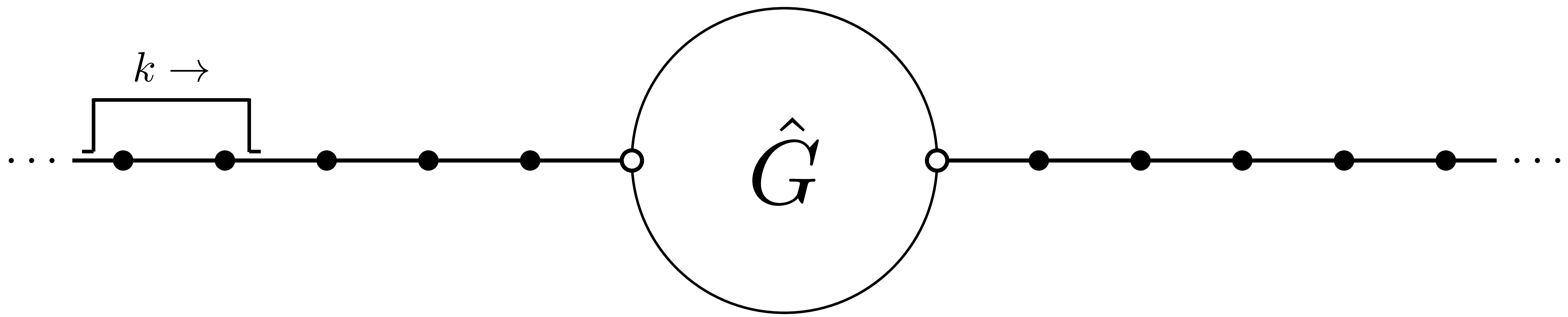


Propagation speed: $$\left|\frac{\mathrm{d}E}{\mathrm{d}k}\right| = 2|\sin k|$$

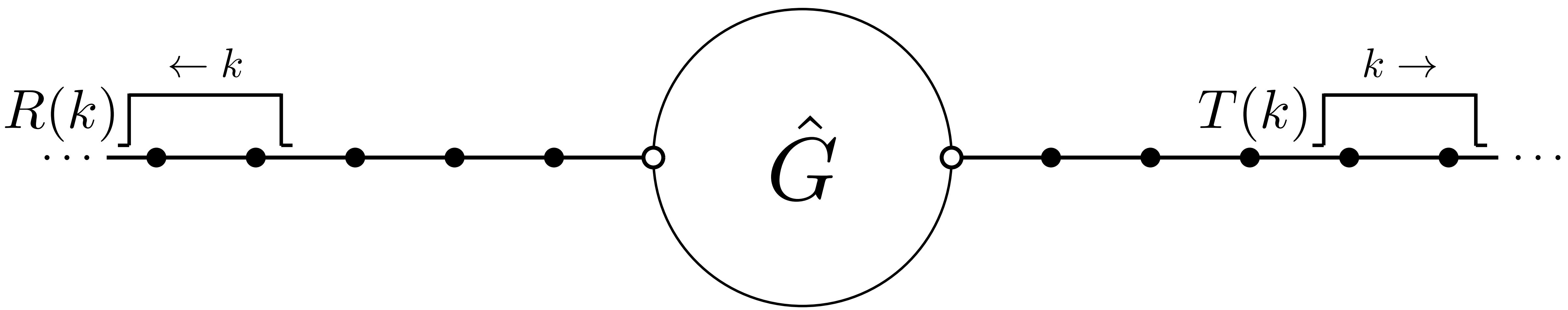
Scattering on graphs

Now consider adding semi-infinite lines to two vertices of an arbitrary finite graph.

Before:

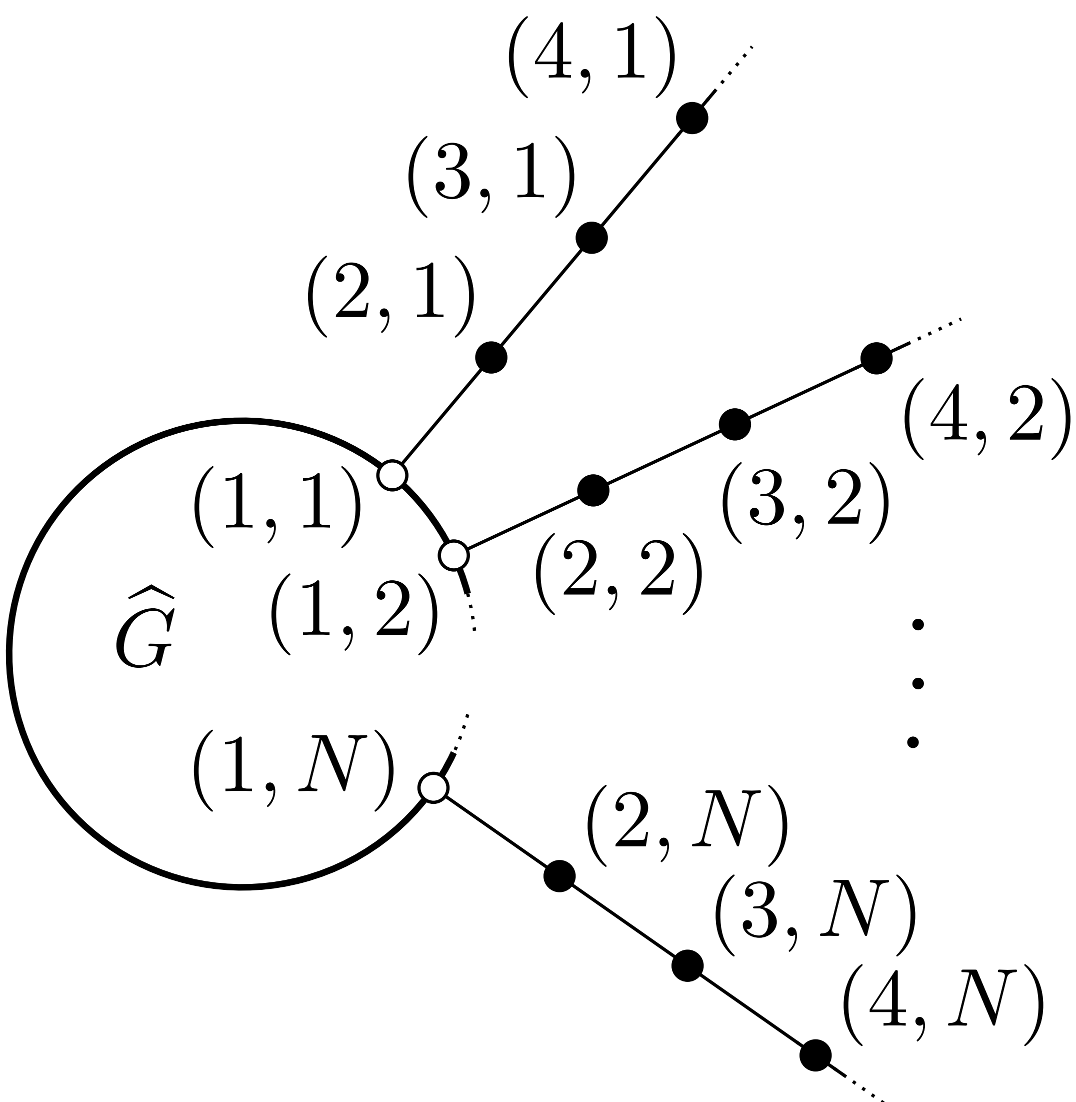


After:



The S-matrix

This generalizes to any number N of semi-infinite paths attached to any finite graph.



Incoming wave packets of momentum near k are mapped to outgoing wave packets (of the same momentum) with amplitudes corresponding to entries of an $N \times N$ unitary matrix $S(k)$, called the S-matrix.

Single-particle universality

arXiv:0806.1972

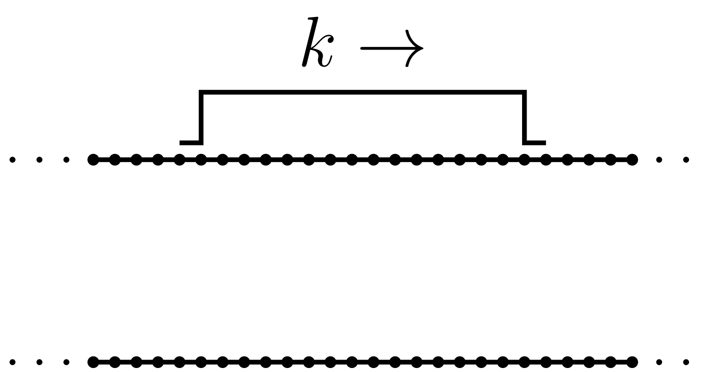
Physical Review Letters 102, 180501 (2009)

Encoding a qubit

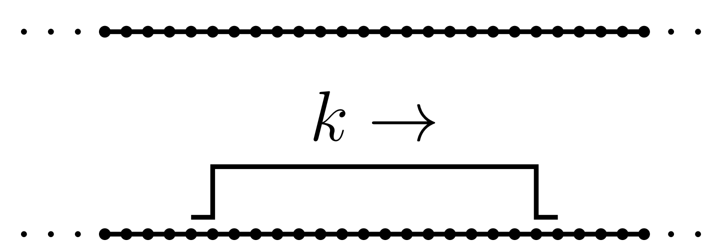
Encode quantum circuits into graphs.

Computational basis states correspond to paths ("quantum wires").

For one qubit, use two wires ("dual-rail encoding"):



encoded $|0\rangle$



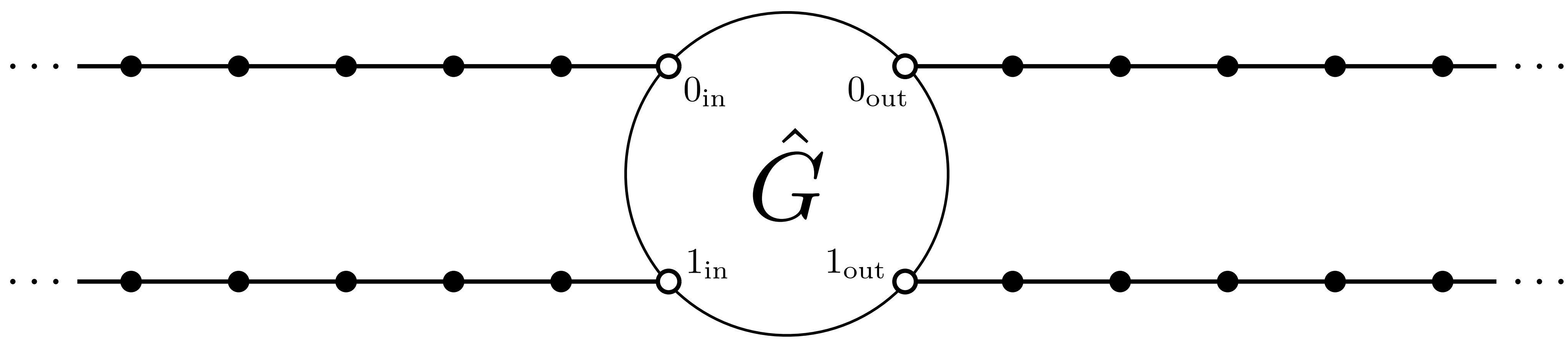
encoded $|1\rangle$

Fix some value of the momentum (e.g., $k = \pi/4$).

Quantum information propagates from left to right at constant speed.

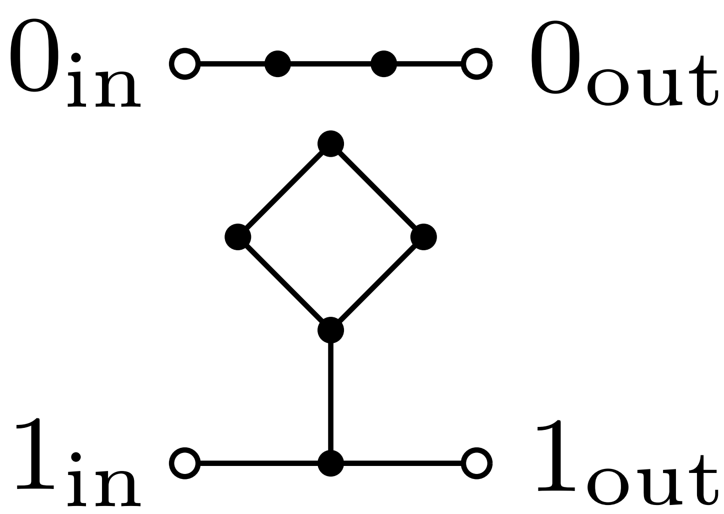
Implementing a gate

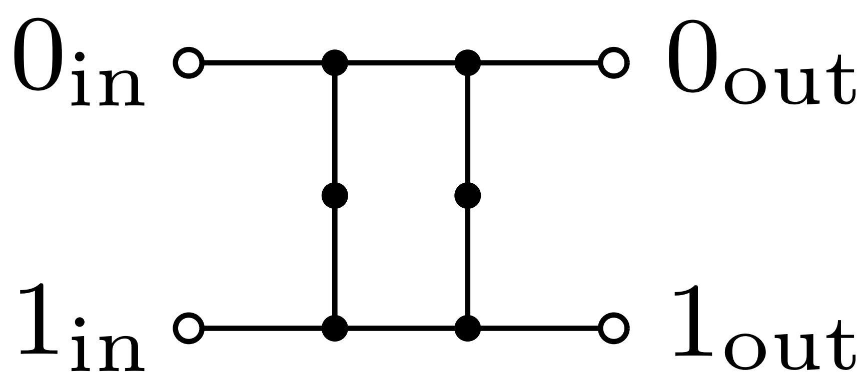
To perform a gate, design a graph whose S-matrix implements the desired transformation U at the momentum used for the encoding.



$$S(k) = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$$

Universal set of single-qubit gates





$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\mathrm{i}} \end{pmatrix}$$

$$-\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix}$$

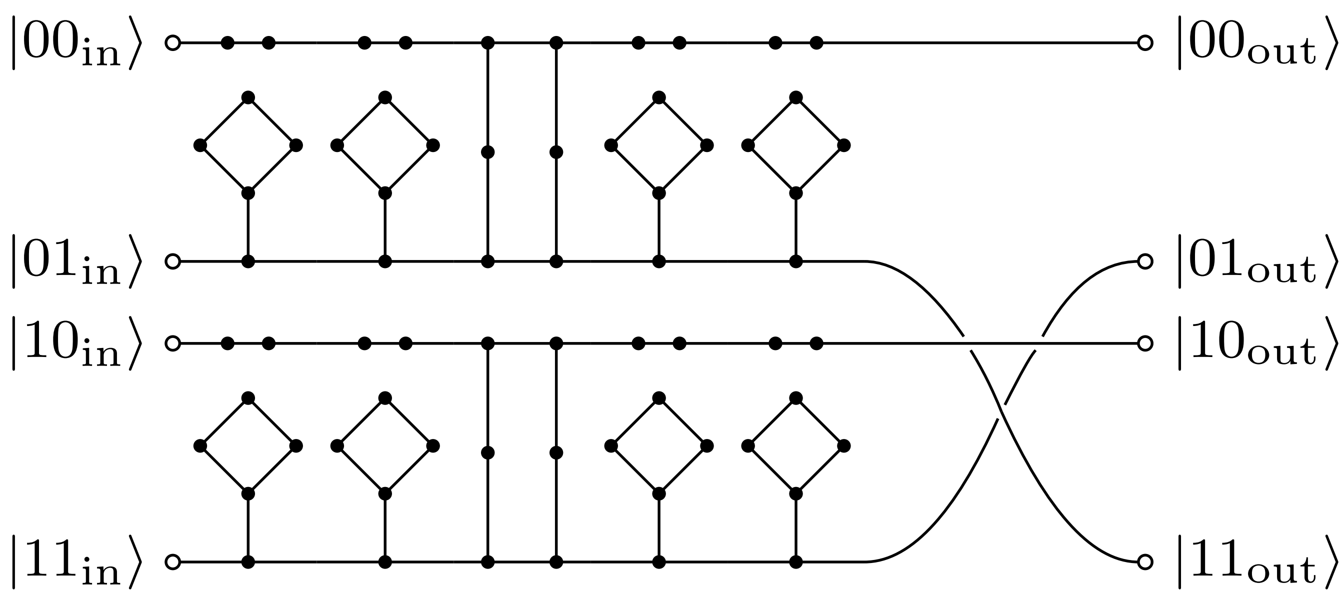
momentum for logical states: $k = \pi/4$

Single-particle universality construction

With an appropriate encoding of n-qubit states, two-qubit gates are trivial.

Implement sequences of gates by concatenation.

Any n-qubit circuit can be simulated by some graph. The number of vertices is (necessarily) exponential in n.



Multi-particle universality

Multi-particle quantum walk

With m distinguishable particles:

states: $|v_1, \ldots, v_m\rangle \qquad v_i \in V(G)$

Hamiltonian: $H_G^{(m)} = \sum_{i=1}^{m} \sum_{(u,v)\in E(G)} |u\rangle\langle v|_i + \mathcal{U}$

Indistinguishable particles:

bosons: symmetric subspace

fermions: antisymmetric subspace

Many possible interactions:

on-site: $\mathcal{U} = J \sum_{v\in V(G)} \hat{n}_v(\hat{n}_v - 1) \qquad \hat{n}_v = \sum_{i=1}^{m} |v\rangle\langle v|_i$

nearest-neighbor: $\mathcal{U} = J \sum_{(u,v)\in E(G)} \hat{n}_u \hat{n}_v$

Logical encoding

Encode each qubit in a single-particle state as before:

encoded $|0\rangle$ encoded $|1\rangle$

For n qubits, use n particles on $2n$ paths.

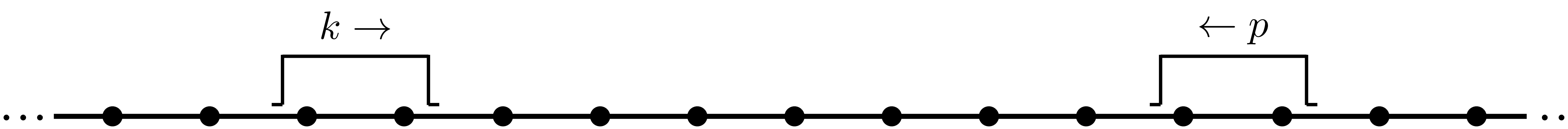
Perform single-qubit gates as before.
How to perform interactions?

Two-particle scattering

In general, multi-particle scattering is complicated.

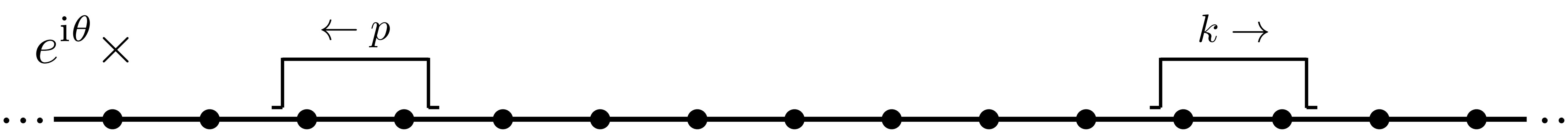
But scattering of indistinguishable particles on an infinite path is simple.

Before:



After:

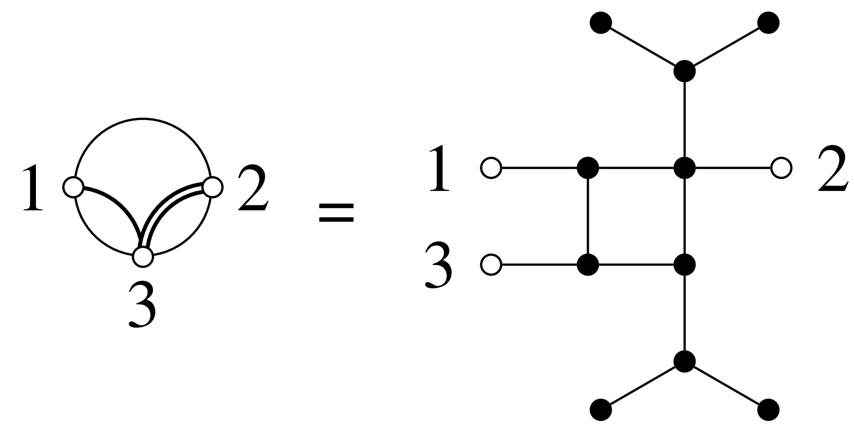
$e^{\mathrm{i}\theta} \times$



Phase θ depends on momenta and interaction details.

Momentum switch

To selectively induce the two-particle scattering phase, we route particles depending on their momentum.

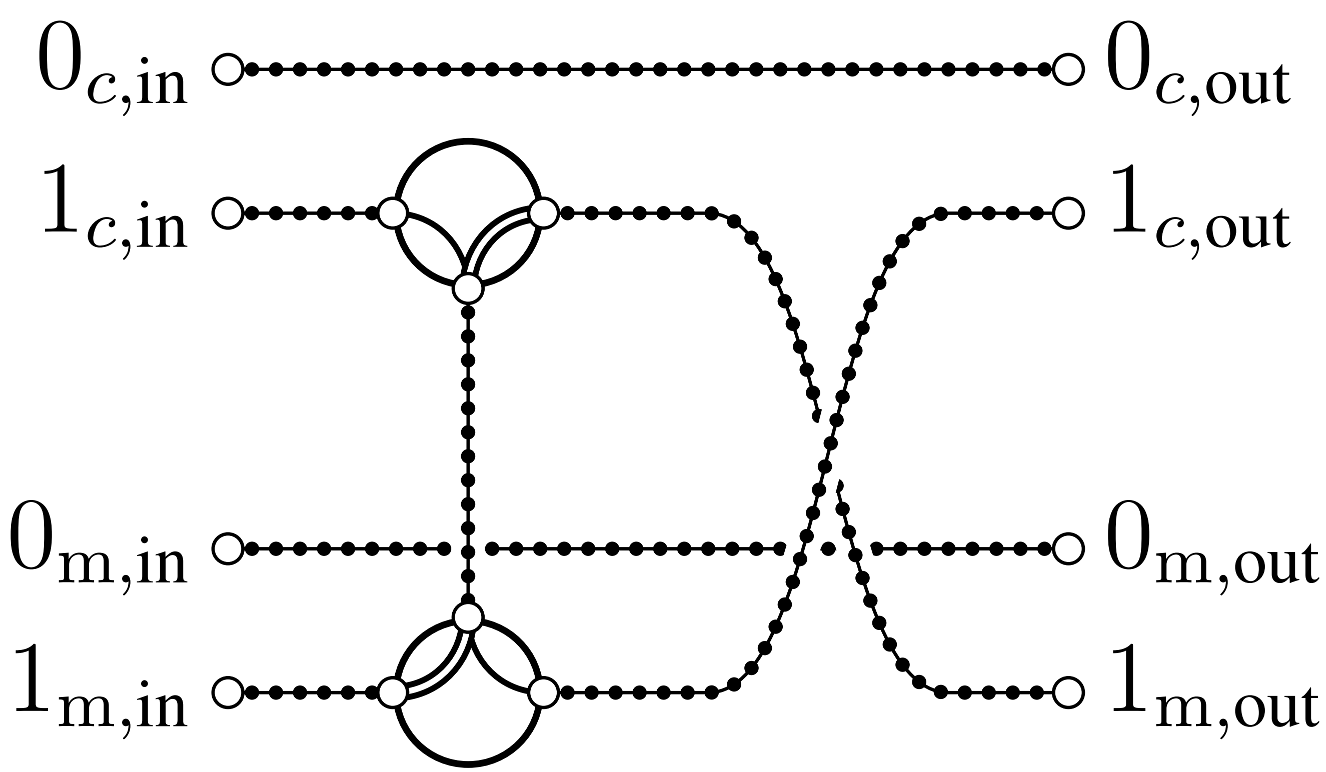


Particles with momentum $\pi/4$ follow the single line.

Particles with momentum $\pi/2$ follow the double line.

Controlled phase gate

Computational qubits have momentum $\pi/4$. Introduce a "mediator qubit" with momentum $\pi/2$. We can perform an entangling gate with the mediator qubit.



$$C\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}$$

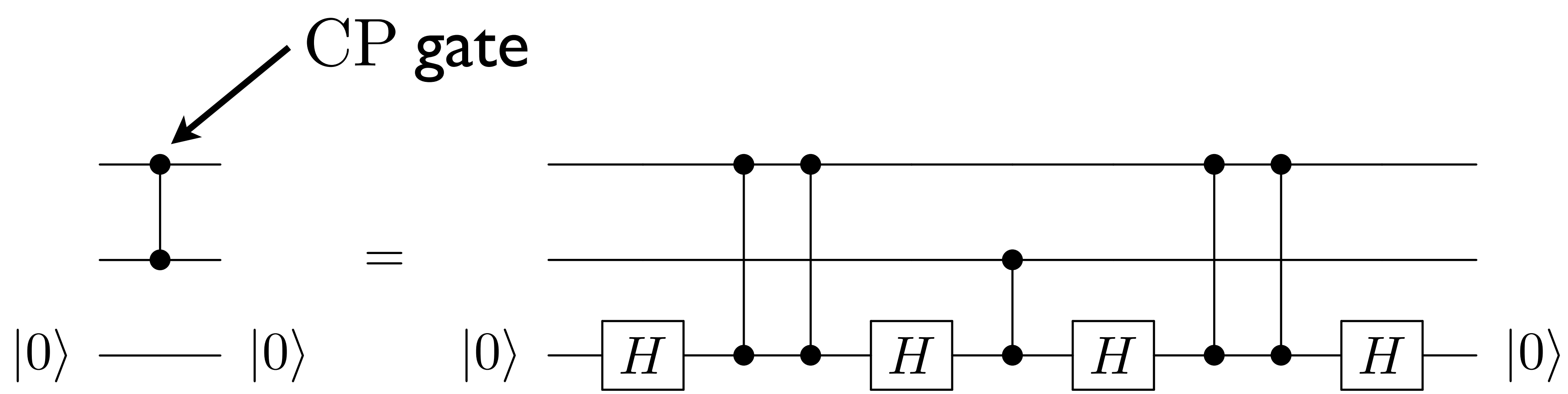
Canonical form of a circuit

For almost all θ, we can use $\mathrm{C}\theta$ to do $\mathrm{CP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}$.

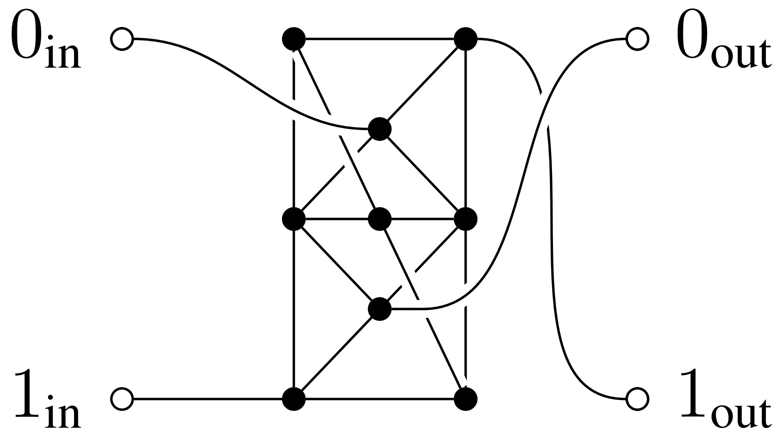
Any circuit can be implemented using

- Single-qubit gates on computational qubits
- CP gates between a computational and the mediator qubit
- Hadamard gates on the mediator qubit

Proof:

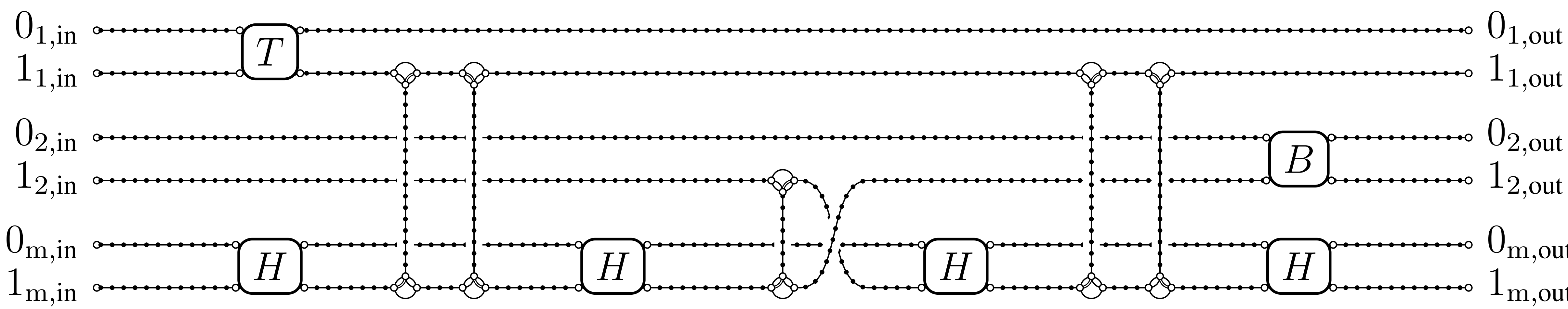


Hadamard on mediator qubit



[Blumer-Underwood-Feder 11]

Example

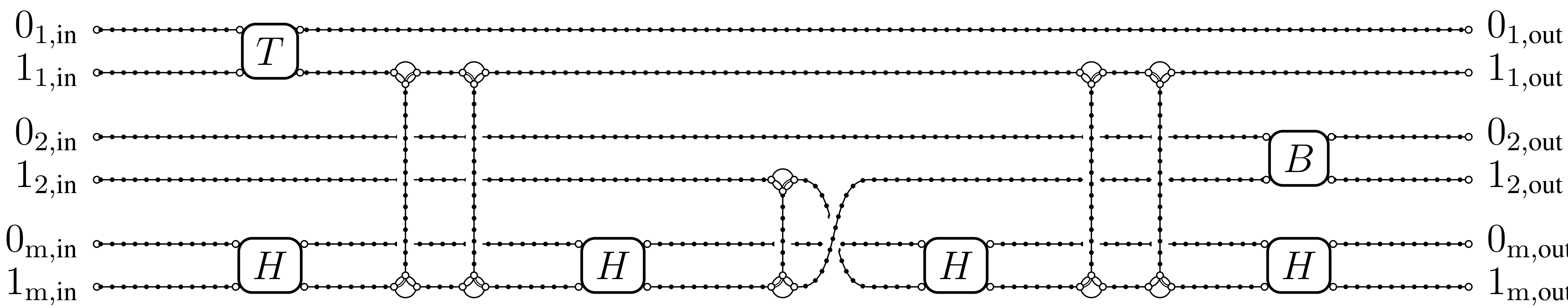


$0_{1,\mathrm{in}}$
$1_{1,\mathrm{in}}$
$0_{2,\mathrm{in}}$
$1_{2,\mathrm{in}}$
$0_{\mathrm{m,in}}$
$1_{\mathrm{m,in}}$
T
H
H
H
B
H
$0_{1,\mathrm{out}}$
$1_{1,\mathrm{out}}$
$0_{2,\mathrm{out}}$
$1_{2,\mathrm{out}}$
$0_{\mathrm{m,out}}$
$1_{\mathrm{m,out}}$

Error bound

Initial state: each particle is a square wave packet of length L

Consider a g-gate, n-qubit circuit:



$2(n+1)$ paths, $O(gL)$ vertices on each path

Evolution time $O(gL)$

Total # of vertices $O(ngL)$

Theorem: The error can be made arbitrarily small with $L = \text{poly}(n, g)$.

Example: For Bose-Hubbard model, $L = O(n^{12} g^4)$ suffices.

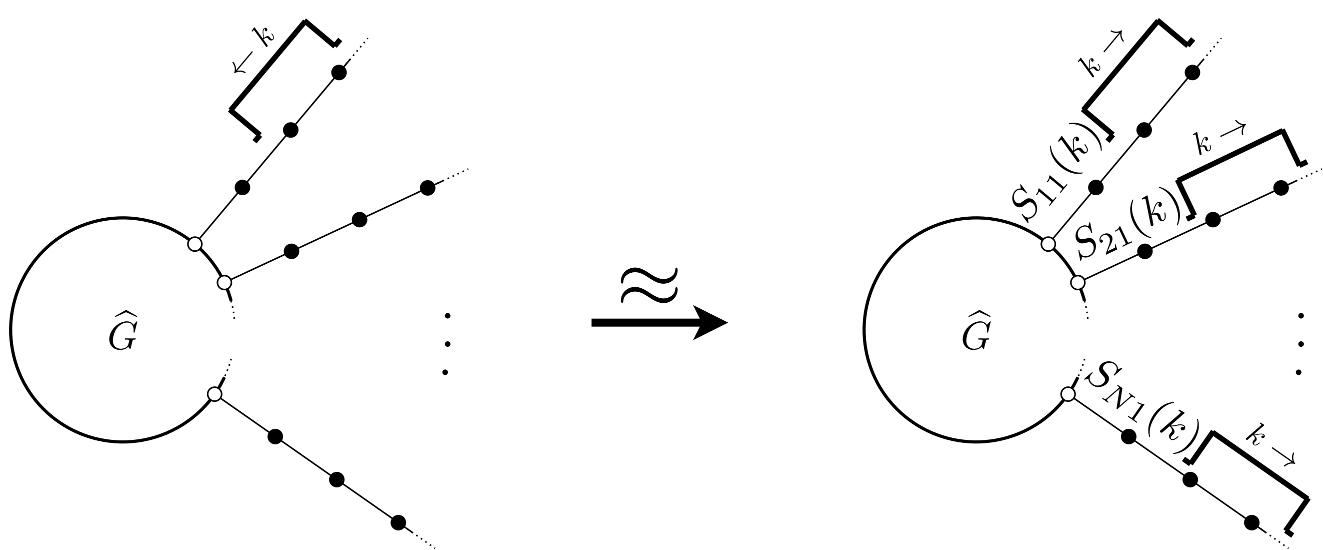
Proof ideas

Approximating wave packet scattering

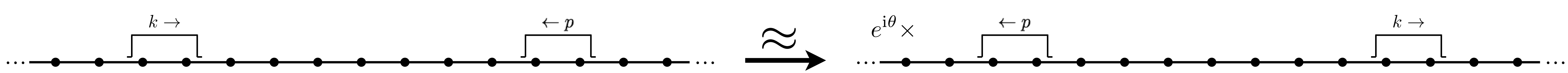
Analysis of single-particle construction: method of stationary phase

Instead, we directly prove that, with long enough incoming square wave packets (length L), the outgoing wave packets are well-approximated by the effect of the S-matrix (error $O(L^{-1/4})$).

Theorem 1:



Theorem 2:



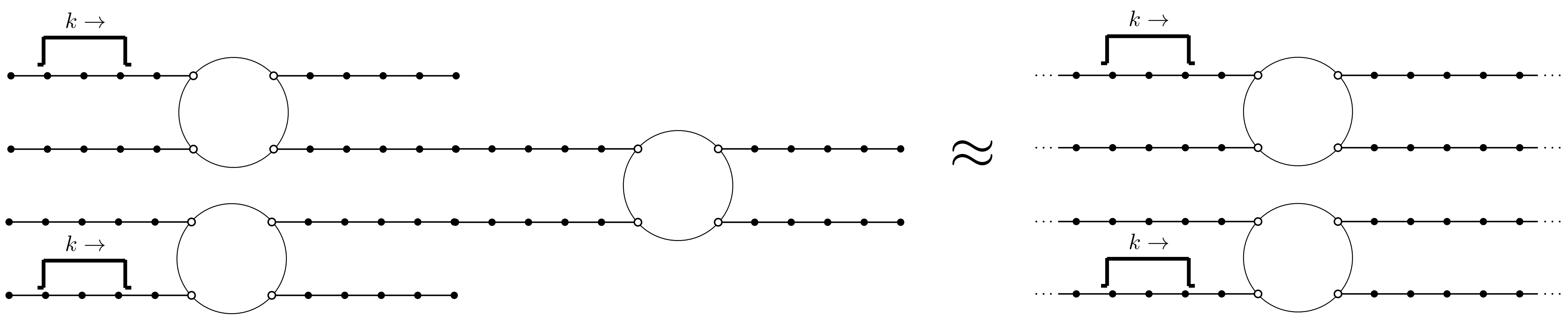
Truncation

Truncation Lemma: If, for all times of interest, the state is well-approximated by one that is well-localized to some region, then changing how the Hamiltonian acts outside that region has little effect.

This lets us

- approximate finite graphs by infinite ones
- approximate the evolution by piecewise scattering through separate gate gadgets
- focus exclusively on one- and two-particle scattering

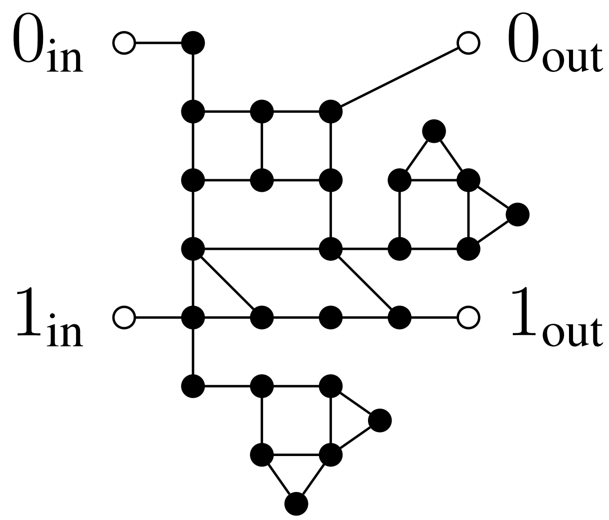
Example: For small enough times,



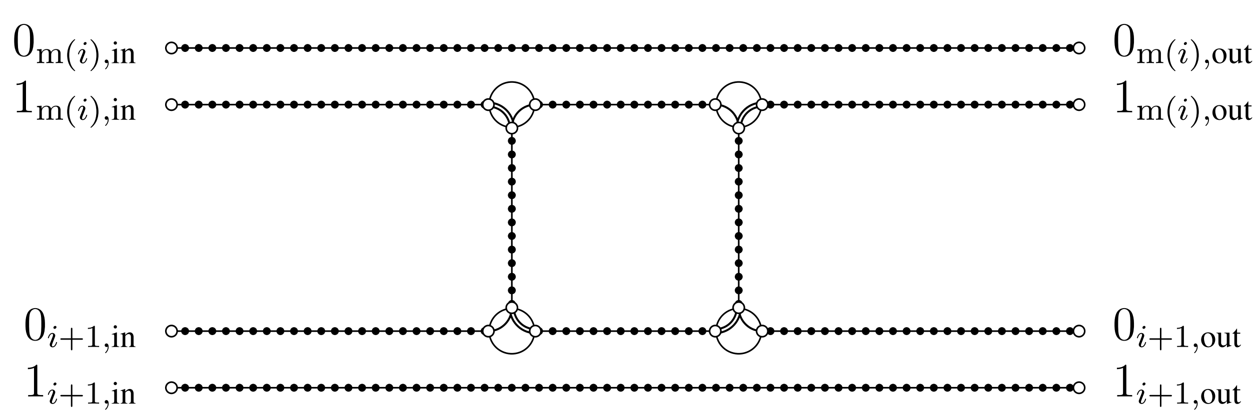
Refinements and extensions

Planarity

Planar Hadamard:

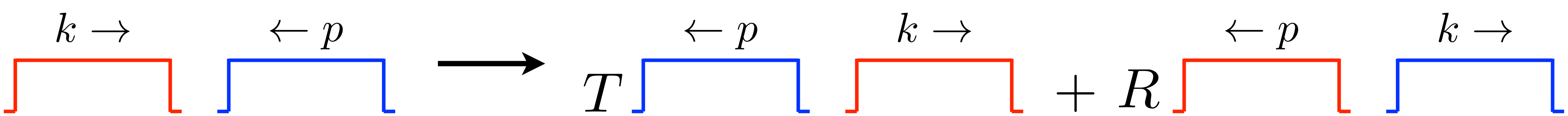


Put a mediator between every pair of computational qubits and unwind pairs of $\mathrm{C}\theta$ gates:



Distinguishable particles

For two distinguishable particles on a long path, reflected and transmitted states are distinct:



Then the effect of scattering on a long path is not just to accumulate a phase.

But with an appropriate choice of parameters, can ensure (say) $R = 0$.

Open questions

- Improved error bounds
- Simplified initial state
- Are generic interactions universal for distinguishable particles?
- New quantum algorithms
- Experiments
- Fault tolerance